Lined Waveguide*

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The existing approximate analysis of wave propagation in lined waveguide is, under practical conditions, limited to linings thinner than 0.025 per cent of the waveguide diameter. A more exact analysis is presented here for the straight and curved waveguide and for all practical linings. In the case of anisotropic or sandwiched linings, the boundary value problem is formulated using wall impedances. The single isotropic lining is taken as an example to prove this formulation useful for typical cases.

The exact analysis shows that neither the thickness nor the permittivity of the lining can increase the phase difference between TM_{11} and TE_{01} beyond a certain limit. The curvature coupling between these two waves is enhanced slightly by the lining.

I. INTRODUCTION

Round waveguide with dielectric lining shows promise as a communication medium.[1] The circular electric wave loss in perfectly straight and round metallic waveguide decreases steadily with frequency. Any deformations of the cross section or curvature of the guide axis degrade these ideal transmission characteristics.

The TM_{1n} waves in particular are coupled by curvature to TE_{0n}, and since they propagate with nearly equal phase velocity, there will be large mode conversion. A dielectric lining close to the wall changes the TM_{1n} waves appreciably with almost no change to TE_{0n}. The phase velocities are now different and, despite curvature coupling, mode conversion stays small.

When the lining is made lossy it will serve still another purpose.[2] Circular electric wave loss is increased only very little, while all other waves suffer an effective dielectric loss. This mode filtering loss will reduce the degrading effects of mode conversion and reconversion.

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To serve both purposes best, the lining should be made anisotropic or sandwiched from different materials.[3] The circular electric wave loss will then remain very low, yet all other waves will suffer high loss and also curvature loss will stay small.

Wave propagation in straight and curved lined waveguide has been analyzed elsewhere.[1,4] Likewise, imperfections in the lining and cross-sectional deformations have been calculated.[4,5] However, the lining was always assumed to be very thin, and so far only a first-order approximation has been found.

On the other hand, it has been shown both theoretically and experimentally that these approximations do not in general hold for any practical linings.[1] Linings which are designed optimally change wave propagation much more than could be described by a first-order approximation.

An analysis of wave propagation in lined waveguide will be presented here which is sufficiently general and accurate to hold for all practical cases. Sandwiched and anisotropic linings will also be considered.

Circular electric wave transmission is most strongly degraded in curved waveguide. Therefore, the lined waveguide will be assumed to have curvature. Cross-sectional deformations and imperfections of the lining will be analyzed with corresponding accuracy in another paper.[6]

II. NORMAL MODE FIELDS

Normal modes of straight round waveguide with a single isotropic lining have been analyzed before.[1] To adapt this analysis to an investi-

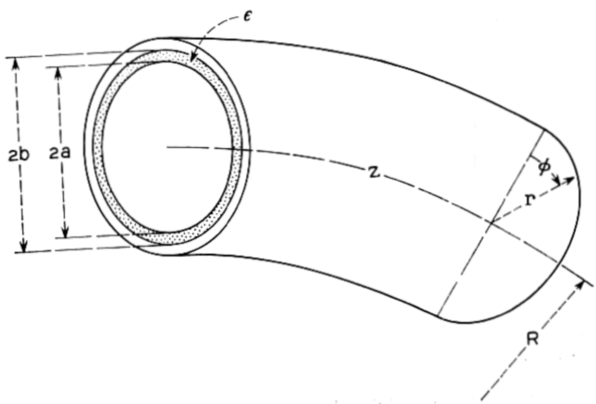

Fig. 1 — Lined waveguide with curvature.

gation of bends in lined waveguide and of waveguides with an irregular lining, the boundary value problem will be repeated and generalized here.

The waveguide structure to be considered is shown in Fig. 1. The dielectric lining will later on be assumed to be heterogeneous or anisotropic or irregular. At present, however, a single uniform and homogeneous lining is assumed.

The electromagnetic field in the waveguide can be derived from two sets of scalar functions T_n and T_n' given by:

$$\left.\begin{aligned} T_n &= N_n\, J_p(\chi_n r)\, \sin p\varphi \\ T_n' &= N_n\, J_p(\chi_n r)\, \cos p\varphi \end{aligned}\right\} \quad \text{for} \quad 0 < r < a \tag{1}$$

and

$$T_n = N_n \frac{\chi_n^2}{\chi_n^{e^2}} J_p(k_n) \frac{H_p^{(2)}(\chi_n^e r) - cH_p^{(1)}(\chi_n^e r)}{H_p^{(2)}(k_n^e) - cH_p^{(1)}(k_n^e)} \sin p\varphi$$

$$\text{for} \quad a < r < b \tag{2}$$

$$T_n' = N_n \frac{\chi_n^2}{\chi_n^{e^2}} J_p(k_n) \frac{H_p^{(2)}(\chi_n^e r) - c'H_p^{(1)}(\chi_n^e r)}{H_p^{(2)}(k_n^e) - c'H_p^{(1)}(k_n^e)} \cos p\varphi .$$

