Maximization of the Fundamental Power in Nonlinear Capacitance Diodes

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In this paper we consider the problem of determining the maximum fundamental power in a nonlinear capacitance diode, when the charge waveform has a given periodicity and (i) varies between prescribed maximum and minimum values, (ii) has a prescribed maximum and a prescribed maximum slope. Under (i) the maximum obtainable fundamental power is first determined. The charge waveform is then further restricted to contain no higher than second harmonics, so that the diode is being used as a frequency doubler, and the maximum power transfer is determined. The maximum power transfer is also determined under (ii). Particular diodes considered are the abrupt-junction and the graded-junction ones, with operation in the forward conduction region being permitted.

I. ENGINEER'S SUMMARY

This section of the paper is a summary which stresses some of the contents of the introduction and summary that follow. It is hoped that this will make it easier for the engineer who is involved in parametric amplifier and varactor design to deduce the relevant applications of the results contained in this paper.

In the first instance it should be emphasized that an idealized problem, based on a mathematical model, is considered. The nonlinear capacitor is assumed to be isolated from any external circuits, and we do not discuss how the power is fed into or taken from the device. Clearly there will be some power lost in the external circuit, and the maximum obtainable fundamental power determined in this paper is only a theoretical maximum, but it would seem to be worthwhile to understand this theoretical maximum. When the maximum power transfer from the first to the second harmonic is considered, the charge waveform, and hence the current, giving this maximum is determined. Clearly there is some relative phase between the first and second harmonics in the current, and the reactance of the output circuit must be adjusted so as to obtain this relative phase.

It is also important to stress that some of the results obtained hold for a general, i.e., arbitrary single-valued, voltage-charge relationship, and are accordingly applicable to any particular such voltage-charge relationship in which the engineer may be interested. We have, for simplicity, considered just the abrupt-junction and the graded-junction diodes as special cases, and have idealized the voltage-charge relationship in the forward conduction region, but other particular diodes can be considered as special cases of the general results. We discuss below the results which are pertinent to the general voltage-charge relationship.

Firstly, we have derived the functional form of the charge waveform (of given periodicity and varying between prescribed values) which gives the maximum power in the fundamental. The charge waveform is composed (see (33) below) of intervals in which it takes on either the maximum or minimum prescribed value, or else follows a certain curve. The form of the curve depends on the voltage-charge relationship and involves parameters which are functionals of the charge waveform throughout the entire period, and hence are not known a priori. These parameters have to be determined for each particular voltage-charge relationship, by solving simultaneous transcendental equations. It is also necessary to allow for finite jumps in the charge waveform, and (36) below must hold at such a jump. Of course, a jump is not physically realizable, since it would correspond to an infinite current, and this makes it evident that the maximum is a theoretical one, quite apart from losses in the external circuit. It does, however, provide an upper bound on the maximum realizable fundamental power.

In view of the fact that the maximum fundamental power has to be determined separately for each specific diode, we derive upper and lower bounds for the maximum fundamental power, (11) to (13), which apply to a general voltage-charge relationship. For a wide class, the ratio of the upper to the lower bound is 1.54. It turns out that, for the particular diodes considered, the lower bound is quite close to the actual value. Further use is made of the charge waveform giving this lower bound, when the power transfer from the fundamental to the second harmonic is considered, subject to the charge waveform containing no higher than second harmonics. A good approximation to the maximum power transfer is obtained by taking the Fourier approximation, up to second harmonics, and suitably normalizing so that the approximating charge waveform has the prescribed maximum and minimum values.

In connection with maximizing the power transfer from the fundamental to the second harmonic, we consider the diode to be a harmonic generator, there being input power in the fundamental only. In order to make the mathematical problem more tractable, it is supposed that the entire output is in the second harmonic. Equations (18) and (19) simply state that the maximum power output in the second harmonic, when there is input power in the fundamental only, is not greater than the maximum obtainable fundamental power without such restrictions, and is not less than the maximum fundamental power when there is no output or input power in the third and higher harmonics. It is assumed here that the charge waveform is continuous. We have already discussed the maximum obtainable fundamental power.

The problem of determining the maximum fundamental power when there is no output or input power in the third and higher harmonics is still not very tractable, without additional restrictions on the charge waveform, and it is thus further supposed that the charge waveform contains no higher than second harmonics. The maximum subject to this additional restriction is obviously not greater than the maximum without it. The significant point about this restriction is that there is then no power output or input in the third and higher harmonics, whatever the voltage-charge relationship. We thus determine a canonical representation of the charge waveform which contains no higher than second harmonics and has prescribed maximum and minimum values. By suitable choice of the time origin, this representation contains just two parameters which lie in a bounded region.

Now, it is a straightforward matter to compute numerically the fundamental power for any given voltage-charge relationship and a given charge waveform. The numerical maximization of this power with respect to the two parameters in the above canonical representation is also a straightforward process. Thus it is clear that the above procedure has general applicability. We add that in the numerical maximization process, the two parameters which give the approximating charge waveform (obtained from the charge waveform giving the good lower bound to the maximum obtainable fundamental power) are used for starting values.

Consideration is also given to the current-limited diode, in which the charge waveform has a prescribed maximum value and a prescribed maximum slope (corresponding to maximum current magnitude). Again, we determine a two-parameter canonical representation for the charge waveform containing no higher than second harmonics, and the numerical maximization of the fundamental power, for any given voltage-charge relationship, proceeds along the same lines as in the previous case, except that we no longer have predetermined starting values for the two parameters. Lack of space has prevented inclusion of the determination of the functional form of the charge waveform which gives the maximum obtainable fundamental power (without restriction on the harmonic content of the charge waveform) in the current-limited case.

II. INTRODUCTION AND SUMMARY

2.1 *Introduction*

We will be concerned with various nonlinear capacitance diodes, these being characterized by a nonlinear voltage-charge relationship. Specific examples are the abrupt-junction diode and the graded-junction diode, which are composed of diffused p-n junctions. In the former case the voltage difference, v, across the diode is proportional to the square of the stored charge (per unit area), q, i.e., $v \propto q^2$, while in the latter case $v \propto q^{\frac{3}{2}}$, provided, in both cases, that $q \geqq 0$, which implies that operation of the diode does not take place in the forward conduction region. Now as electric field strength and barrier width increase, creation of electron-hole pairs through secondary impact ionization by both holes and electrons leads to avalanche multiplication, resulting finally in an effectively infinite increase of current with added applied voltage, and this is termed reverse breakdown. There is thus a maximum voltage $v_{\max}$, and a corresponding maximum value $q_{\max}$ of the charge density (which may be related to $v_{\max}$ through the actual voltage-charge relationship), above which it is not desirable to operate the diode.

We define the normalized voltage V and the normalized charge Q by

$$V = \frac{v}{v_{\max}}; \qquad Q = \frac{q}{q_{\max}}. \tag{1}$$

Hence the normalized voltage-charge relationships for the abrupt-junction and graded-junction diodes, operated in the region between forward conduction and reverse breakdown, are

$$V = \begin{cases} Q^2, & \text{(abrupt)} \\ Q^{\frac{3}{2}}, & \text{(graded)} \end{cases}, \quad 0 \leqq Q \leqq 1. \tag{2}$$

It is also possible to operate the diodes partially in the forward conduction region, corresponding to $Q < 0$. The voltage is not very dependent on the charge in this region and as an idealization we may assume that it is zero throughout. A physical restriction is placed on the maximum possible current magnitude, in that the electron velocity is limited by lattice scattering. Throughout most of our analysis we replace this condition by a limitation on the minimum charge, so that

$$q \geqq q_{\min} = -m(q_{\max}). \quad (3)$$

Thus, in the forward conduction region,

$$V = 0, \qquad -m \leqq Q \leqq 0. \quad (4)$$

We do, however, give some consideration to the current-limited diode in which, instead of (3),

$$| i | \leqq i_{\max} . \quad (5)$$

We will consider charge waveforms that are periodic in time, t, with angular frequency ω. We define the normalized time x and the normalized current I by

$$x = \omega t; \qquad I = \frac{i}{\omega q_{\max}} . \quad (6)$$

Thus $Q(x)$ is periodic in x with period 2π and, since $i = dq/dt$,

$$I = \frac{dQ}{dx} = Q'(x). \quad (7)$$

The average real and reactive powers (per unit area) in the nth harmonic, p_n and r_n , are given by

$$(p_n + jr_n) = \frac{1}{2}\left(\frac{\omega}{\pi}\right)^2 \left(\int_0^{2\pi/\omega} i\, e^{-j\omega nt}\, dt\right)\left(\int_0^{2\pi/\omega} v\, e^{j\omega nt}\, dt\right). \quad (8)$$

We define the normalized real and reactive powers in the nth harmonic, P_n and R_n , by

$$(P_n + jR_n) = \frac{2\pi^2(p_n + jr_n)}{\omega q_{\max} v_{\max}} . \quad (9)$$

We will be concerned with the maximization of the real fundamental power, under various conditions, and summarize the results below. We note that P_n is not affected by a time shift in the charge waveform, but it is reversed in sign by a time reversal of the waveform.

2.2 *The Maximum Obtainable Fundamental Power, When the Charge Waveform is Subject to Bounded Variation*

The functional form of the charge waveform which, subject to the restriction $-m \leqq Q(x) \leqq 1$, maximizes the fundamental power, P_1 , is found for the general voltage-charge relationship, $V = V(Q)$. The specific form is determined for diodes of interest and the corresponding value of max P_1, the maximum obtainable fundamental power, calculated. Thus, for the abrupt-junction diode operated in the region between forward conduction and reverse breakdown, (2), max $P_1 = 0.687$ and the charge waveform $Q(x)$ giving rise to this value is depicted in Fig. 1. The corresponding value of the reactive fundamental power is $R_1 = 2.43$. For the graded-junction diode, operated in the region between forward conduction and reverse breakdown, it is found that max $P_1 = 0.408$, with $R_1 = 2.48$. The charge waveform giving rise to these values is depicted in Fig. 2. The abrupt-junction diode is also considered when the region of operation includes forward conduction. Thus, from (2) and (4), $V(Q) = [\max(0,Q)]^2$, $-m \leqq Q(x) \leqq 1$. Fig. 4 depicts max P_1 and the corresponding R_1 as functions of m. The charge waveform $Q(x)$ which gives these values when $m = 1$ is shown in Fig. 5. The somewhat idealized voltage-charge relationship given by $V(Q) = \max(0,Q)$, $-m \leqq Q(x) \leqq 1$, $m > 0$, may be treated analytically. It is found in this case that

$$\max P_1 = \frac{3\sqrt{3}}{2}\, m; \qquad R_1 = \frac{3}{2}\,(m + 2). \tag{10}$$

The charge waveform giving these values is composed of $Q(x) = 1$, 0 and $-m$ in consecutive intervals of x of length $2\pi/3$.

It is observed that the charge waveform which gives rise to max P_1, for the various diodes, contains at least one discontinuity (or jump) in a period. A jump, of course, is not physically realizable, since it would correspond to an infinite current, so max P_1 cannot actually be attained.

Finally, upper and lower bounds are obtained on the maximum obtainable fundamental power, max P_1, for the general voltage-charge relationship $V = V(Q)$, with $-m \leqq Q(x) \leqq 1$. Thus, it is shown that

$$\frac{3\sqrt{3}}{2}\, L \leqq \max P_1 \leqq 4(1 + m)U, \tag{11}$$

where

$$L = \max_{-m \leqq (\sigma,\rho,\tau) \leqq 1} [(\rho - \tau)V(\sigma) + (\tau - \sigma)V(\rho) + (\sigma - \rho)V(\tau)]; \tag{12}$$

and

$$U = \min_{\lambda} \{ \max_{-m \leqq \sigma \leqq 1} [\lambda\sigma - V(\sigma)] - \min_{-m \leqq \sigma \leqq 1} [\lambda\sigma - V(\sigma)]\}. \tag{13}$$

Moreover, it is shown that

$$L \leqq (1 + m)U \leqq 2L. \tag{14}$$

The bounds in (14) cannot be improved without restriction on $V(Q)$, but if $[(\rho + m)V(1) - (m+1)V(\rho) + (1-\rho)V(-m)]$ does not change sign in $-m \leqq \rho \leqq 1$, then $L = (1+m)U$ and the ratio of the upper to the lower bound in (11) becomes 1.54. The class of voltage-charge relationships

$$V(Q) = [\max (0,Q)]^{\nu}, \quad -m \leqq Q \leqq 1; \qquad m \geqq 0, \quad \nu \geqq 1, \tag{15}$$

which includes the particular diodes considered, satisfies the above condition, and in this case

$$L = m + \left(1 - \frac{1}{\nu}\right)[(1+m)\nu]^{-1/(\nu-1)}. \tag{16}$$

For the particular cases considered, the lower bound in (11) is fairly close to max P_1.

A lower bound is also obtained, for a general voltage-charge relationship $V = V(Q)$, with $-m \leqq Q(x) \leqq 1$, for P_1 such that $P_1 + P_2 = 0$. It is shown that

$$\max [P_1 \mid P_1 + P_2 = 0] \geqq (1.87)L. \tag{17}$$

2.3 *The Maximization of the Power Transfer in a Frequency Doubler, With Bounded Charge Waveform*

Here we are interested in maximizing the power transfer from the fundamental to the second harmonic, when the diode is being used as a harmonic generator. Thus there must be input power at the fundamental frequency only, i.e., $P_1 > 0$ and $P_n \leqq 0$, $n \geqq 2$. In order to make the problem more tractable we suppose that the entire power output is put is in the second harmonic, so that $P_n = 0$, $n \geqq 3$. It follows that $P_1 + P_2 = 0$, provided that the charge waveform is continuous, since then $\sum_{n=1}^{\infty} P_n = 0$. We observe that

$$\begin{aligned} &\max [-P_2 \mid P_n \leqq 0, \quad n \geqq 3] \\ &\qquad \leqq \max [P_1 \mid P_n \leqq 0, \quad n \geqq 3] \leqq \max P_1, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \max [-P_2 \mid P_n \leqq 0, \quad n \geqq 3] &\geqq \max [-P_2 \mid P_n = 0, \quad n \geqq 3] \\ &= \max [P_1 \mid P_n = 0, \quad n \geqq 3]. \end{aligned} \tag{19}$$

Even the problem of determining max $[P_1 \mid P_n = 0, n \geqq 3]$, that is, max P_1 subject to $P_n = 0$, $n \geqq 3$, is not very tractable, without additional restrictions on the charge waveform. Thus, it is supposed that the charge, and hence the current, contains no higher than second harmonics. The conditions $P_1 + P_2 = 0$ and $P_n = 0$, $n \geqq 3$, are then identically satisfied, independently of the voltage-charge relationship $V = V(Q)$.

Now, a change in the time origin does not affect the power transfer. Hence, the canonical representation of a charge waveform which contains no higher than second harmonics and is such that $Q(\pi) = Q_{\min} = -m$ and $Q(2 \tan^{-1} s) = Q_{\max} = 1$, is constructed. In addition to the parameter s there is the parameter y which is subject to the restriction $0 \leqq y \leqq (1 - s^2)$, which of course also implies that $s^2 \leqq 1$. It is found that $P_1 = 0$ on $y = 0$ and on $y = (1 - s^2)$, independently of the voltage-charge relationship. Moreover, $P_1(s,y) = -P_1(-s,y)$ and in particular $P_1 = 0$ on $s = 0$ also, so that it is sufficient to consider only the region $-1 \leqq s \leqq 0$, $0 \leqq y \leqq (1 - s^2)$ and to maximize $| P_1 |$. The abrupt-junction diode, operated in the region between forward conduction and reverse breakdown, may be treated analytically, and it is found that the maximum power transfer is 0.281, as compared with the maximum obtainable fundamental power of 0.687. The corresponding reactive fundamental powers are 1.46 and 2.43, and the charge waveform giving the maximum power transfer is depicted in Fig. 6, which should be compared with Fig. 1.

In order to determine the maximum power transfer for a general voltage-charge relationship, recourse must be made to numerical computation. However, a prior step is the determination of a charge waveform which provides a reasonable approximation to the maximum power transfer, and hence provides starting values for s and y in the numerical maximization process. A good lower bound was obtained for the maximum obtainable fundamental power. Furthermore, for a wide class of voltage-charge relationships $V = V(Q)$, the charge waveform $Q(x)$ giving this lower bound satisfies $Q_{\max} = 1$ and $Q_{\min} = -m$. The class of voltage-charge relationships (15) falls within this class. Thus it would seem feasible that a reasonable approximation to the maximum power transfer will be obtained by taking the Fourier approximation, up to the second harmonics, of the charge waveform giving the good lower bound for the maximum obtainable fundamental power, and suitably shifting and expanding (or contracting) the Fourier approximation so that the resulting charge waveform $\tilde{Q}(x)$ satisfies $\tilde{Q}_{\max} = 1$ and $\tilde{Q}_{\min} = -m$. This is the procedure adopted and, for the abrupt-junction diode, operated in the region between forward conduction and reverse breakdown, it actually yields the charge waveform that gives the maximum power transfer.

The results of the numerical maximization process are tabulated in Section 5.4. Tables II and III, for the cases $\nu = 2$ and $\nu = \frac{3}{2}$, in (15), show the values of $max\ P_1$, the maximum power transfer, and the corresponding values of R_1 and R_2, the reactive powers in the fundamental and second harmonic, $I_{\max}$, the maximum normalized current magnitude, and $(b^2 + c^2)$ and $(d^2 + e^2)$, the squares of the amplitudes of the first and second harmonics in the charge waveform, for several values of m. Tables IV and V show the values of $-s$ and y which give $max\ P_1$ and also $y^{(1)}$ and $P_1^{(1)}$, the value of P_1 corresponding to the starting values $y^{(1)}$ and $-s^{(1)} = 1/\sqrt{3}$. It is interesting to observe how close $P_1^{(1)}$ is to $max\ P_1$, particularly for the smaller values of m. Table VI compares $max\ P_1$ with the maximum obtainable fundamental power, $\max P_1$, in the case $\nu = 2$, for several values of m. It is also worth noting that in the case $\nu = \frac{3}{2}$, $m = 0$ we have $max\ P_1 = 0.162$, whereas $\max P_1 = 0.408$.

2.4 *The Maximization of the Power Transfer in a Frequency Doubler, for the Current-Limited Diode*

We finally turn our attention to the current-limited diode in which (5), instead of (3), holds. Thus, from (5) to (7),

$$| Q'(x) | \leqq \frac{i_{\max}}{(\omega q_{\max})} = \frac{\kappa}{\omega}. \tag{20}$$

For the P^+N abrupt-junction diode of germanium[1,2]

$$\begin{aligned} v_{\max} &\simeq 1.03 \times 10^{13}(N)^{-0.725} \text{ volts}, \\ i_{\max} &\simeq 1.6 \times 10^{-12} N \text{ amps/cm}^2, \end{aligned} \tag{21}$$

where N is the donor concentration in cm^{-3}. But, from the voltage-charge relationship,

$$q_{\max}{}^2 = 2e\,\epsilon\, N v_{\max}, \tag{22}$$

where e denotes electron charge. Hence,

$$q_{\max} \simeq 2.16 \times 10^{-9}(N)^{0.1375} \text{ coulombs/cm}^2, \tag{23}$$

and

$$\kappa = \frac{i_{\max}}{q_{\max}} \simeq 0.74 \times 10^{-3}(N)^{0.8625} \text{ sec}^{-1}. \tag{24}$$

For $N = 2 \times 10^{16}$, a reasonable value, $\kappa \approx 10^{11}\ \text{sec}^{-1}$, which is in the range of angular frequencies of interest.

We consider the problem of maximizing the power transfer from the fundamental to the second harmonic, when the diode is being used as a frequency doubler, and, as previously, the additional assumption is made that the charge waveform $Q(x)$, and hence the current, contains no higher than second harmonics. The first step is the construction of the canonical representation of $Q(x)$ such that $Q_{max} = 1$, $Q'(\pi) = Q'_{min} = -k$ and $Q'(2 \tan^{-1} s) = Q'_{max} \leqq k$. In addition to the parameter s there is the parameter y which is subject to the restriction $0 \leqq y \leqq \frac{1}{2}(1 - s^2)$, which of course also implies $s^2 \leqq 1$. It is found that $P_1 = 0$ on $y = \frac{1}{2}(1 - s^2)$, independently of the voltage-charge relationship. Since, if $\tilde{Q}(x) = Q(\pi - x)$, then $\tilde{Q}_{max} = 1$, $\tilde{Q}'_{max} = k$ and $\tilde{Q}'_{min} \geqq -k$, it is sufficient to consider the above canonical representation and to maximize $|P_1|$, in order to maximize P_1 subject to $Q_{max} = 1$, $|Q'|_{max} = k$. We denote this maximum by $\Pi(k)$. For the abrupt-junction diode operated in the region between forward conduction and reverse breakdown, the determination of $\Pi(k)$ is carried out analytically for k sufficiently small that $Q_{min} \geqq 0$. It is found that $\Pi(k) = 0.731k^3$, for $0 \leqq k \leqq 0.681$. Combining this result with that obtained when the charge waveform is subject just to bounded variation, $0 \leqq Q(x) \leqq 1$, it is shown that, from the viewpoint of maximizing the actual fundamental real power p_1, the optimum operating frequency lies in the range

$$1.299 \leqq \frac{(\omega q_{max})}{i_{max}} \leqq 1.468, \tag{25}$$

and that

$$1 \leqq \frac{54(\max p_1)}{(i_{max} v_{max})} \leqq \frac{2(4)^{\frac{1}{3}}}{3} < 1.06. \tag{26}$$

For the abrupt-junction diode which is allowed to operate partly in the forward conduction region, the maximization of the power transfer is determined by numerical computation. For the values of s and y which give $\max |P_1|$, i.e., $\Pi(k)$, the reactive powers R_1 and R_2, and Q_{min}, i.e., $-M(k)$, were calculated, the results being given in Table VII (Section 6.4). It is shown that $\max P_1$ subject to $Q_{max} \leqq 1$ and $|Q'|_{max} \leqq k$ is attained with $Q_{max} = 1$ and $|Q'|_{max} = k$. For $k < 0.681$ it can also be attained with $1.468k \leqq Q_{max} < 1$ and $|Q'|_{max} = k$. Optimizing with respect to the frequency it appears that $20(\max p_1) \sim i_{max} v_{max}$. Thus a considerable improvement is obtained by permitting operation in the forward conduction region. The optimum frequency in this case is roughly one-fifth that in the case when operation is not allowed in the forward conduction region, although close to $\max p_1$ may be obtained at one-third the frequency.

In conclusion, we add that lack of space has necessitated the omission of several aspects of this problem, and in particular of the determination of the maximum obtainable fundamental power when the periodic charge waveform is restricted only to have bounded slope.

III. THE CHARGE WAVEFORM WHICH, SUBJECT TO BOUNDED VARIATION, MAXIMIZES THE POWER IN THE FUNDAMENTAL HARMONIC

3.1 *The Functional Form of the Charge Waveform*

From (1), (6), (7), (8) and (9),

$$P_n + jR_n = \left(\int_0^{2\pi} Q'(x)\, e^{-jnx}\, dx\right)\left(\int_0^{2\pi} V[Q(x)]\, e^{jnx}\, dx\right). \quad (27)$$

It is noted that P_n is not affected by a time shift in the charge waveform $Q(x)$, but it is reversed in sign by a time reversal of the waveform. On the other hand, R_n is not affected by either a time shift or a time reversal in the charge waveform. Integrating by parts the first integral in (27), and remembering that $Q(x)$ is periodic with period 2π, and then separating real and imaginary parts,

$$P_n = n(\alpha_n\delta_n - \beta_n\gamma_n); \qquad R_n = n(\alpha_n\gamma_n + \beta_n\delta_n), \quad (28)$$

where

$$\begin{aligned} \alpha_n &= \int_0^{2\pi} Q(x)\sin nx\, dx; & \beta_n &= \int_0^{2\pi} Q(x)\cos nx\, dx; \\ \gamma_n &= \int_0^{2\pi} V[Q(x)]\sin nx\, dx; & \delta_n &= \int_0^{2\pi} V[Q(x)]\cos nx\, dx. \end{aligned} \quad (29)$$

From (28) and (29) we may express P_n as a double integral,

$$\frac{1}{n} P_n = \int_0^{2\pi}\int_0^{2\pi} Q(x)V[Q(y)]\sin n(x - y)\, dx\, dy. \quad (30)$$

To find the functional form of $Q(x)$ which, subject to the restriction

$$-m \leqq Q(x) \leqq 1, \quad (31)$$

maximizes P_1, we set

$$Q(x) = [(1 + m)\operatorname{sech} R(x) - m], \quad (32)$$

so that the inequalities in (31) are satisfied. A variational procedure applied to (30) then shows that for stationary values of P_1, we have, for each x,

$$Q(x) = -m, \quad or \quad Q(x) = 1, \quad or$$
$$V'[Q(x)] = \frac{(\gamma_1 \cos x - \delta_1 \sin x)}{(\alpha_1 \cos x - \beta_1 \sin x)}, \tag{33}$$

where α_1, β_1, γ_1, and δ_1 are as defined in (29). This, then, is the functional form of $Q(x)$ which maximizes P_1. Evaluation of the integrals in (29) will lead to four equations for the four unknowns α_1, β_1, γ_1, and δ_1. Note, however, that

$$\begin{aligned}\frac{d}{dx}\left[\frac{(\gamma_1 \cos x - \delta_1 \sin x)}{(\alpha_1 \cos x - \beta_1 \sin x)}\right] &= -\frac{(\alpha_1\delta_1 - \beta_1\gamma_1)}{(\alpha_1 \cos x - \beta_1 \sin x)^2} \\ &= \frac{-P_1}{(\alpha_1 \cos x - \beta_1 \sin x)^2},\end{aligned} \tag{34}$$

from (28), is of one sign. Since we are not interested in $P_1 = 0$, which case arises in particular if $Q(x) = \text{const}$, it follows that allowance must be made for discontinuities in $Q(x)$, since we require that $Q(x)$ be periodic. Supposing that $Q(x)$ is discontinuous at $x = \varphi$, we obtain a condition by integrating the equation

$$V'[Q(x)]Q'(x) = \frac{(\gamma_1 \cos x - \delta_1 \sin x)}{(\alpha_1 \cos x - \beta_1 \sin x)} Q'(x), \tag{35}$$

from $x = \varphi - 0$ to $x = \varphi + 0$. This gives

$$[V[Q(x)]]_{\varphi-0}^{\varphi+0} = \frac{(\gamma_1 \cos \varphi - \delta_1 \sin \varphi)}{(\alpha_1 \cos \varphi - \beta_1 \sin \varphi)} [Q(x)]_{\varphi-0}^{\varphi+0}. \tag{36}$$