The T functions satisfy the wave equation:

$$\nabla^2 T = \frac{1}{e_1 e_2}\left[\frac{\partial}{\partial u}\left(\frac{e_2}{e_1}\frac{\partial T}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{e_1}{e_2}\frac{\partial T}{\partial v}\right)\right] = -\chi^2 T \tag{3}$$

in a general orthogonal curvilinear coordinate-system (u,v,w), in which the element of length is:

$$ds^2 = e_1^2 du^2 + e_2^2 dv^2 + e_3^2 dw^2. \tag{4}$$

The curved waveguide may be described in these coordinates when, according to Fig. 1:

$$u = r, \qquad v = \varphi, \qquad w = z \tag{5}$$

$$e_1 = 1, \qquad e_2 = r, \qquad e_3 = 1 + \xi \tag{6}$$

where

$$\xi = \frac{r}{R}\cos\varphi . \tag{7}$$

The field components are written in terms of the field functions (1) and (2)

$$
\begin{aligned}
E_u &= \sum_n V_n \left[\frac{\partial T_n}{e_1 \partial u} + d_n \frac{\partial T_n'}{e_2 \partial v}\right] \\
E_v &= \sum_n V_n \left[\frac{\partial T_n}{e_2 \partial v} - d_n \frac{\partial T_n'}{e_1 \partial u}\right] \\
H_u &= -\sum_n I_n \left[\frac{\partial T_n}{e_2 \partial v} - d_n \frac{h_n^2}{k^2} \frac{\partial T_n'}{e_1 \partial u}\right] \epsilon \\
H_v &= \sum_n I_n \left[\frac{\partial T_n}{e_1 \partial u} + d_n \frac{h_n^2}{k^2} \frac{\partial T_n'}{e_2 \partial v}\right] \epsilon .
\end{aligned}
\tag{8}
$$

Maxwell's equations are:

$$\frac{1}{e_2 e_3}\left[\frac{\partial}{\partial v}(e_3 E_w) - \frac{\partial}{\partial w}(e_2 E_v)\right] = -j\omega\mu_0 H_u \tag{9}$$

$$\frac{1}{e_3 e_1}\left[\frac{\partial}{\partial w}(e_1 E_u) - \frac{\partial}{\partial u}(e_3 E_w)\right] = -j\omega\mu_0 H_v \tag{10}$$

$$\frac{1}{e_1 e_2}\left[\frac{\partial}{\partial u}(e_2 E_v) - \frac{\partial}{\partial v}(e_1 E_u)\right] = -j\omega\mu_0 H_w \tag{11}$$

$$\frac{1}{e_2 e_3}\left[\frac{\partial}{\partial v}(e_3 H_w) - \frac{\partial}{\partial w}(e_2 H_v)\right] = j\omega\epsilon\epsilon_0 E_u \tag{12}$$

$$\frac{1}{e_3 e_1}\left[\frac{\partial}{\partial w}(e_1 H_u) - \frac{\partial}{\partial u}(e_3 H_w)\right] = j\omega\epsilon\epsilon_0 E_v \tag{13}$$

$$\frac{1}{e_1 e_2}\left[\frac{\partial}{\partial u}(e_2 H_v) - \frac{\partial}{\partial v}(e_1 H_u)\right] = j\omega\epsilon\epsilon_0 E_w . \tag{14}$$

μ_0 and ϵ_0 are permeability and permittivity of free space. ϵ is the relative permittivity of the respective cross-sectional part of the waveguide.

Substituting from (8) into (11) and (14) and taking advantage of (3) the longitudinal field components are obtained:

$$
\begin{aligned}
H_w &= j\omega\epsilon\epsilon_0 \sum_n V_n d_n \frac{\chi_n^2}{k^2} T_n' \\
E_w &= j\omega\mu_0 \sum_n I_n \epsilon \frac{\chi_n^2}{k^2} T_n
\end{aligned}
\tag{15}
$$

where $k = \omega\sqrt{\epsilon\epsilon_0\mu_0}$ is the intrinsic propagation constant of the medium in a particular cross-sectional part of the guide. ϵ and k have constant but different values for the different cross-sectional parts of the guide.

In (8), this dependence on the cross-sectional part of the guide is not only true for ϵ and k, but until we learn more, also for d_n.

The quantities d_n, c, c' and the separation constants χ_n and χ_n^e must be chosen so that the boundary conditions of the lined waveguide are satisfied.

These boundary conditions are, at the surface of the lining:

$$E_w^i = E_w^e \tag{16}$$

$$H_w^i = H_w^e \tag{17}$$

$$E_v^i = E_v^e \tag{18}$$

$$H_v^i = H_v^e \tag{19}$$

and at the metal surface

$$E_w = 0 \tag{20}$$

$$E_v = 0. \tag{21}$$

The superscrips i and e indicate the internal and external field components at the surface of the lining.

To satisfy (20):

$$c = \frac{H_p^{(2)}(\rho k_n^e)}{H_p^{(1)}(\rho k_n^e)} \tag{22}$$

where

$$\rho = \frac{b}{a} \tag{23}$$

and

$$\begin{aligned} k_n &= \chi_n a \\ k_n^e &= \chi_n^e a. \end{aligned} \tag{24}$$

To satisfy (21)

$$c' = \frac{H_p^{(2)'}(\rho k_n^e)}{H_p^{(1)'}(\rho k_n^e)}. \tag{25}$$

The prime at the Bessel functions denotes differentiation with respect to the argument.

The condition of E_w being continuous across the surface of the lining is satisfied by virtue of the formulation of the T-functions in (1) and (2). To satisfy (17)

$$d_n^i = d_n^e.$$

d_n is therefore independent of the cross-sectional area of the guide and needs no superscript.