3.2 *The Charge Waveform for the Abrupt-Junction Diode*

In normalized form the voltage-charge relationship for the abrupt-junction diode operated in the region between forward conduction and reverse breakdown is

$$V(Q) = Q^2, \qquad 0 \leqq Q(x) \leqq 1, \tag{37}$$

so that $m = 0$ in (31). We make use of the fact that P_1 is invariant under the transformation $Q(x) \rightarrow Q(x - \theta)$, and choose θ so that $\beta_1 = 0$, since this leads to a simplification of the analysis. Let us define a and b by the equations

$$\gamma_1 = 2a\alpha_1, \quad \delta_1 = 2b\alpha_1; \qquad \beta_1 = 0. \tag{38}$$

Then, from (28),

$$P_1 = 2b\alpha_1^2. \tag{39}$$

It is clear that max $P_1 > 0$, and hence that $b > 0$. The functional form of $Q(x)$ for max P_1 is, from (33), (37) and (38),

$$Q(x) = 0, \qquad or \qquad Q(x) = 1, \qquad or \qquad Q(x) = (a - b \tan x). \quad (40)$$

Rejecting combinations which lead to $P_1 = 0$, we are led to the conclusion that, within a cycle, $Q(x) = 1$ for an interval, it then follows the curve $Q(x) = (a - b \tan x)$ and then $Q(x) = 0$ for an interval, after which it jumps from 0 to 1 and the cycle is repeated.

Let φ be a value of x at which a jump in $Q(x)$ from 0 to 1 occurs. Then (36), (37), and (38) give

$$\tan \varphi = \frac{(2a - 1)}{2b}. \quad (41)$$

Thus we obtain max P_1 by taking

$$Q(x) = \begin{cases} 1, & \text{for } \varphi < x \leqq \pi + \tan^{-1}[(a-1)/b]; \\ (a - b \tan x), & \text{for } \pi + \tan^{-1}[(a-1)/b] \leqq x \\ & \qquad \leqq \pi + \tan^{-1}(a/b); \\ 0, & \text{for } \pi + \tan^{-1}(a/b) \leqq x < 2\pi + \varphi, \end{cases} \quad (42)$$

where

$$-\frac{\pi}{2} < \tan^{-1}[(a-1)/b] < \varphi < \tan^{-1}(a/b) < \frac{\pi}{2}, \quad (43)$$

and

$$Q(x + 2\pi) = Q(x), \qquad \text{all } x. \quad (44)$$

Now α_1, β_1, γ_1, and δ_1 may be calculated from (29), (37), (42) and (44). Substitution into (38) then leads to

$$\begin{aligned} (2a - 1)\cos\varphi &= 2b\{[(a-1)^2 + b^2]^{\frac{1}{2}} - (a^2 + b^2)^{\frac{1}{2}}\}; \\ 2b\cos\varphi + \sin\varphi + 3b\tau &= \{(a+1)[(a-1)^2 + b^2]^{\frac{1}{2}} \\ &\qquad - a(a^2 + b^2)^{\frac{1}{2}}\}; \\ \sin\varphi &= \{[(a-1)^2 + b^2]^{\frac{1}{2}} - (a^2 + b^2)^{\frac{1}{2}}\}, \end{aligned} \quad (45)$$

where

$$\tau = b[\tanh^{-1}\{a(a^2 + b^2)^{-\frac{1}{2}}\} - \tanh^{-1}\{(a-1)[(a-1)^2 + b^2]^{-\frac{1}{2}}\}]. \quad (46)$$

It would appear that we now have one too many conditions on a, b and φ because of the relationship in (41), which was obtained from the jump condition at $x = \varphi$, but it is observed that the first and last equations in (45) are consistent with (41). Since $|\varphi| < \pi/2$ and $b > 0$, (41) gives

$$\cos\varphi = 2b[(2a-1)^2 + 4b^2]^{-\frac{1}{2}};$$
$$\sin\varphi = (2a-1)[(2a-1)^2 + 4b^2]^{-\frac{1}{2}}. \tag{47}$$

Substituting into the first equation in (45), we obtain

$$(2a-1)[(2a-1)^2 + 4b^2]^{-\frac{1}{2}} = \{[(a-1)^2 + b^2]^{\frac{1}{2}} - (a^2 + b^2)^{\frac{1}{2}}\}. \tag{48}$$

A solution to (48) is $a = \frac{1}{2}$ and, moreover, this is the only solution since if a $> \frac{1}{2}$ the L.H.S. > 0 and the R.H.S. < 0, and vice versa. Thus,

$$a = \tfrac{1}{2}, \qquad \varphi = 0. \tag{49}$$

The second equation in (45), using the definition of τ given in (46), now leads to an equation for b, namely

$$3b^2 \tanh^{-1}[(1 + 4b^2)^{-\frac{1}{2}}] = [\tfrac{1}{4}(1 + 4b^2)^{\frac{1}{2}} - b], \tag{50}$$

and (39) and the expression for α_1 give

$$P_1 = 2b\{1 + 2b \tanh^{-1}[(1 + 4b^2)^{-\frac{1}{2}}]\}^2 = \frac{1}{18b}[2b + (1 + 4b^2)^{\frac{1}{2}}]^2, \tag{51}$$

using (50). Equation (50) was solved numerically and it was found that

$$b = 0.14136; \qquad \max P_1 = 0.6868. \tag{52}$$

The shape of $Q(x)$ which gives this maximum value of P_1 is shown in Fig. 1. From (28) and (38) the corresponding reactive fundamental

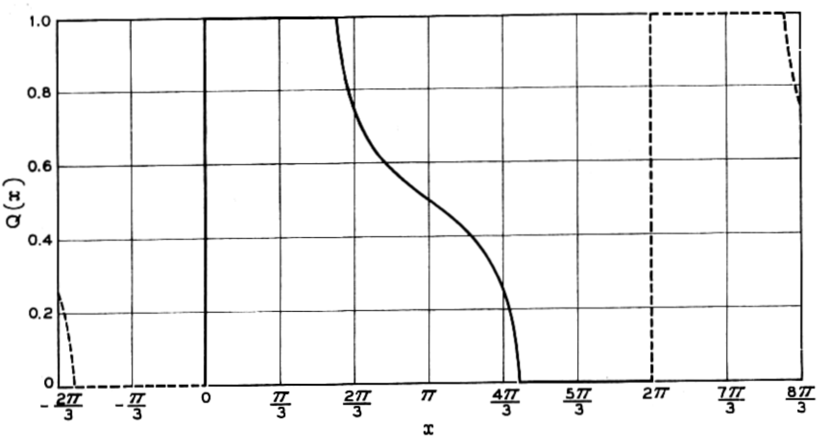

Fig. 1 — Charge waveform for maximum obtainable fundamental power in abrupt-junction diode operated in the region between forward conduction and reverse breakdown.

power is given by

$$R_1 = 2a\alpha_1^{\,2} = \frac{P_1}{(2b)} = 2.429. \tag{53}$$

Note that the reactive power is about three and a half times as large as the real power.

3.3 *The Charge Waveform for the Graded-Junction Diode*

We now turn our attention to the second diode of interest, namely the graded-junction diode, and suppose that it is operated in the region between forward conduction and reverse breakdown. In normalized form the voltage-charge relationship is

$$V(Q) = Q^{\frac{3}{2}}, \qquad 0 \leqq Q(x) \leqq 1. \tag{54}$$

The determination of the maximum obtainable fundamental power, max P_1, is carried out along the same lines as for the abrupt-junction diode, although the details are more involved. The analytical form of the charge waveform $Q(x)$ which gives max P_1 is

$$Q(x) = \begin{cases} 1, & \text{for} \quad \psi < x \leqq \pi + \tan^{-1}[(a-1)/b]; \\ (\mathrm{a} - b \tan x)^2, & \text{for} \quad \pi + \tan^{-1}[(a-1)/b] \leqq x \\ & \qquad \leqq \pi + \tan^{-1}(a/b); \\ 0, & \text{for} \quad \pi + \tan^{-1}(a/b) \leqq x < 2\pi + \psi, \end{cases} \tag{55}$$

where

$$-\frac{\pi}{2} < \tan^{-1}[(a-1)/b] < \psi < \tan^{-1}(a/b) < \frac{\pi}{2}, \tag{56}$$

and (44) holds. Here

$$\gamma_1 = \frac{3}{2}\, a\alpha_1; \qquad \delta_1 = \frac{3}{2}\, b\alpha_1; \qquad \beta_1 = 0, \tag{57}$$

which leads to three equations for a, b and ψ. These equations are consistent with the jump condition (36) which gives

$$\tan \psi = (3a - 2)/(3b). \tag{58}$$

Elimination of ψ leads to two equations for a and b which were solved numerically, giving

$$b = 0.11098; \qquad a = 0.67375. \tag{59}$$

These values lead to

$$\max P_1 = 0.4084; \qquad R_1 = 2.479. \tag{60}$$

The corresponding charge waveform $Q(x)$ is depicted in Fig. 2.

3.4 *The Abrupt-Junction Diode When the Region of Operation Includes Forward Conduction*

In this case the normalized voltage-charge relationship is, from (2) and (4),

$$V(Q) = [\max(0,Q)]^2, \qquad -m \leqq Q(x) \leqq 1, \qquad m > 0. \tag{61}$$

As previously, we translate $Q(x)$ so that $\beta_1 = 0$ and again define a and b by (38), so that (39) for P_1 also holds. From (33), (38), and (61), the functional form of $Q(x)$ for max P_1 is

$$\begin{aligned} Q(x) &= -m, \qquad \text{or} \qquad Q(x) = 1, \\ \text{or} \qquad \max\,[0,Q(x)] &= (a - b \tan x). \end{aligned} \tag{62}$$

Thus we are led to the conclusion that within a cycle $Q(x) = 1$ for an interval, it then follows the curve $Q(x) = (a - b \tan x)$ until the point at which $Q(x) = 0$ where it jumps to the value $-m$, and after $Q(x) = -m$ for an interval it jumps to the value 1 and the cycle is repeated. Thus in this idealized case there are two discontinuities in $Q(x)$ in one cycle. Note that according to (36), together with (38), the jump of

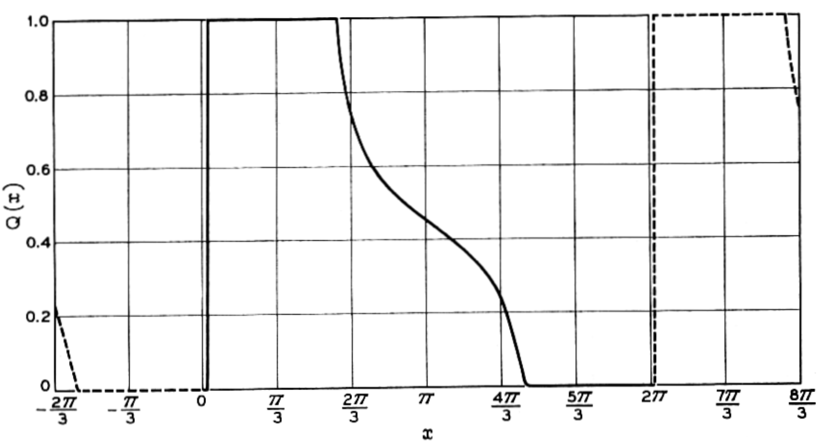

Fig. 2 — Charge waveform for maximum obtainable fundamental power in graded-junction diode operated in the region between forward conduction and reverse breakdown.