To satisfy (18)

$$\frac{1}{d_n} = \frac{k_n k_n^{e2}}{pk^2a^2(\epsilon - 1)}\left[\frac{J_p'(k_n)}{J_p(k_n)} - \frac{k_n}{k_n^e}\frac{H_p^{(2)'}(k_n^e) - c'H_p^{(1)'}(k_n^e)}{H_p^{(2)}(k_n^e) - c'H_p^{(1)}(k_n^e)}\right]. \tag{26}$$

The remaining condition (19) leads to the following (characteristic) equation of the lined waveguide:

$$\begin{aligned}\left[\frac{J_p'(k_n)}{J_p(k_n)} - \frac{k_n}{k_n^e}\frac{H_p^{(2)'}(k_n^e) - c'H_p^{(1)'}(k_n^e)}{H_p^{(2)}(k_n^e) - c'H_p^{(1)}(k_n^e)}\right]& \\ \cdot\left[\frac{J_p'(k_n)}{J_p(k_n)} - \epsilon\frac{k_n}{k_n^e}\frac{H_p^{(2)'}(k_n^e) - cH_p^{(1)'}(k_n^e)}{H_p^{(2)}(k_n^e) - cH_p^{(1)}(k_n^e)}\right]& \\ = p^2(\epsilon - 1)^2\frac{h_n^2a^2k^2a^2}{k_n^2k_n^{e4}}.&\end{aligned} \tag{27}$$

The characteristic equation (27), together with

$$\begin{aligned}k_n^2 &= (\omega^2\epsilon_0\mu_0 - h_n^2)a^2 \\ k_n^{e2} &= (\omega^2\epsilon\epsilon_0\mu_0 - h_n^2)a^2\end{aligned} \tag{28}$$

determines the separation constants k_n and k_n^e.

The transverse field components of any two different modes are orthogonal to each other in that[7]

$$\begin{aligned}\frac{1}{V_nI_m}\int_S (E_{tn} \times H_{tm})\, dS& \\ = \int_S \epsilon\Bigg[\left(\frac{\partial T_n}{e_1\partial u} + d_n\frac{\partial T_n'}{e_2\partial v}\right)\left(\frac{\partial T_m}{e_1\partial u} + d_m\frac{h_m^2}{k^2}\frac{\partial T_m'}{e_2\partial v}\right)& \\ + \left(\frac{\partial T_n}{e_2\partial v} - d_n\frac{\partial T_m}{e_1\partial u}\right)\left(\frac{\partial T_m}{e_2\partial v} - d_m\frac{h_m^2}{k^2}\frac{\partial T_m'}{e_1\partial u}\right)\Bigg]\, dS = \delta_{nm}.&\end{aligned} \tag{29}$$

The integration is to be extended over the entire cross section of the guide. The quantity δ_{nm} is the Kronecker delta. To satisfy (29) for $n = m$ requires N_n to have a certain value, which is to be determined from (29).

III. GENERALIZED TELEGRAPHIST'S EQUATIONS FOR CURVED WAVEGUIDE

All quantities in (8) and (15) have now been determined except the current and voltage coefficients I_n and V_n. To find relations for them,

(8) is substituted for the field components into Maxwell's equations. Then

$$-e_3\left(\frac{\partial T_m}{e_2\partial v} - d_m\frac{h_m^2}{k^2}\frac{\partial T_m'}{e_1\partial u}\right)\epsilon \text{ times } (9)$$

is added to

$$e_3\left(\frac{\partial T_m}{e_1\partial u} + d_m\frac{h_m^2}{k^2}\frac{\partial T_m'}{e_2\partial v}\right)\epsilon \text{ times } (10)$$

and the result is integrated over the cross section. Using the orthonormality condition (29), the wave equation (3), and the boundary conditions (16), (19), and (20), one obtains:

$$\begin{aligned}\frac{dV_m}{dw} + j\frac{h_m^2}{\omega\epsilon_0}I_m = j\omega\mu_0\sum_n I_n\Bigg\{&\int_S \xi\,\frac{\chi_n^2\chi_m^2}{k^2}\,\epsilon^2 T_nT_m\,dS \\ &- \int_S \xi\epsilon^2\left[\left(\frac{\partial T_n}{e_2\partial v} - d_n\frac{h_n^2}{k^2}\frac{\partial T_n'}{e_1\partial u}\right)\left(\frac{\partial T_m}{e_2\partial v} - d_m\frac{h_n^2}{k^2}\frac{\partial T_m'}{e_1\partial u}\right)\right. \\ &\left.+\left(\frac{\partial T_n}{e_1\partial u} + d_n\frac{h_n^2}{k^2}\frac{\partial T_n'}{e_2\partial v}\right)\left(\frac{\partial T_m}{e_1\partial u} + d_m\frac{h_m^2}{k^2}\frac{\partial T_m'}{e_2\partial v}\right)\right]dS\Bigg\}.\end{aligned} \tag{30}$$

Similarly

$$-e_3\left(\frac{\partial T_m}{e_1\partial u} + d_m\frac{\partial T_m'}{e_2\partial v}\right)\text{ times } (12)$$

is added to

$$-e_3\left(\frac{\partial T_m}{e_2\partial v} - d_m\frac{\partial T_m'}{e_1\partial u}\right)\text{ times } (13)$$

and the result is integrated over the cross section:

$$\begin{aligned}\frac{dI_m}{dw} + j\omega\epsilon_0 V_m = j\omega\epsilon_0\sum_n V_n\Bigg\{&\int_S \xi\epsilon\,d_nd_m\frac{\chi_n^2\chi_m^2}{k^2}\,T_n'T_m'\,dS \\ &- \int_S \xi\epsilon\left[\left(\frac{\partial T_n}{e_2\partial v} - d_n\frac{\partial T_n'}{e_1\partial u}\right)\left(\frac{\partial T_m}{e_2\partial v} - d_m\frac{\partial T_m'}{e_1\partial u}\right)\right. \\ &\left.+\left(\frac{\partial T_n}{e_1\partial u} + d_n\frac{\partial T_n'}{e_2\partial v}\right)\left(\frac{\partial T_m}{e_1\partial u} + d_m\frac{\partial T_m'}{e_2\partial v}\right)\right]dS\Bigg\}.\end{aligned} \tag{31}$$

Equations (30) and (31) are the generalized telegraphist's equations for the curved waveguide with a dielectric lining.

Introducing traveling waves

$$V_m = \sqrt{K_m}(a_m + b_m)$$
$$I_m = \frac{1}{\sqrt{K_m}}(a_m - b_m) \tag{32}$$

with

$$K_m = \frac{h_m}{\omega \epsilon_0}$$

the more convenient form in terms of amplitudes of forward (a_m) and backward (b_m) traveling waves is obtained.

$$\frac{da_m}{dw} + jh_m a_m = j\sum_n (c_{mn}^{+} a_n + c_{mn}^{-} b_n)$$
$$\frac{db_m}{dw} - jh_m b_m = -j\sum_n (c_{mn}^{+} b_n + c_{mn}^{-} a_n). \tag{33}$$

The coupling coefficients in (33) are

$$\begin{aligned}
c_{mn}^{\pm} = {} & \\
& \pm \frac{\omega^2 \mu_0 \epsilon_0}{2\sqrt{h_m h_n}} \int_S \xi \epsilon^2 \left[\left(\frac{\partial T_n}{e_1 \partial u} + d_n \frac{h_n^2}{k^2} \frac{\partial T_n'}{e_2 \partial v} \right) \left(\frac{\partial T_m}{e_1 \partial u} + d_m \frac{h_m^2}{k^2} \frac{\partial T_m'}{e_2 \partial v} \right) \right. \\
& \qquad \left. + \left(\frac{\partial T_n}{e_2 \partial v} - d_n \frac{h_n^2}{k^2} \frac{\partial T_n'}{e_1 \partial u} \right) \left(\frac{\partial T_m}{e_2 \partial v} - d_m \frac{h_m^2}{k^2} \frac{\partial T_m'}{e_1 \partial u} \right) \right] dS \\
& \qquad - \frac{\sqrt{h_m h_n}}{2} \int_S \xi \epsilon \left[\left(\frac{\partial T_n}{e_1 \partial u} + d_n \frac{\partial T_n'}{e_2 \partial v} \right) \left(\frac{\partial T_m}{e_1 \partial u} + d_m \frac{\partial T_m'}{e_2 \partial v} \right) \right. \\
& \qquad \left. + \left(\frac{\partial T_n}{e_2 \partial v} - d_n \frac{\partial T_n}{e_1 \partial u} \right) \left(\frac{\partial T_m}{e_2 \partial v} - d_m \frac{\partial T_m'}{e_1 \partial u} \right) \right] dS \\
& \pm \frac{\omega^2 \mu_0 \epsilon_0}{2\sqrt{h_m h_n}} \int_S \xi \epsilon^2 \frac{\chi_n^2 \chi_m^2}{k^2} T_n T_m \, dS \\
& \qquad + \frac{\sqrt{h_m h_n}}{2} \int_S \xi \epsilon \, d_n d_m \frac{\chi_n^2 \chi_m^2}{k^2} T_n' T_m' \, dS.
\end{aligned} \tag{34}$$

To analyze circular electric wave propagation, it is sufficient to consider only coupling between circular electric and other waves. Let m denote the TE_{om} wave; then

$$T_m = 0$$

and (34) reduces to

$$
\begin{aligned}
c_{mn}^{\pm} = \pm \frac{d_m h_n^2}{2\sqrt{h_m h_n}} \int_S \xi\epsilon \left(\frac{\partial T_n}{e_2 \partial v} - d_n \frac{h_n^2}{k^2} \frac{\partial T_n'}{e_1 \partial u}\right) \frac{\partial T_m'}{e_1 \partial u}\, dS \\
+ \frac{1}{2}\sqrt{h_m h_n}\, d_m \left[\int_S \xi\epsilon \left(\frac{\partial T_n}{e_2 \partial v} - d_n \frac{\partial T_n'}{e_1 \partial u}\right) \frac{\partial T_m'}{e_1 \partial u}\, dS \right. \\
\left. + \frac{d_n}{\omega^2 \mu_0 \epsilon_0} \int_S \xi \chi_n^2 \chi_m^2 T_n T_m \, dS \right].
\end{aligned}
\tag{35}
$$

To find the z-dependence of the wave-amplitudes a_m and b_m for certain initial conditions, requires the solution of the generalized telegraphist's equations (33).