$Q(x)$ from 0 to $-m$ occurs at $x = \psi$ where $(a - b \tan \psi) = 0$, since $V(0) = 0$ and $V(-m) = 0$. Hence we obtain max P_1 by taking

$$Q(x) = \begin{cases} 1, & \text{for} \quad \varphi < x \leqq \pi + \tan^{-1}[(a-1)/b]; \\ (a - b\tan x), & \text{for} \quad \pi + \tan^{-1}[(a-1)/b] \leqq x \\ & \qquad < \pi + \tan^{-1}(a/b); \\ -m, & \text{for} \quad \pi + \tan^{-1}(a/b) < x < 2\pi + \varphi, \end{cases} \tag{63}$$

where (43) and (44) hold. In this case the jump condition at $x = \varphi$, gives

$$\tan \varphi = \frac{[2a(1 + m) - 1]}{2b(1 + m)}. \tag{64}$$

The calculation of α_1, β_1, γ_1, and δ_1, and substitution into (38), leads to three equations for a, b, and φ, which are consistent with (64). The elimination of φ leads to two equations for a and b, which quantities of course are functions of m. It was found to be possible to eliminate m analytically from these two equations, so that instead of solving the two simultaneous equations for a and b for given values of m, the single relation between a and b which did not involve m was solved for b for given values of a. Thus a parametric solution was obtained in the form $b = b(a)$, $m = m(a)$. From this a and b were plotted graphically against m and the results are shown in Fig. 3. It was shown analytically that

$$1 < 4a(1 + m) \leqq 2, \tag{65}$$

the upper bound being attained for $m = 0$ and the lower bound being approached for $m \to \infty$. Also, as $m \to \infty$ it is found that

$$b \sim \sqrt{3}a; \qquad \max P_1 \sim \frac{3\sqrt{3}}{2}m; \qquad R_1 \sim \frac{3}{2}m, \tag{66}$$

where R_1 is the reactive power in the fundamental. Fig. (4) shows max P_1 and the corresponding R_1 as functions of m. It is interesting to note that the ratio (max P_1)/R_1 increases with increasing m from its initial value of 0.28, its asymptotic value being $\sqrt{3}$, from (66). The charge waveform $Q(x)$ giving rise to max P_1 is shown, for $m = 1$, in Fig. 5.

3.5 *The Charge Waveform for an Idealized Voltage-Charge Relationship*

We now consider a special voltage-charge relationship which may be handled analytically. Thus we suppose that the capacitance has a finite

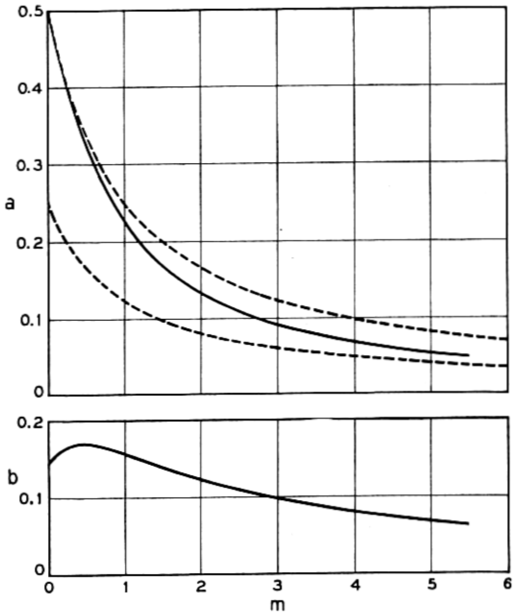

Fig. 3 — Parameters in charge waveform for maximum obtainable fundamental power in abrupt-junction diode operated partly in forward conduction region, vs. minimum charge.

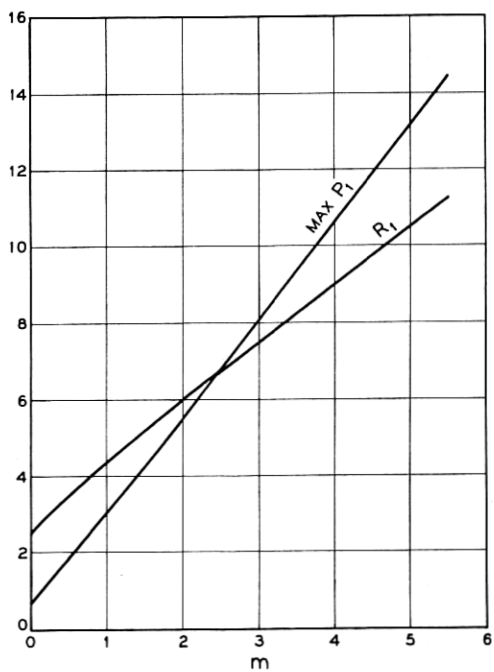

Fig. 4 — Maximum obtainable fundamental power, and corresponding fundamental reactive power, in abrupt-junction diode operated partly in forward conduction region, vs. minimum charge.

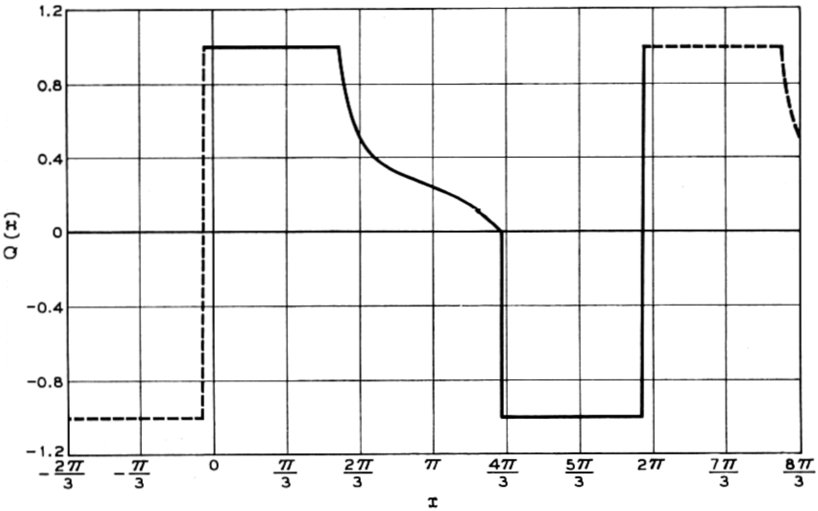

Fig. 5 — Charge waveform for maximum obtainable fundamental power in abrupt-junction diode operated partly in forward conduction region.

constant value for reverse bias and is infinite for forward bias, and hence in normalized form

$$V(Q) = \max\,(0,Q), \qquad -m \leqq Q(x) \leqq 1; \qquad m > 0. \tag{67}$$

Since $V'(Q)$ is constant except possibly at $Q = 0$, where it is indeterminate, we deduce from (33) that $Q(x)$ has one of the values 1, 0, and $-m$ at each point. Omitting further details, it is found that max P_1 is given by

$$Q(x) = \begin{cases} 1, & 0 < x < 2\pi/3; \\ 0, & 2\pi/3 < x < 4\pi/3; \\ -m, & 4\pi/3 < x < 2\pi. \end{cases} \tag{68}$$

Also,

$$\max P_1 = \frac{3\sqrt{3}}{2}\,m; \qquad R_1 = \frac{3}{2}\,(m + 2). \tag{69}$$

Note that, as might be expected, these values are asymptotically, as $m \to \infty$, the same as for the voltage-charge relationship in (61), as is seen from (66).

IV. BOUNDS ON THE MAXIMUM OBTAINABLE FUNDAMENTAL POWER

4.1 *Lower Bounds*

We now derive some lower bounds for the maximum obtainable power in the fundamental, for a general voltage-charge relationship, by the simple expedient of choosing specific charge waveforms. Any P_1 which we obtain is, of course, a lower bound for max P_1. Thus we consider the charge waveforms

$$Q(x) = \begin{cases} \sigma, & \text{on } \Gamma_1 ; \\ \rho, & \text{on } \Gamma_2 ; \\ \tau, & \text{on } \Gamma_3 , \end{cases} \tag{70}$$

where each $\Gamma_j (j = 1,2,3)$ is a finite collection of nonintersecting intervals, open at the left and closed at the right, and furthermore

$$\Gamma_j \cap \Gamma_k = 0, \quad j \neq k; \qquad \bigcup_{j=1}^{3} \Gamma_j = (0,2\pi]. \tag{71}$$

From (28), (29), (70) and (71),

$$\begin{aligned} P_n = nL(\sigma,\rho,\tau) \Bigg[&\left(\int_{\Gamma_1} \cos nx \, dx \right) \left(\int_{\Gamma_2} \sin nx \, dx \right) \\ &- \left(\int_{\Gamma_1} \sin nx \, dx \right) \left(\int_{\Gamma_2} \cos nx \, dx \right) \Bigg], \end{aligned} \tag{72}$$

where

$$L(\sigma,\rho,\tau) = [(\rho - \tau)V(\sigma) + (\tau - \sigma)V(\rho) + (\sigma - \rho)V(\tau)]. \tag{73}$$

The significant point here is that we can choose the intervals Γ_1 and Γ_2 to make P_1 as large as possible, for the waveform class of (70), independently of the functional form of the voltage-charge relationship, $V = V(Q)$. This is still true if we wish to make P_1 as large as possible subject to the condition $P_1 + P_2 = 0$, say, since the factor containing V, namely $L(\sigma,\rho,\tau)$, occurs in each P_n. Note, from (73), that $L(\sigma,\rho,\tau)$ vanishes unless σ, ρ, and τ are unequal. Also, if (σ,ρ,τ) undergo a cyclic permutation then $L(\sigma,\rho,\tau)$ is unaltered, but if (σ,ρ,τ) undergo an anticyclic permutation then $L(\sigma,\rho,\tau)$ is reversed in sign. We suppose that the charge waveform has bounded variation as in (31) and define

$$L = \max_{-m \leqq (\sigma,\rho\,\tau) \leqq 1} [L(\sigma,\rho,\tau)] \geqq 0. \tag{74}$$

Also from (72) it is seen that P_1 changes sign if Γ_1 and Γ_2 are interchanged, which is equivalent to an anticyclic permutation of (σ,ρ,τ).

Thus we are interested in making the modulus (or magnitude) of the bracketed expression following the factor $L(\sigma,\rho,\tau)$ in (72) as large as possible, in order to obtain as large as possible a lower bound for max P_1. We will restrict ourselves to special Γ_j and find the maximum modulus of the bracketed expression in (72) for these subclasses. In particular, we consider

$$\Gamma_1 = (0,\lambda]; \qquad \Gamma_2 = (\mu,\nu], \quad 0 < \lambda \leqq \mu < \nu < 2\pi. \tag{75}$$

Then, from (72),

$$P_n = \frac{1}{n} L(\sigma,\rho,\tau) F(n\lambda,n\mu,n\nu), \tag{76}$$

where

$$\begin{aligned} F(\lambda,\mu,\nu) &= [\sin(\nu - \lambda) - \sin(\mu - \lambda) + \sin\mu - \sin\nu] \\ &= 4 \sin[(\nu - \mu)/2] \sin(\lambda/2) \sin[(\nu + \mu - \lambda)/2]. \end{aligned} \tag{77}$$

We first set $\lambda = \mu$ and determine μ and ν to maximize $F(\mu,\mu,\nu)$ which from (75) and (77) is seen to be positive. The stationary values of $[\sin(\nu - \mu) + \sin\mu - \sin\nu]$ are given by

$$\cos\mu = \cos(\mu - \nu) = \cos\nu. \tag{78}$$

Hence $F(\mu,\mu,\nu)$ is a maximum for $\mu = 2\pi/3$, $\nu = 4\pi/3$ and from (74), (76), and (77) the corresponding maximum of P_1 is

$$P_1 = \frac{3\sqrt{3}}{2} L. \tag{79}$$

Now for the voltage-charge relationship (37) it is readily verified that $L(\sigma,\rho,\tau)$, as defined in (73), has a maximum value of $\frac{1}{4}$ which is attained for $\sigma = 1$, $\rho = \frac{1}{2}$, $\tau = 0$, and hence in this case we obtain the value $P_1 = 0.650$, which is quite close to the value of max P_1 given in (52).