They are a system of simultaneous first-order and linear differential equations and can be solved by standard methods. From this point of view, (33) with (35) and all the preceding definitions represent the formal solution to propagation of circular electric waves in round waveguide with a single uniform lining. But this formal solution is still to be reduced to a practical form accessible for numerical evaluation. Also, heterogeneous or anisotropic linings have yet to be considered.

IV. A FIELD APPROXIMATION IN THE LINING

Before proceeding any further, an approximation will be made, which is justified for all practical linings in round waveguide for circular electric wave transmission. In all practical cases, the lining is thin compared to the radius of the guide

$$\rho - 1 \equiv \delta \ll 1. \tag{36}$$

Furthermore, it can be seen from (35) that there is only coupling between TE_{om} waves and waves of first circumferential order, i.e. $p = 1$. For these waves it may be safely assumed that

$$p^2 \ll \chi_n^{e2} r^2 \tag{37}$$

for all

$$a < r < b$$

within the lining, since $|\chi_n^e r| \geqq |k_n^e|$ and, according to (28),

$$k_n^{e2} = k^2 a^2 \left[\epsilon - \frac{h_n^2}{k^2}\right].$$

Under conditions (36) and (37), the wave functions (2) for the lining may be simplified, by replacing the Hankel-functions by their asymptotic expressions. The result is:

$$
\begin{aligned}
T_n &= N_n \frac{k_n^2}{k_n^{e2}} J_p(k_n) \frac{\sin\left(\rho - \frac{r}{a}\right) k_n^e}{\sin(\rho - 1)k_n^e} \sin p\varphi \\
T_n' &= N_n \frac{k_n^2}{k_n^{e2}} J_p(k_n) \frac{\cos\left(\rho - \frac{r}{a}\right) k_n^e}{\cos(\rho - 1)k_n^e} \cos p\varphi .
\end{aligned}
\tag{38}
$$

The characteristic equation may now also be simplified to:

$$
\left(y_n - \frac{1}{k_n^e} \tan \delta\, k_n^e\right)\left(y_n + \frac{\epsilon}{k_n^e} \cot \delta\, k_n^e\right) = p^2(\epsilon - 1)^2 \frac{h_n^2 k^2 a^4}{k_n^4 k_n^{e4}} \tag{39}
$$

where

$$
y_n = \frac{J_p'(k_n)}{k_n J_p(k_n)} . \tag{40}
$$

Likewise, using (39) and the simplified form of (26), the factor d_n may be written:

$$
d_n = \frac{k_n^2 k_n^{e2}}{p(\epsilon - 1)h_n^2 a^2} \left(\frac{\epsilon}{k_n^e} \cot \delta\, k_n^e + y_n\right). \tag{41}
$$

The orthonormality condition (29) determines the normalization factor N_n. Using the asymptotic expressions (38) for the field functions within the lining, the integration in (29) results in:

$$
\begin{aligned}
&\frac{\pi}{2} N_n^2 k_n^2 J_p^2(k_n) \left\{\left(1 + d_n \frac{2h_n^2}{k^2}\right)\left(1 - \frac{p^2}{k_n^2} + k_n^2 y_n^2 + 2y_n\right)\right. \\
&\qquad - 2\, d_n \frac{p}{k_n^2}\left[\left(1 + \frac{h_n^2}{k^2}\right) - \left(1 + \frac{h_n^2}{\epsilon k^2}\right)\frac{\epsilon k_n^4}{k_n^{e4}}\right] \\
&+ \left.\frac{k_n^2}{2k_n^{e3}}\left(\epsilon \frac{2\delta k_n^e + \sin 2\delta k_n^e}{\sin^2 \delta\, k_n^e} + d_n^2 \frac{h_n^2}{k^2} \frac{2\delta k_n^e - \sin 2\delta\, k_n^e}{\cos^2 \delta\, k_n^e}\right)\right\} = 1.
\end{aligned}
\tag{42}
$$

For circular electric waves with $p = 0$, the integration results in:

$$
\begin{aligned}
&\pi N_m^2\, d_m^2 k_m^2 J_0^2(k_m) \frac{h_m^2}{k^2}\left[1 + k_m^2 y_m^2 + 2y_m\right. \\
&\qquad\qquad \left. + \frac{k_m^2}{k_m^{e3}} \frac{2\delta k_m^e - \sin 2\delta\, k_m^e}{2 \cos^2 \delta\, k_m^e}\right] = 1.
\end{aligned}
\tag{43}
$$

The asymptotic expressions (38) have also been used to calculate the coupling coefficients (35) for circular electric waves. Instead of y_n, another abbreviation

$$x_n = \frac{J_1(k_n)}{k_n\, J_0(k_n)} \tag{44}$$

has been used to facilitate numerical evaluation:

$$\begin{aligned} c_{mn}^{\pm} = {} & \frac{\pi}{2}\sqrt{h_m h_n}\,\frac{k_m^{\,2}\,d_m N_n N_m}{k_m^2 - k_n^2}\,J_1(k_n)\,J_0(k_m)\,\frac{a}{R}\cdot\Bigg\{\left(1 \pm \frac{h_m}{h_n}\right)\Bigg[1 + d_n \\ & + d_n\,\frac{h_n^{\,2}}{k^2}\left(\frac{2k_n^{\,2}}{k_n^{\,2} - k_m^{\,2}} - \frac{1}{x_n}\right) - \left(1 + d_n\,\frac{k_n^{\,2} + k_m^{\,2}\left[1 - 2\,\frac{\chi_n^{\,2}}{k^2}\right]}{k_n^{\,2} - k_m^{\,2}}\right)\frac{x_m}{x_n} \\ & - d_n k_n^{\,2} x_m + \epsilon\,\frac{k_n^{\,2}}{k_n^{\,e2}}\,\frac{k_m^{\,2} - k_n^{\,2}}{k_n^{\,e2} - k_m^{\,e2}}\left(1 - \frac{k_n^{\,e}}{k_m^{\,e}}\,\frac{\tan k_m^{\,e}\,\delta}{\tan k_n^{\,e}\,\delta}\right. \\ & \left. - d_n k_n^{\,e}\left[1 - \frac{k_n^{\,2}}{\epsilon k^2 a^2}\right]\tan k_n^{\,e}\,\delta + d_n\,\frac{k_n^{\,e2}}{k_m^{\,e}}\,\tan k_m^{\,e}\,\delta\right)\Bigg] \\ & + d_n\,\frac{\chi_n^{\,2}}{k^2}\Bigg[\left(1 \pm \frac{h_m}{h_n}\,\frac{k_n^{\,e2}}{k_m^{\,e2}}\right)\frac{k_n^{\,2} - k_m^{\,2}}{k_n^{\,e2} - k_m^{\,e2}}\,k_m^{\,e}\,\tan k_m^{\,e}\,\delta \\ & \qquad + \left(k_m^{\,2} \pm \frac{h_m}{h_n}\,k_n^{\,2}\right)x_m \pm \frac{h_m}{h_n}\left(\frac{x_m}{x_n} - 1\right)\Bigg]\Bigg\}. \end{aligned} \tag{45}$$

Equations (39) to (45) reduce the problem to as simple analytical expressions as seems to be possible at this time. Any further simplification would only be brought about by replacing the trigonometric functions and the Bessel-functions by their Taylor series expansions for small arguments, respectively arguments k_n close to the roots of Bessel-functions for the empty guide. Such simplification, however, would lead to a first-order approximation for very thin lining, which has been studied in detail elsewhere.[1]

The present aim is for a better approximation. Therefore, numerical methods starting with expressions (39) through (45) will have to do the rest.

To this end the characteristic equation (39), which in implicit form determines the separation-constant k_n, will first have to be solved. For a lossless or low-loss lining, the relative permittivity is real or may be assumed real. Then all the quantities in (39) are real. An iterative procedure for this solution has been found earlier.[1]

For the sake of completeness it will be repeated here using the present symbols. Equation (39) may be written as:

$$\cot \delta k_n^e = \frac{F}{2}\left(1 \pm \sqrt{1 + \frac{4}{F^2\epsilon}}\right) \tag{46}$$

where

$$F = \frac{1}{k_n^e y_n}\left[1 + (\epsilon - 1)\frac{2p^2 h_n^2 k^2 a^4}{\epsilon\, k_n^4\, k_n^{e2}}\right] - \frac{k_n^e y_n}{\epsilon}. \tag{47}$$

A value for the relative permittivity ϵ of the lining is now specified. For a free-space propagation constant $k = 2\pi/\lambda$, a wave propagation constant h_n is assumed and in calculating

$$\begin{aligned} k_n^2 &= k^2a^2 - h_n^2a^2 \\ k_n^{e2} &= \epsilon k^2a^2 - h_n^2a^2 \end{aligned} \tag{48}$$

for lack of knowing the true radius of the lining, a is replaced by b, the radius of the guide. Using (47), a first approximation for the relative thickness is found. In general, according to the two signs of the square root in (46), there will be two such values of relative thickness which will lead to the same propagation constant h_n.

The first approximation for δ is used to correct the radius a of the lining in (47) and (48). The calculation is then repeated. Since for small values of δ, a change in δ affects the right-hand side only slightly, this method converges rapidly.

For typical values of b/λ and ϵ the phase constants of four normal modes have been plotted in Fig. 2.* The modes shown in Fig. 2 degenerate into TE_{11} TM_{11} TE_{12} and TM_{12} when the lining is very thin. Of all the modes, these four are most strongly coupled to TE_{01} by curvature. The broken lines represent first-order approximations as they have been found earlier.[1] Note that the first-order approximations hold only for extremely thin linings. In the case of the TE_{11} wave, in particular, the lining should be less than 0.05 per cent of the waveguide radius. Here the first-order approximations are of no use whatsoever.

Note also that the TM_{11} phase constant does not increase as expected from the first-order approximation. The curve levels off, and eventually

* These and most of the other numerical results have been obtained by H. P. Kindermann.[8]