We now consider the maximization of $F(\lambda,\mu,\nu)$ subject to the condition

$$F(\lambda,\mu,\nu) + \tfrac{1}{2}F(2\lambda,2\mu,2\nu) = 0, \tag{80}$$

corresponding to $P_1 + P_2 = 0$. Using the second part of (77), (80) becomes

$$1 + 4\cos[(\nu - \mu)/2]\cos(\lambda/2)\cos[(\nu + \mu - \lambda)/2] = 0, \tag{81}$$

supposing that $F(\lambda,\mu,\nu) \neq 0$. It is interesting to note that (81) cannot be satisfied with $\lambda = \mu$. It is found that $F(\lambda,\mu,\nu)$ is maximized, subject to (75) and (81), by

$$\lambda = 2(\pi - \theta); \qquad \mu = \theta; \qquad \nu = (2\pi - \theta), \tag{82}$$

where

$$\cos\theta = -(\tfrac{1}{2})^{\frac{2}{3}}, \qquad (2\pi/3 < \theta < \pi), \tag{83}$$

and the corresponding value of P_1, with $P_1 + P_2 = 0$, is

$$P_1/(4L) = [1 - (\tfrac{1}{2})^{\frac{4}{3}}]^{\frac{3}{2}} = 0.468. \tag{84}$$

4.2 *An Upper Bound, and its Relationship to a Lower Bound*

In Appendix A we give the derivation of an upper bound, for a general voltage-charge relationship, on the maximum obtainable fundamental power, using the fact that the charge waveform is of bounded variation, (31). It is shown that

$$\max P_1 \leqq 4(1 + m)U, \tag{85}$$

where

$$U = \min_{\lambda}\{\max_{-m\leqq\sigma\leqq 1}[\lambda\sigma - V(\sigma)] - \min_{-m\leqq\sigma\leqq 1}[\lambda\tau - V(\sigma)]\}. \tag{86}$$

In the previous section we showed, by example, that

$$\max P_1 \geqq \frac{3\sqrt{3}}{2}L, \tag{87}$$

where L is defined by (73) and (74). From Appendix A, we have

$$1 \leqq (1 + m)U/L \leqq 2, \tag{88}$$

and these bounds cannot be improved without restriction on the voltage-charge relationship. However, there is a large class of voltage-charge relationships for which the lower bound is attained, namely those for which $[(\rho + m)V(1) - (m + 1)V(\rho) + (1 - \rho)V(-m)]$ does not change sign in $-m \leqq \rho \leqq 1$. From (85) and (87) it follows that

$$\frac{3\sqrt{3}}{2} \leqq \frac{\max P_1}{L} \leqq 4, \qquad \text{if} \qquad L = (1 + m)U. \tag{89}$$

Also, for the above class, L in (74) is given with $\sigma = 1$, $\tau = -m$, or vice versa, and U in (86) is given with $\lambda = [V(1) - V(-m)]/(1 + m)$. A class of voltage-charge relationships of interest is

$$\mathrm{V}(Q) = [\max\,(0,Q)]^{\nu}, \quad -m \leqq Q \leqq 1; \qquad m \geqq 0, \quad \nu \geqq 1, \tag{90}$$

of which we have already considered the cases $\nu = 2$, $\nu = 1$, and $\nu = \frac{3}{2}$ (with $m = 0$). It is readily seen that this class satisfies the above condition, and hence

$$L = \max_{-m \leqq \rho \leqq 1} \{(\rho + m) - (1 + m)[\max(0,\rho)]^\nu\};$$

$$U = \max_{-m \leqq \rho \leqq 1} \left\{\frac{\rho}{(1+m)} - [\max(0,\rho)]^\nu\right\} - \min_{-m \leqq \rho \leqq 1} \left\{\frac{\rho}{(1+m)} - [\max(0,\rho)]^\nu\right\}. \tag{91}$$

Thus,

$$L = m + \left(1 - \frac{1}{\nu}\right)[(1+m)\nu]^{-1/(\nu-1)} = (1+m)U, \tag{92}$$

and the bounds on $(\max P_1)/L$ in (89) hold. From (69) and (92) it is seen that the lower bound is exact for the case $\nu = 1$, $m > 0$. For $\nu = \frac{3}{2}$ and $m = 0$, $3\sqrt{3}\,L/2 = 0.385$ as compared with $\max P_1 = 0.408$. For $\nu = 2$, $L = (2m+1)^2/[4(1+m)]$, and Table I shows the ratio $2(\max P_1)/(3\sqrt{3}L)$ for several values of m, and it is noted that the lower bound improves with increasing m.

TABLE I—($\nu = 2$)

m	0	0.589	1.20	1.89	3.07	5.50
$\dfrac{2(\max P_1)}{3\sqrt{3}L}$	1.058	1.042	1.027	1.018	1.012	1.006

V. THE MAXIMIZATION OF THE POWER TRANSFER FROM THE FUNDAMENTAL TO SECOND HARMONIC, WITH BOUNDED CHARGE WAVEFORM

5.1 *The Canonical Representation of the Charge Waveform*

We wish to consider the problem of maximizing the power transfer from the fundamental to the second harmonic, when the charge waveform contains no higher than second harmonics, so that

$$Q(x) = a + b\sin x + c\cos x + d\sin 2x + e\cos 2x. \tag{93}$$

We also impose the conditions

$$Q_{\max} = 1; \qquad Q_{\min} = -m. \tag{94}$$

Note that it does not follow a priori that the maximum power transfer subject to (94) is equal to the maximum subject to $Q_{\max} \leqq 1$, $Q_{\min} \geqq$

$-m$. We observe, however, that for the voltage-charge relationship (90),

$$\begin{aligned}\max\,[P_1 \mid Q_{\max} = r,\, Q_{\min} = -q] \\ = r^{(\nu+1)} \max\,[P_1 \mid Q_{\max} = 1,\, Q_{\min} = -q/r], \qquad 0 < r \leqq 1,\end{aligned} \tag{95}$$

as may be deduced from (28) and (29). Thus it is sufficient to determine $\max\,[P_1 \mid Q_{\max} = 1, Q_{\min} = -m]$, that is, $\max P_1$ subject to the conditions of (94), for a range of values of m.

A canonical representation of $Q(x)$ is found in Appendix B. In addition to the two conditions in (94) it is supposed, by a suitable choice of time origin, that

$$Q(\pi) = Q_{\min} = -m. \tag{96}$$

Thus the five coefficients in (93) are given in terms of two parameters and it is found that

$$\begin{aligned} a &= [(c - e) - m]; \qquad\qquad b = 2d = (1 + m)sy; \\ c &= (1 + m)[\tfrac{1}{2}(1 - s^4) - s^2 y]; \\ e &= \tfrac{1}{4}(1 + m)[y(1 - s^2) - \tfrac{1}{2}(1 + s^2)^2]. \end{aligned} \tag{97}$$

The parameter s arises from the equation

$$Q(2 \tan^{-1} s) = Q_{\max} = 1. \tag{98}$$

The parameter y is subject to the condition

$$0 \leqq y \leqq (1 - s^2), \tag{99}$$

which of course also implies that $s^2 \leqq 1$. Thus we have a two-parameter canonical representation of $Q(x)$, and these two parameters lie in a bounded region. Moreover, it is shown in Appendix B that, independently of the voltage-charge relationship $V = V(Q)$,

$$P_1 \,|_{y=0} = 0; \qquad P_1 \,|_{y=(1-s^2)} = 0, \tag{100}$$

so that P_1 vanishes on the boundary of this region. Also it is seen, from (93) and (97), that changing the sign of s is equivalent to the transformation $Q(x) \to Q(2\pi - x)$, and hence

$$P_1(-s,y) = -P_1(s,y); \qquad P_1 \,|_{s=0} = 0. \tag{101}$$

5.2 *The Abrupt-Junction Diode*

We now consider the abrupt-junction diode operated in the region between forward conduction and reverse breakdown. From (28), (29), (37), (93) and (97), with $m = 0$, it follows that

$$P_1 = -\frac{\pi^2}{4} s(1 + s^2)^2 y[(1 - s^2) - y]. \tag{102}$$

The maximum of (102) subject to (99) is

$$\max P_1 = \frac{4\pi^2}{81\sqrt{3}} = 0.2814, \tag{103}$$

being given by $s = -(1/\sqrt{3})$, $y = \frac{1}{3}$. Thus a charge waveform giving max P_1 is

$$\bar{Q}(x) = \frac{1}{2} + \frac{1}{3\sqrt{3}}\left[2 \sin\left(x + \frac{2\pi}{3}\right) + \sin 2\left(x + \frac{2\pi}{3}\right)\right], \tag{104}$$

and the corresponding fundamental reactive power is found to be

$$R_1 = \frac{4\pi^2}{27} = 1.462. \tag{105}$$

$Q(x) = \bar{Q}(x - (2\pi/3))$ is depicted in Fig. 6. It is interesting to compare (52) and (103), and Figs. 1 and 6. We comment that the above results may be obtained quite elegantly, without using the canonical representation of the charge waveform.

5.3 *A Charge Waveform Which Provides an Approximation to the Maximum Power Transfer*

In Section IV we obtained a lower bound to the maximum obtainable fundamental power, (87), and it was seen to be a close bound in the particular cases considered. The charge waveform giving this lower bound is one which has values σ, ρ, and τ on consecutive intervals of

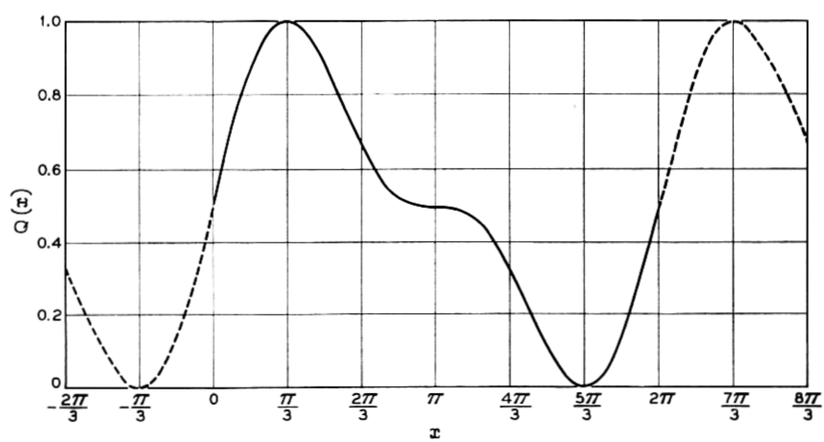

Fig. 6 — Charge waveform for maximum power transfer from fundamental to second harmonic in abrupt-junction diode operated in the region between forward conduction and reverse breakdown.

x of length $2\pi/3$. Here σ, ρ, and τ are those values which, subject to $-m \leqq (\sigma,\rho,\tau) \leqq 1$, maximize $L(\sigma,\rho,\tau)$, as defined in (73). It was also pointed out that if $[(\rho + m)V(1) - (1 + m)V(\rho) + (1 - \rho)V(-m)]$, i.e., $L(1,\rho,-m)$, does not change sign in $-m \leqq \rho \leqq 1$, then $L(\sigma,\rho,\tau)$ is maximized with $\sigma = 1$, $\tau = -m$ (or vice versa) and a suitable value of ρ. The class of voltage-charge relationships given in (90) satisfies this condition and then

$$\rho = [(1 + m)\nu]^{-1/(\nu-1)}. \tag{106}$$

Now the Fourier coefficients, up to the second harmonic as in(93), of the charge waveform giving the close lower bound to the maximum obtainable fundamental power, are

$$\begin{aligned} a &= \frac{1}{3}(\sigma + \rho + \tau); \qquad b = 2d = \frac{3}{2\pi}(\sigma - \tau); \\ c &= -2e = \frac{\sqrt{3}}{2\pi}(\sigma + \tau - 2\rho). \end{aligned} \tag{107}$$

We will restrict ourselves to that class of voltage-charge relationships, $V = V(Q)$, for which $L(\sigma,\rho,\tau)$ in (73) attains its maximum, subject to $-m \leqq (\sigma,\rho,\tau) \leqq 1$, when

$$\sigma = 1, \qquad \tau = -m, \qquad -m < \rho < 1. \tag{108}$$

It would seem feasible that we might obtain a reasonable approximation to the maximum power transfer from the fundamental to the second harmonic, by suitably shifting and expanding (or contracting) the above Fourier approximation, so that (94) is satisfied. Setting $\sigma = 1$, $\tau = -m$ in (107) and carrying out this procedure, we obtain the approximating charge waveform

$$\begin{aligned} \tilde{Q}(x) = \frac{1}{3}(1 + \rho - m) &+ \frac{\sqrt{3}}{9}(1 + m)(2 \sin x + \sin 2x) \\ &+ \frac{1}{9}(1 - m - 2\rho)(2 \cos x - \cos 2x). \end{aligned} \tag{109}$$

If we define $\bar{Q}(x) = \tilde{Q}[x + (2\pi/3)]$, then (96) is satisfied and in the canonical representation of $\bar{Q}(x)$, (97), we have

$$s = -\frac{1}{\sqrt{3}}, \qquad y = \frac{2(1 - \rho)}{3(1 + m)}. \tag{110}$$

For the abrupt-junction diode operated in the region between forward conduction and reverse breakdown, $\rho = \frac{1}{2}$, setting $m = 0$, $\nu = 2$ in

(106). Hence, from (110), $s = -(1/\sqrt{3})$ and $y = 1/3$, so that, from the previous section, the approximating charge waveform is actually the one which gives the maximum power transfer.

5.4 *The Numerical Computation of the Maximum Power Transfer, for Particular Diodes*

We have already obtained a two-parameter canonical representation of the charge waveform containing no higher than second harmonics and satisfying (94). The two parameters s and y lie in the bounded region given by (99), and P_1 vanishes, independently of the voltage-charge relationship, on the boundary of this region. Also, since P_1 is antisymmetric in s, it is sufficient to consider only half the region and to maximize $| P_1 |$. The maximization was carried out numerically for particular diodes, by means of the iterative process of fitting a quadric surface. As a starting point $s^{(1)},y^{(1)}$ in the process, that point corresponding to the approximating charge waveform, derived in the previous section, was used.

The results of the numerical computations for the voltage-charge relationship of (90), with $\nu = 2$ and $\nu = \frac{3}{2}$, and several values of m, are tabulated below. Tables II and III give the values of the maximum power transfer, *max* P_1, together with the corresponding values of the

TABLE II—($\nu = 2$)

m	*max* P_1	R_1	R_2	$I_{\max}$	$(b^2 + c^2)$	$(d^2 + e^2)$
0	0.2814	1.462	0.7310	0.7698	0.1482	0.0370
$\frac{1}{2}$	0.7773	1.966	1.060	1.160	0.3289	0.0865
1	1.284	2.300	1.300	1.549	0.5947	0.1573
2	2.198	2.921	1.561	2.310	1.451	0.3484
$\frac{7}{2}$	3.366	3.854	1.679	3.422	3.616	0.7414
5	4.371	4.788	1.642	4.515	6.869	1.250
7	5.544	6.020	1.474	5.951	12.92	2.097
9	6.586	7.228	1.230	7.372	20.95	3.096

TABLE III—($\nu = \frac{3}{2}$)

m	*max* P_1	R_1	R_2	$I_{\max}$	$(b^2 + c^2)$	$(d^2 + e^2)$
0	0.1623	1.514	0.7389	0.7684	0.1499	0.0366
$\frac{1}{2}$	0.6782	2.137	1.182	1.162	0.3257	0.0878
1	1.246	2.428	1.529	1.560	0.5635	0.1652
2	2.271	3.023	1.882	2.330	1.367	0.3682
$\frac{7}{2}$	3.575	4.034	2.006	3.445	3.479	0.7703
5	4.691	5.115	1.907	4.533	6.710	1.277

reactive powers in the fundamental and second harmonic, R_1 and R_2, and the maximum current I_{max} associated with the charge waveform $Q(x)$, that is the maximum value of $|Q'(x)|$. It is worth noting that R_2 does not continue to increase with m. Also included are the squares of the amplitudes of the first and second harmonics in the charge waveform, $(b^2 + c^2)$ and $(d^2 + e^2)$. These, together with the real and reactive powers, determine the normalized impedances. Tables IV and V give the values of $-s$ and y which given $max\ P_1$, and also $y^{(1)}$ and $P_1^{(1)}$, the value of P_1 corresponding to $y^{(1)}$ and $-s^{(1)} = 1/\sqrt{3} = 0.5774$. It is interesting to note how close $P_1^{(1)}$ is to $max\ P_1$, except for the larger values of m. Table VI compares $max\ P_1$ with the maximum obtainable fundamental power, max P_1, as obtained in Section III, for the case $\nu = 2$ and several values of m. It is also worth comparing the value of $max\ P_1 = 0.162$ for the case $\nu = \frac{3}{2}$, $m = 0$ with the corresponding value of max $P_1 = 0.408$.

TABLE IV—($\nu = 2$)

m	$-s$	$y^{(1)}$	y	$P_1^{(1)}$	$max\ P_1$
0	0.5774	0.3333	0.3333	0.2814	0.2814
$\frac{1}{2}$	0.5839	0.2963	0.2942	0.7770	0.7773
1	0.5848	0.2500	0.2426	1.283	1.284
2	0.5716	0.1852	0.1782	2.192	2.198
$\frac{7}{2}$	0.5465	0.1317	0.1301	3.307	3.366
5	0.5246	0.1019	0.1046	4.199	4.371
7	0.5008	0.0781	0.0844	5.197	5.544
9	0.4816	0.0633	0.0717	6.047	6.586

TABLE V—($\nu = \frac{3}{2}$)

m	$-s$	$y^{(1)}$	y	$P_1^{(1)}$	$max\ P_1$
0	0.5742	0.3704	0.3704	0.1622	0.1623
$\frac{1}{2}$	0.5871	0.3566	0.3562	0.6775	0.6782
1	0.5977	0.2963	0.2829	1.241	1.246
2	0.5875	0.2112	0.1989	2.262	2.271
$\frac{7}{2}$	0.5591	0.1449	0.1419	3.518	3.575
5	0.5331	0.1097	0.1130	4.536	4.691

TABLE VI—($\nu = 2$)

m	0	$\frac{1}{2}$	1	2	$\frac{7}{2}$	5
$max\ P_1$	0.281	0.777	1.28	2.20	3.37	4.37
max P_1	0.687	1.83	3.02	5.50	9.33	13.15

5.5 *On the Power Transfer From the Fundamental to the Third Harmonic*

Breitzer et al[3] were also concerned with the abrupt-junction diode operated in the region between forward conduction and reverse breakdown and considered charge waveforms containing no higher than third harmonics. They treated in detail the power transfer from the fundamental to the third harmonic, subject to $P_2 = 0$, and obtained a maximum value of

$$P_1 = (0.0242)\pi^2 = 0.238 = -P_3 , \tag{111}$$

making allowance for the difference in notation. This value of P_1 arose from two distinct charge waveforms. One was

$$Q(x) = (0.5) + (0.310) \sin x + (0.168) \sin 2x + (0.155) \sin 3x, \tag{112}$$

and the other was quite close to this. We saw previously how by taking the Fourier approximation, containing up to second harmonics, of a charge waveform which gives a good lower bound for max P_1 subject only to restrictions on $Q_{\max}$ and $Q_{\min}$, and suitably shifting and expanding (or contracting) so that the restrictions on $Q_{\max}$ and $Q_{\min}$ are satisfied by the approximating charge waveform, we could obtain a good approximation to the maximum power transfer from the first to second harmonic, when no higher than second harmonics are allowed. In the case of the abrupt-junction diode operated in the region between forward conduction and reverse breakdown, which is the diode that we will consider in this section, it was found that the charge waveform so derived was precisely one that gives the maximum power transfer.

Now, it is found that the best mean square approximation containing up to third harmonics, and subject to $P_2 = 0$, to the charge waveform which gives the good lower bound to the maximum obtainable fundamental power is

$$\bar{Q}(x) = \left[\frac{1}{2} + \frac{3}{\pi} f(x)\right], \tag{113}$$

where

$$\mathrm{f}(x) = [(0.4) \sin x + (0.25) \sin 2x + (0.2) \sin 3x]. \tag{114}$$

We shift and contract $\bar{Q}(x)$ by setting

$$Q(x) = \frac{1}{2}\left[1 + \frac{f(x)}{M}\right]; \qquad M = \max\,[f(x)], \tag{115}$$

so that $Q_{\max} = 1$ and $Q_{\min} = 0$. For this charge waveform,

$$P_1 = (0.0075)\pi^2/M^3 = -P_3; \qquad P_2 = 0. \tag{116}$$

It is found that

$$M = 0.680; \qquad P_1 = 0.235, \tag{117}$$

and $Q(x)$, as given by (114) and (115) is plotted in Fig. 7(a). The value of P_1 in (117) is very close to the maximum value obtained by Breitzer et al, (111), and it is interesting to compare Fig. 7(a) with Fig. 7(b) which depicts $Q(x)$ as given by (112).

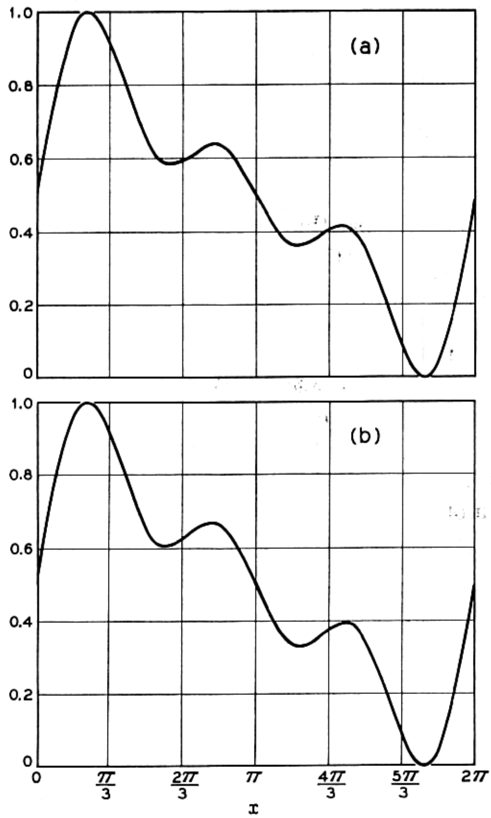

Fig. 7 — Charge waveforms giving (a) approximately, and (b) exactly, the maximum power transfer from fundamental to third harmonic in abrupt-junction diode operated in the region between forward conduction and reverse breakdown.

VI. THE MAXIMIZATION OF THE POWER TRANSFER FROM THE FUNDAMENTAL TO THE SECOND HARMONIC, FOR THE CURRENT-LIMITED DIODE

6.1 *The Canonical Representation of the Charge Waveform*

We are concerned with charge waveforms as in (93) and impose the restrictions

$$Q_{\max} = 1, \qquad | Q' |_{\max} = k. \tag{118}$$

We observe that, for voltage-charge relationships of the form given by (90),

$$\begin{aligned} &\max [P_1 \mid Q_{\max} = p, | Q' |_{\max} = l] \\ &\quad = p^{(\nu+1)} \max \left[P_1 \mid Q_{\max} = 1, | Q' |_{\max} = \frac{l}{p} \right], \qquad 0 < p \leqq 1 . \end{aligned} \tag{119}$$

In Appendix C we determine a canonical representation of $Q(x)$, subject to the conditions

$$Q_{\max} = 1; \qquad Q'_{\max} \leqq k; \qquad Q'_{\min} = -k, \tag{120}$$

by making use of the canonical representation obtained in Section 5.1, when the charge waveform has prescribed maximum and minimum values. Note that if $\tilde{Q}(x) = Q(\pi - x)$, where $Q(x)$ satisfies the conditions of (120), then

$$\tilde{Q}_{\max} = 1; \qquad \tilde{Q}'_{\max} = k; \qquad \tilde{Q}'_{\min} \geqq -k. \tag{121}$$

From Appendix C, the five coefficients in (93) are given in terms of two parameters s and y. It is found that

$$b = \frac{kw}{(w - z)}; \qquad 2d = \frac{kz}{(w - z)}; \qquad c = -\frac{ksy}{(w - z)} = 4e, \tag{122}$$

where

$$w = [\tfrac{1}{2}(1 - s^4) - s^2 y]; \qquad z = \tfrac{1}{4}[y(1 - s^2) - \tfrac{1}{2}(1 + s^2)^2], \tag{123}$$

and that

$$a = [1 - \max (b \sin x + c \cos x + d \sin 2x + e \cos 2x)], \tag{124}$$

which in general has to be determined numerically. The waveform is translated so that

$$Q'(\pi) = -k = Q'_{\min}. \tag{125}$$

The parameter s arises from the equation

$$Q'(2 \tan^{-1} s) = Q'_{\max} \leqq k, \tag{126}$$

and the parameter y is subject to the condition

$$0 \leqq y \leqq \tfrac{1}{2}(1 - s^2), \tag{127}$$

which of course also implies that $s^2 \leqq 1$. It is shown in Appendix C that

$$P_1 \equiv 0 \quad \text{for} \quad y = \tfrac{1}{2}(1 - s^2). \tag{128}$$

In order to maximize P_1 subject to (118), it is sufficient, in view of the correspondence between $Q(x)$ and $\tilde{Q}(x) = Q(\pi - x)$ given by (120) and (121), to use the above canonical representation and to maximize $| P_1 |$.

6.2 *The Abrupt-Junction Diode*

We now consider the abrupt-junction diode operated in the region between forward conduction and reverse breakdown, for which the voltage-charge relationship is $V(Q) = Q^2$, $0 \leqq Q \leqq 1$. We first maximize $| P_1 |$ subject to the conditions of (120), and suppose that k is sufficiently small that $Q_{\min} \geqq 0$. Using the canonical representation obtained in the previous section, P_1 may be expressed in terms of s and y. Omitting the details, it is found that $| P_1 |$ is maximized, subject to the restriction (127), for $s^2 = \frac{1}{3}$, $y = 0$. The charge waveform giving this maximum is

$$Q(x) = 1 + k[S(x) - S_{\max}], \tag{129}$$

where

$$S(x) = \frac{(4 \sin x - \sin 2x)}{6}. \tag{130}$$

It is readily verified that $S_{\max} = g = -S_{\min}$, where

$$g = \frac{(1 + \sqrt{3})}{2\sqrt{2}(3)^{\frac{1}{4}}} = 0.734. \tag{131}$$

Thus $Q_{\min} = (1 - 2gk)$, so that $Q_{\min} \geqq 0$ for $2gk \leqq 1$. This $Q(x)$ actually gives a negative value of P_1, so that $\tilde{Q}(x) = Q(\pi - x)$ maximizes P_1, and it is found that

$$\max P_1 = \frac{2\pi^2}{27} k^3 = 0.7311k^3, \qquad \text{for} \quad k \leqq \frac{1}{2g} = 0.681. \tag{132}$$

Fig. 8 depicts $S(\pi - x)$.

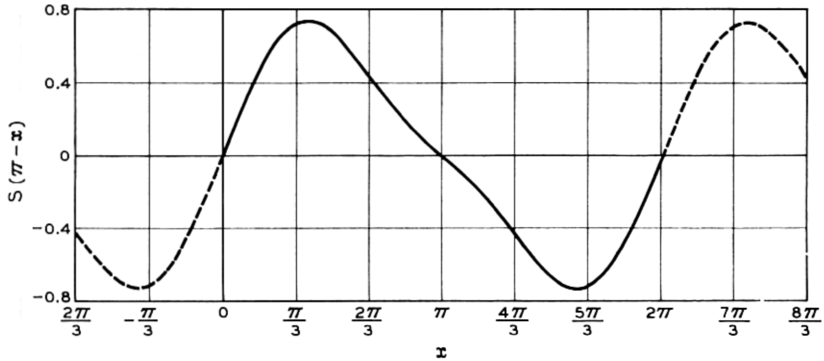

Fig. 8 — Shifted and normalized charge waveform for maximum power transfer from fundamental to second harmonic in current-limited abrupt-junction diode, with maximum current less than a critical value.