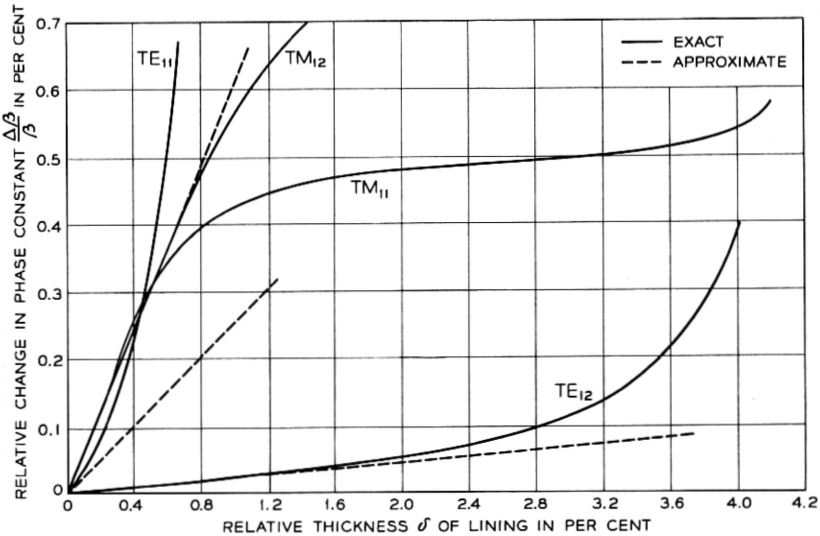

Fig. 2 — Change in phase constant of normal modes in lined waveguide, $b/\lambda = 4.70$, $\epsilon = 2.5$ — solid line exact; broken line approximate.

a heavier lining will not change the TM_{11} phase. According to Fig. 3, at higher frequencies the TM_{11} phase levels off at even lower values. Also, a higher permittivity will not change these relations.

This is, of course, very unfortunate since to reduce curvature losses the TM_{11} phase should differ most from the TE_{01} phase.

Having solved the characteristic equation, curvature coupling is

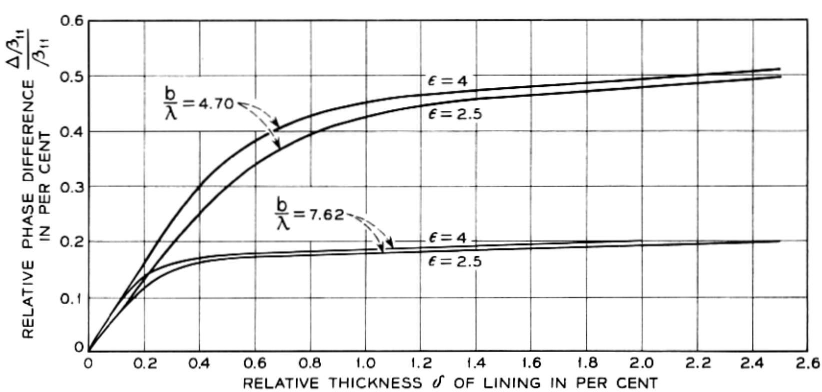

Fig. 3 — The phase constant of TM_{11} is, over a wide range, nearly independent of δ and ϵ.

obtained by substituting numerical values into (45). For the TE_{01} characteristics first-order approximations may be substituted. Even a very heavy lining does not change these characteristics very much. For example, with $\delta = 0.03$ and $\epsilon = 2.5$ the relative change in phase constant of TE_{01} according to this approximation is only:

$$\frac{\Delta\beta_m}{\beta_m} = 2.10^{-4}.$$

The coefficients of curvature coupling between TE_{01} and TE_{11}, TM_{11} and TE_{12} are plotted in Fig. 4. Note again that any first-order approximations hold only for extremely thin linings.

The coupling between TE_{01} and TM_{11} is at first increased by the lining and then stays almost constant. The increase in TE_{01}-TM_{11} coupling disagrees with another first-order approximation.[4] The present result has to be considered correct, however, since the corresponding shielded helix waveguide curvature coupling is about equally enhanced.[9]

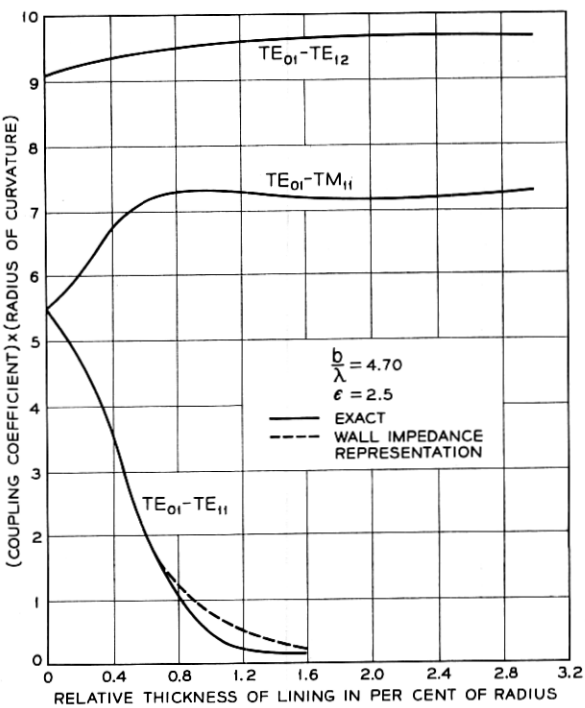

Fig. 4 — Coefficient of curvature coupling in lined waveguide $b/\lambda = 4.70$, $\epsilon = 2.5$ — solid line exact; broken line is wall impedance representation.

Curvature coupling between TE_{01} and TE_{11} decreases substantially in lined waveguide as is shown by the solid line in Fig. 4. The broken line will be referred to later. TE_{01}-TE_{12} coupling is nearly independent of the lining.

V. A WALL IMPEDANCE REPRESENTATION

The preceding analysis of lined waveguide considers only the simplest case, of a single isotropic and uniform lining. Yet this analysis could only be reduced to analytical expressions which are still quite involved and require a lot of computation for numerical evaluation. It is even more difficult to analyze a waveguide with a more complicated lining by the same methods. Further simplifications are necessary to facilitate the analysis of anisotropic or heterogeneous linings.