The fundamental reactive power corresponding to max P_1 is

$$R_1 = \frac{8\pi^2}{9}(1 - gk)k^2. \tag{133}$$

But, for the voltage-charge relationship $V(Q) = Q^2$, the addition of a constant to the charge waveform does not affect P_1. Hence, if instead of requiring $Q_{\max} = 1$ we just require $0 \leqq Q(x) \leqq 1$, we have

$$R_1 = \frac{8\pi^2}{9} ak^2 = 8.78ak^2; \qquad gk \leqq a \leqq (1 - gk). \tag{134}$$

6.3 *The Optimum Operating Frequency*

So far, no discussion has been made of the angular frequency ω of the actual periodicity of the charge waveform. We here consider this factor in the case of the abrupt-junction diode operated in the region between forward conduction and reverse breakdown. Now the physical limitation placed on the maximum current magnitude takes the form

$$| Q'(x) | \leqq \frac{\kappa}{\omega}, \tag{135}$$

from (20). Also, the actual fundamental power p_1 is, from (9), proportional to ωP_1. We thus consider the maximization of ωP_1 as ω varies, where the charge waveform $Q(x)$, containing no higher than second harmonics, is subject to $0 \leqq Q(x) \leqq 1$ and the condition in (135). We make use of results from Section V, as well as from the previous section.

Thus, we define

$$\max [P_1 \mid Q_{\max} = 1, Q_{\min} = -m] = P(m), \tag{136}$$

and let the value of $|Q'|_{\max}$ for the charge waveform which gives $P(m)$ be denoted by $K(m)$. Then, remembering that the addition of a constant to the charge waveform does not affect P_1, since $V(Q) = Q^2$, we obtain from (95) and (103),

$$P(m) = \frac{4\pi^2}{81\sqrt{3}} (1 + m)^3, \tag{137}$$

and also, from (104),

$$K(m) = (1 + m)K(0) = \frac{4}{3\sqrt{3}} (1 + m). \tag{138}$$

Similarly, we define

$$\max [P_1 \mid Q_{\max} = 1, |Q'|_{\max} = k] = \Pi(k), \tag{139}$$

and let the value of $Q_{\min}$ for charge waveforms which give $\Pi(k)$ be denoted by $-M(k)$. Then, from the previous section,

$$\Pi(k) = \frac{2\pi^2}{27} k^3; \qquad M(k) = -(1 - 2gk), \tag{140}$$

where g is given by (131).

Now if $Q(x)$ is subject to just the restriction $0 \leqq Q(x) \leqq 1$, then $\max P_1 = P(0)$, from (137). But, from (138), if $(\omega/\kappa) \leqq (3\sqrt{3})/4$ then the $Q(x)$ which give this value of $\max P_1$ satisfy (135). Hence,

$$\left(\frac{\omega}{\kappa}\right) \max P_1 = 0.2814 \left(\frac{\omega}{\kappa}\right), \qquad 0 \leqq \left(\frac{\omega}{\kappa}\right) \leqq 1.299. \tag{141}$$

Note that if $(\omega/\kappa) > 1.299$, then this gives an upper bound on $(\omega/\kappa) \max P_1$. Also, if $(\omega/\kappa) > 1.299$, then $\max P_1 \geqq P(m)$ if $K(m) = (\kappa/\omega)$, and hence, from (137) and (138),

$$\left(\frac{\omega}{\kappa}\right) \max P_1 \geqq 0.617 \left(\frac{\kappa}{\omega}\right)^2, \qquad \left(\frac{\omega}{\kappa}\right) \geqq 1.299. \tag{142}$$

From (140), setting $k = (\kappa/\omega)$, we have

$$\left(\frac{\omega}{\kappa}\right) \max P_1 = 0.731 \left(\frac{\kappa}{\omega}\right)^2, \qquad \left(\frac{\omega}{\kappa}\right) \geqq 1.468, \tag{143}$$

and if $0 \leqq (\omega/\kappa) < (1.468) = 2g$, then this provides an upper bound on $(\omega/\kappa) \max P_1$. Also, if $0 \leqq (\omega/\kappa) < 2g$, then $\max P_1 \geqq \Pi[1/(2g)]$, from (140). Hence,

$$\left(\frac{\omega}{\kappa}\right) \max P_1 \geqq 0.231 \left(\frac{\omega}{\kappa}\right), \qquad 0 \leqq \left(\frac{\omega}{\kappa}\right) \leqq 1.468. \tag{144}$$

Fig. 9 shows (ω/κ) max P_1 as a function of (ω/κ). For

$$1.299 \leqq \left(\frac{\omega}{\kappa}\right) \leqq 1.468, \tag{145}$$

the curve lies between the dashed lines. Thus, from the viewpoint of maximizing the actual fundamental real power, the optimum operating frequency, when the diode is not allowed to operate in the forward conduction region, lies in the range given by (145). Also, we can assert that

$$0.3655 = \frac{\pi^2}{27} \leqq \max\left[\left(\frac{\omega}{\kappa}\right) P_1\right] \leqq \frac{2(4)^{\frac{1}{3}}\pi^2}{81} = 0.387. \tag{146}$$

6.4 *Maximization of the Power Transfer, When the Region of Operation Includes Forward Conduction*

In a previous section we obtained a canonical representation of a charge waveform $Q(x)$, containing no higher than second harmonics, for which $Q_{\max} = 1$, $Q'_{\max} \leqq k$ and $Q'_{\min} = -k$. This canonical representation is given by (93), (122), (123) and (124), and involves two parameters s and y which lie in a bounded domain given by $0 \leqq y \leqq \frac{1}{2}(1 - s^2)$. It was shown that, independently of the voltage-charge relationship, $P_1 = 0$ on $y = \frac{1}{2}(1 - s^2)$. Moreover, it was seen that in order to maximize P_1 subject to $Q_{\max} = 1$ and $|Q'|_{\max} = k$, it is sufficient to consider this canonical representation and to maximize $|P_1|$.

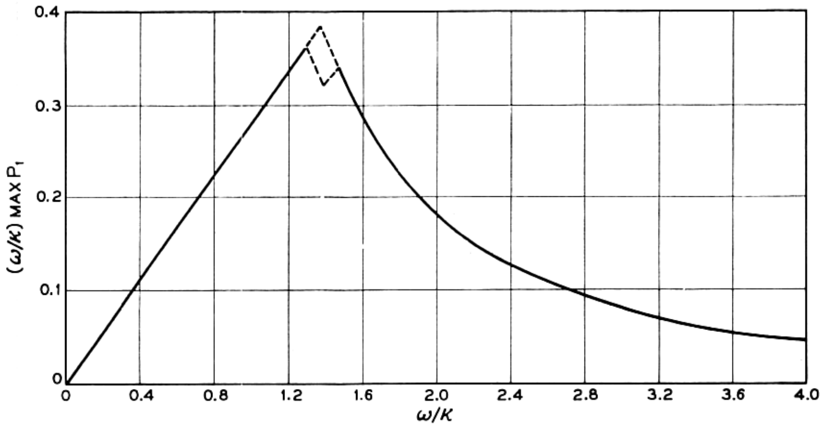

Fig. 9 — Maximum power transfer from fundamental to second harmonic in current-limited abrupt-junction diode operated in the region between forward conduction and reverse breakdown, vs. frequency.

The maximization was carried out analytically for the abrupt-junction diode when k is sufficiently small that the diode does not operate in the forward conduction region. We treat here, by means of numerical computation, the abrupt-junction diode when partial operation in the forward conduction region takes place, the normalized voltage-charge relationship being given by (61).

Again, the maximization process was that of fitting a quadric surface, and this time it was also necessary to calculate a in (124) numerically. Further, it was desirable to first compute the value of P_1 over a rough grid, and then to pick appropriate values $s^{(1)}$ and $y^{(1)}$, as a starting point in the maximization process. Thus, for several values of k, max $| P_1 |$, i.e., $\Pi(k)$ in the notation of (139), was computed in the manner described above. For the values of s and y which gave max $| P_1 |$, the corresponding values of R_1 and R_2, the reactive powers in the fundamental and second harmonic, and of $Q_{\min}$, i.e., $-M(k)$ in the notation of the previous section, were calculated, together with $(b^2 + c^2)$ and $(d^2 + e^2)$, the squares of the amplitudes of the first and second harmonics in the charge waveform. The results of the numerical computations are tabulated in Table VII. We note that the values of P_1 corresponding to the given values of s and y are negative. If $Q(x)$ is the charge waveform corresponding to s and y, (93), (122), (123), and (124), then the positive value of P_1, that is $\Pi(k)$, is obtained from the charge waveform $\tilde{Q}(x) = Q(\pi - x)$, or any translation thereof.

Now, from (119) with $\nu = 2$, and from (139),

$$\max [P_1 \mid Q_{\max} = p, | Q' |_{\max} = l] = p^3 \Pi \left(\frac{l}{p} \right), \qquad 0 < p \leqq 1. \quad (147)$$

For $Q_{\max} \leqq 0$ we have $P_1 \equiv 0$, from (61). We may write

$$\frac{p^3 \Pi \left(\frac{l}{p} \right)}{\Pi(k)} = \frac{\left(\frac{l}{p} \right)^{-3} \Pi \left(\frac{l}{p} \right)}{l^{-3} \Pi(l)} \cdot \frac{\Pi(l)}{\Pi(k)} . \quad (148)$$

The quantity $k^{-3}\Pi(k)$ is depicted in Fig. 10(a), and it is seen to be a nonincreasing function of k. It follows, from (147) and (148), since $\Pi(k)$ is a strictly increasing function of k, that max P_1 subject to $Q_{\max} \leqq 1$ and $| Q' |_{\max} \leqq k$ is attained with $Q_{\max} = 1$ and $| Q' |_{\max} = k$. For $k < 1/(2g) = 0.681$, it can also be attained with $2gk \leqq Q_{\max} < 1$ and $| Q' |_{\max} = k$. We comment that for the voltage-charge relationship $V(Q) = \max(0,Q)$, max P_1 subject to $Q_{\max} \leqq 1$ and $| Q' |_{\max} \leqq k$ is not attained with $Q_{\max} = 1$, for sufficiently small k, since in this case $P_1 \equiv 0$ if $Q_{\min} \geqq 0$.

Let us now consider the frequency factor, as we did at the end of the previous section, so that (135) holds. Hence, setting $k = (\kappa/\omega)$.

$$\max\left[\frac{\omega}{\kappa} P_1\right] = \max\left[\frac{\Pi(k)}{k}\right]. \tag{149}$$

The curve in Fig. 10(b) depicts $\Pi(k)/k$ and it is seen to be an increasing function of k in the range shown, although it is to be expected that it tends to zero as $k \to \infty$. It appears that max $[\Pi(k)/k] \sim 1$, so that, from (146), a considerable improvement is obtained if the diode is permitted to operate in the forward conduction region. We must bear in mind, however, that we have idealized the voltage-charge relationship in the forward conduction region.

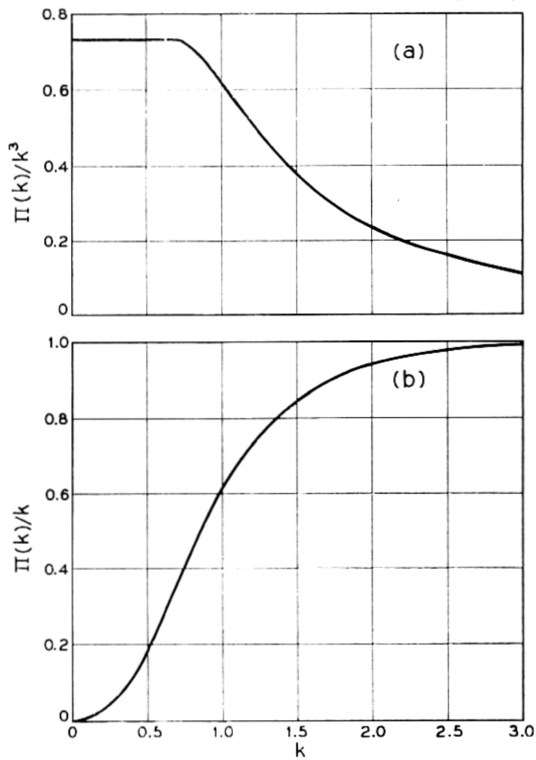

Fig. 10 — Maximum power transfer from fundamental to second harmonic divided by (a) the cube of the maximum current, and (b) the maximum current, for current-limited abrupt-junction diode with operation in forward conduction region permitted, vs. the maximum current.

TABLE VII

k	$\Pi(k)$	R_1	R_2	$M(k)$
0.75	0.3058	2.167	0.3033	0.0896
1.0	0.6159	2.440	0.5075	0.3980
1.5	1.265	2.840	0.8274	1.027
2.25	2.169	3.483	1.058	2.013
3.0	2.979	4.153	1.135	3.024
k	$-s$	y	$(b^2 + c^2)$	$(d^2 + e^2)$
0.75	0.5988	0.0018	0.2404	0.0169
1.0	0.6647	0.0288	0.3653	0.0393
1.5	0.7035	0.0834	0.7159	0.1124
2.25	0.7058	0.1354	1.613	0.2775
3.0	0.6996	0.1663	3.006	0.5039

VII. ACKNOWLEDGMENTS

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APPENDIX A

From (28) and (29), for any λ (which we take to be real),

$$P_1 = \left(\int_0^{2\pi} \{\lambda Q(x) - V[Q(x)]\} \sin x \, dx\right)\left(\int_0^{2\pi} Q(x) \cos x \, dx\right)$$

$$- \left(\int_0^{2\pi} \{\lambda Q(x) - V[Q(x)]\} \cos x \, dx\right)\left(\int_0^{2\pi} Q(x) \sin x \, dx\right) \quad (150)$$

$$= \Delta \int_0^{2\pi} \{\lambda Q(x) - V[Q(x)]\} \sin (x - \theta) \, dx,$$

where

$$\Delta \sin \theta = \int_0^{2\pi} Q(x) \sin x \, dx; \qquad \Delta \cos \theta = \int_0^{2\pi} Q(x) \cos x \, dx. \quad (151)$$

Hence,

$$\Delta = \int_0^{2\pi} Q(x) \cos (x - \theta) \, dx. \quad (152)$$

Now, $\max P_1 = \max | P_1 |$. Since

$$\left| \int_0^{2\pi} f(x) \sin(x - \varphi)\, dx \right| \leqq 2(\max f - \min f), \tag{153}$$

(31), (150) and (152) lead to (85) and (86) in Section 4.2. We next derive the inequalities (88), where L is defined by (73) and (74). Now,

$$\begin{aligned} L &\geqq \max_{-m \leqq \sigma \leqq 1} L(\sigma,1,-m) \\ &= \max_{-m \leqq \sigma \leqq 1} [(1 + m)V(\sigma) - (m + \sigma)V(1) + (\sigma - 1)V(-m)] \\ &= -(1 + m) \min_{-m \leqq \sigma \leqq 1} \left\{ \frac{\sigma[V(1) - V(-m)]}{(1 + m)} - V(\sigma) \right\} \\ &\qquad - [mV(1) + V(-m)]. \end{aligned} \tag{154}$$

Also,

$$\begin{aligned} L &\geqq \max_{-m \leqq \rho \leqq 1} L(1,\rho,-m) \\ &= (1 + m) \max_{-m \leqq \rho \leqq 1} \left\{ \frac{\rho[V(1) - V(-m)]}{(1 + m)} - V(\rho) \right\} \\ &\qquad + [mV(1) + V(-m)]. \end{aligned} \tag{155}$$

Hence, from (86), (154) and (155),

$$2L \geqq (1 + m)U. \tag{156}$$

Also, from (73) and (74),

$$\begin{aligned} L = \max_{-m \leqq (\sigma,\rho,\tau) \leqq 1} \{ &(\tau - \rho)[\lambda\sigma - V(\sigma)] + (\sigma - \tau)[\lambda\rho - V(\rho)] \\ &+ (\rho - \sigma)[\lambda\tau - V(\tau)] \}, \end{aligned} \tag{157}$$

for any (real) λ. In view of the remarks preceding (74) we may assume either that $-m \leqq \sigma \leqq \rho \leqq \tau \leqq 1$, or that $-m \leqq \tau \leqq \rho \leqq \sigma \leqq 1$, without loss of generality. In the former case

$$\begin{aligned} &(\tau - \rho)[\lambda\sigma - V(\sigma)] + (\sigma - \tau)[\lambda\rho - V(\rho)] + (\rho - \sigma)[\lambda\tau - V(\tau)] \\ &\leqq (\tau - \rho) \max_{-m \leqq \kappa \leqq 1} [\lambda\kappa - V(\kappa)] + (\sigma - \tau) \min_{-m \leqq \kappa \leqq 1} [\lambda\kappa - V(\kappa)] \\ &\qquad + (\rho - \sigma) \max_{-m \leqq \kappa \leqq 1} [\lambda\kappa - V(\kappa)] \\ &= (\tau - \sigma)\{ \max_{-m \leqq \kappa \leqq 1} [\lambda\kappa - V(\kappa)] - \min_{-m \leqq \kappa \leqq 1} [\lambda\kappa - V(\kappa)]\}. \end{aligned} \tag{158}$$

Hence

$$(\tau - \rho)[\lambda\sigma - V(\sigma)] + (\sigma - \tau)[\lambda\rho - V(\rho)] + (\rho - \sigma)[\lambda\tau - V(\tau)] \leqq (1 + m)\{\max_{-m \leqq \kappa \leqq 1} [\lambda\kappa - V(\kappa)] - \min_{-m \leqq \kappa \leqq 1} [\lambda\kappa - V(\kappa)]\}. \tag{159}$$

Equation (159) may be derived, in a similar manner, when $-m \leqq \tau \leqq \rho \leqq \sigma \leqq 1$. Thus, from (157),

$$L \leqq (1 + m)\{\max_{-m \leqq \kappa \leqq 1} [\lambda\kappa - V(\kappa)] - \min_{-m \leqq \kappa \leqq 1} [\lambda\kappa - V(\kappa)]\}. \tag{160}$$

But this is true for all (real) λ. Hence, from (86)

$$L \leqq (1 + m)U. \tag{161}$$

If we do not restrict the voltage-charge relationship then the bounds given by (156) and (161) cannot be improved. This is demonstrated by considering the (somewhat artificial) relationship

$$V(Q) = \begin{cases} 1, & Q = [(1 + m)\alpha - m]; \\ -1, & Q = [1 - (1 + m)\alpha]; \\ 0, & \text{otherwise}; \quad m > -1, \qquad 0 < \alpha < \tfrac{1}{2}. \end{cases} \tag{162}$$

It may be verified that in this case

$$L = (1 + m) = (1 - \alpha)(1 + m)U. \tag{163}$$

We now find a class of voltage-charge relationships for which the bound in (161) is attained. If $\sigma \geqq \tau$, then, by the definition of U in (86),

$$(\sigma - \tau)U = \leqq \max_{-m \leqq \rho \leqq 1} \{[V(\sigma) - V(\tau)]\rho - (\sigma - \tau)V(\rho)\} - \min_{-m \leqq \rho \leqq 1} \{[V(\sigma) - V(\tau)]\rho - (\sigma - \tau)V(\rho)\}. \tag{164}$$

Let $-m \leqq \tau \leqq \sigma \leqq 1$. Then,

$$\begin{aligned} L &\geqq \max_{-m \leqq \rho \leqq 1} [(\rho - \tau)V(\sigma) + (\tau - \sigma)V(\rho) + (\sigma - \rho)V(\tau)] \\ &= \max_{-m \leqq \rho \leqq 1} \{[V(\sigma) - V(\tau)]\rho - (\sigma - \tau)V(\rho)\} \\ &\qquad + [\sigma V(\tau) - \tau V(\sigma)] \\ &\geqq (\sigma - \tau)U + [\sigma V(\tau) - \tau V(\sigma)] \\ &\qquad + \min_{-m \leqq \rho \leqq 1} \{[V(\sigma) - V(\tau)]\rho - (\sigma - \tau)V(\rho)\} \\ &= (\sigma - \tau)U \\ &\qquad + \min_{-m \leqq \rho \leqq 1} [(\rho - \tau)V(\sigma) + (\tau - \sigma)V(\rho) + (\sigma - \rho)V(\tau)]. \end{aligned} \tag{165}$$

$$Q(\pi/2) = [\tfrac{1}{4}(3 - 2y)(1 + m) - m] = [1 - \tfrac{1}{4}(1 + 2y)(1 + m)], \quad (181)$$

so that $Q_{\max} > 1$ for $s = 0, y < -\frac{1}{2}$ and $Q_{\min} < -m$ for $s = 0, y > \frac{3}{2}$. Hence $Q_{\max} > 1$ for y < 0, and $Q_{\min} < -m$ for $y > (1 - s^2)$. The region of interest, i.e., $Q_{\max} = 1$ and $Q_{\min} = -m$, is given by

$$0 \leqq y \leqq (1 - s^2). \quad (182)$$

We next consider the fundamental power when the charge waveform $Q(x)$ contains no higher than second harmonics. From (28), (29) and (170),

$$P_1 = \pi \int_0^{2\pi} V[Q(x)](b \cos x - c \sin x)\, dx. \quad (183)$$

We determine conditions under which $P_1 \equiv 0$, independently of the voltage-charge relationship $V = V(Q)$. This is clearly the case if $b = 0 = c$, or if $Q(x)$, as given by (170), is a single-valued function of $(b \sin x + c \cos x)$, for then the integrand in (183) is the derivative of a periodic function. Noting that

$$2(b \sin x + c \cos x)^2 = (b^2 + c^2) + 2bc \sin 2x + (c^2 - b^2) \cos 2x, \quad (184)$$

it follows from (170) that the latter condition holds if

$$d = 2\lambda bc; \qquad e = \lambda(c^2 - b^2), \quad (185)$$

for some λ. Combining this condition with $b = 0 = c$,

$$2bce + d(b^2 - c^2) = 0 \Rightarrow P_1 \equiv 0. \quad (186)$$

We now consider the canonical representation of $Q(x)$, with $Q_{\max} = 1$ and $Q_{\min} = -m$, wherein the coefficients in (170) are given by (97). Then condition (186) becomes, upon reduction,

$$y = 0, \qquad \text{or} \qquad y = (1 - s^2), \qquad \text{or} \qquad s = 0 \Rightarrow P_1 \equiv 0. \quad (187)$$

APPENDIX C

We here determine the canonical form of $Q(x)$, as given by (170), such that

$$Q_{\max} = 1; \qquad Q'_{\max} \leqq k; \qquad Q'_{\min} = Q'(\pi) = -k. \quad (188)$$

Now, when the charge waveform $\bar{Q}(x)$ is subject to $\bar{Q}_{\max} = 1$ and $\bar{Q}_{\min} = -m$, the five coefficients corresponding to those in (170) are given in terms of two parameters s and y, from (97), by

$$\bar{a} = [(\bar{c} - \bar{e}) - m]; \qquad \bar{b} = 2\bar{d} = (1 + m)sy;$$
$$\bar{c} = (1 + m)w; \qquad \bar{e} = (1 + m)z, \tag{189}$$

where

$$w = [\tfrac{1}{2}(1 - s^4) - s^2 y]; \qquad z = \tfrac{1}{4}[y(1 - s^2) - \tfrac{1}{2}(1 + s^2)^2]. \tag{190}$$

The charge waveform is translated so that $\bar{Q}(\pi) = -m$, which may be done without loss of generality. The parameter s arises from the condition $\bar{Q}(2 \tan^{-1} s) = 1$, and the parameter y is subject to the condition $0 \leqq y \leqq (1 - s^2)$, which of course also implies $s^2 \leqq 1$. If, in addition, $\bar{a} = 0$, then

$$(1 + m)(w - z) = m, \tag{191}$$

and hence, from (190),

$$2y(1 + 3s^2) = [(1 + s^2)(5 - 3s^2) - 8m/(1 + m)]. \tag{192}$$

Now $0 \leqq y \leqq (1 - s^2)$, but if we require $m \geqq 1$ then

$$0 \leqq y \leqq \tfrac{1}{2}(1 - s^2), \qquad (m \geqq 1). \tag{193}$$

Turning to a charge waveform $Q(x)$, as given by (170), which satisfies the conditions of (188), we may write

$$Q'(x) = \frac{k}{m} \bar{Q}(x), \qquad m \geqq 1, \tag{194}$$

where (191) and (193) hold. Hence,

$$Q'(x) = \frac{k[sy(\sin x + \tfrac{1}{2} \sin 2x) + w \cos x + z \cos 2x]}{(w - z)}. \tag{195}$$

Integrating, and remembering that $Q_{\max} = 1$,

$$Q(x) = \{1 + k[S(x) - S_{\max}]\}, \tag{196}$$

where

$$S(x) = \frac{\left[w \sin x + \dfrac{z}{2} \sin 2x - sy(\cos x + \tfrac{1}{4} \cos 2x)\right]}{(w - z)}. \tag{197}$$

In general, $S_{\max} = \max[S(x)]$ is determined numerically.

We now turn to the fundamental power, P_1, when $Q(x)$ has the above canonical representation. From (170), (186), (190), (196) and (197), we find that $P_1 \equiv 0$, independently of the voltage-charge relationship, if

$$[(1 - s^2) - 2y]\{(1 - s^2)(1 + 3s^2) - s^2[(1 - s^2) - 2y]^2\} = 0. \quad (198)$$

In view of (193), the second factor vanishes only if $s^2 = 1$, $y = 0$ Hence we conclude that

$$P_1 \equiv 0 \qquad \text{for} \qquad y = \tfrac{1}{2}(1 - s^2). \quad (199)$$

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