Such simplifications are brought about when the effects of the lining are described by wall impedances which the lining presents to the waveguide interior. Looking in radial direction, two wall impedances may be defined which are associated with the two different polarizations of the fields:

$$Z_w = -\frac{E_w}{H_v}; \qquad Z_v = \frac{E_v}{H_w}. \tag{49, 50}$$

For a mode n represented by one term n of the series expansions (8) and (15), these wall impedances can be expressed by the field functions (38):

$$Z_w = j\frac{\chi_n^e}{\omega\epsilon\epsilon_0} \frac{1}{\cot k_n^e\delta + d_n \dfrac{p}{k_n^e}\dfrac{h_n^2}{k_e^2}} \tag{51}$$

$$Z_v = j\frac{\omega\mu_0}{\chi_n^e}\left[\tan k_n^e\delta - \frac{p}{d_n k_n^e}\right]. \tag{52}$$

For circular symmetric modes $p = 0$ and

$$Z_{w0} = j\frac{\chi_n^e}{\omega\epsilon\epsilon_0}\tan k_n^e\,\delta \tag{53}$$

$$Z_{v0} = j\frac{\omega\mu_0}{\chi_n^e}\tan k_n^e\,\delta. \tag{54}$$

The characteristic equation can be derived with these wall impedances. Instead of satisfying the boundary conditions (16) through (19), the two conditions (49) and (50) will now be substituted. The ratio

$-(E_w/H_v)$ and E_v/H_w of the field components in the waveguide interior adjacent to the lining will be required to be equal to the wall impedances Z_w and Z_v presented by the lining:

$$-\frac{E_w}{H_v} \equiv \frac{1}{j\omega\epsilon_0 \, a \left(y_n - d_n \dfrac{p}{k_n^2} \dfrac{h_n^2}{k^2} \right)} = Z_w \tag{55}$$

$$\frac{E_v}{H_w} \equiv j \frac{k^2 a^2}{\omega\epsilon_0 a} \left(y_n - \frac{p}{k_n^2 \, d_n} \right) = Z_v \,. \tag{56}$$

After the factor d_n has been eliminated from (55) and (56) one equation, the characteristic equation, remains:

$$\left(y_n + \frac{j}{\omega\epsilon_0 a Z_w} \right) \left(y_n + j \frac{Z_v}{\omega\mu_0 a} \right) = \frac{p^2}{k_n^4} \frac{h_n^2}{k^2} \,. \tag{57}$$

This equation is still exact to the same order as are the wall impedances. If expressions like (51) and (52) for the wall impedance were substituted, the same equation as before, that is (39), would result.

Instead of (51) and (52) wall impedance values for circular symmetric modes as given by (53) and (54) will now be substituted into (57). But for the rest, the circumferential index p will be kept general in (57). One obtains:

$$\left(y_n - \frac{1}{k_n^e} \tan \delta \, k_n^e \right) \left(y_n + \frac{\epsilon}{k_n^e} \cot \delta \, k_n^e \right) = \frac{p^2}{k_n^4} \frac{h_n^2}{k^2} \,. \tag{58}$$

This expression is quite similar to (39).

The left-hand sides of (58) and (39) are identical, and there is only a small difference on the right-hand sides of these equations. The right-hand side of (39), for example, may be written as:

$$\frac{h_n^2 p^2}{k^2 k_n^4} \frac{k^4 a^4 (\epsilon - 1)^2}{k_n^{e4}} = \frac{h_n^2 p^2}{k^2 k_n^4} \frac{(\epsilon - 1)^2}{\left(\epsilon - \dfrac{h_n^2}{k^2} \right)^2} \,. \tag{59}$$

All modes of interest are those which have nearly the same propagation constant h_n as the circular electric wave. Since under practical circumstances the circular electric wave propagates nearly as in free space, h_n will also be nearly equal to k. If under these circumstances the right-hand side of (39) according to (59) is compared with the right-hand side of (58), the difference is found to be very small indeed.

To determine the normal modes in a straight waveguide with a single dielectric lining, it therefore seems well justified to use (58) as characteristic equation instead of (39).

As further confirmation, this approximation (58) has been solved numerically and compared with solutions of (39).

Let δ' be the thickness of the lining. Solving (58) will give a certain phase constant for a particular mode. To obtain the same phase constant from (39), the thickness has to be δ. Plotted in Fig. 5 for the three modes is the relative error $(\delta' - \delta)/\delta$ that results from using (58) instead of the exact form (39).

An analytic expression for this error can be given for a very thin lining, when the modes in lined waveguide may be considered first-order perturbations of modes in metallic waveguide. Under these conditions for TM modes

$$\frac{\delta' - \delta}{\delta} = 0$$

and for TE_{pn} modes

$$\frac{\delta' - \delta}{\delta} = -\frac{k_{n0}^2}{(\epsilon - 1)k^2a^2}\left[2 - \frac{k_{n0}^2}{k^2a^2}\right]$$

where $J_p'(k_{n0}) = 0$.

In the numerical example of Fig. 5 the error is largest for the TE_{12} mode and very thin lining. For the other two modes the error stays generally below 1 per cent.

The wall impedance representation, therefore, holds for single isotropic linings of any practical dimensions.

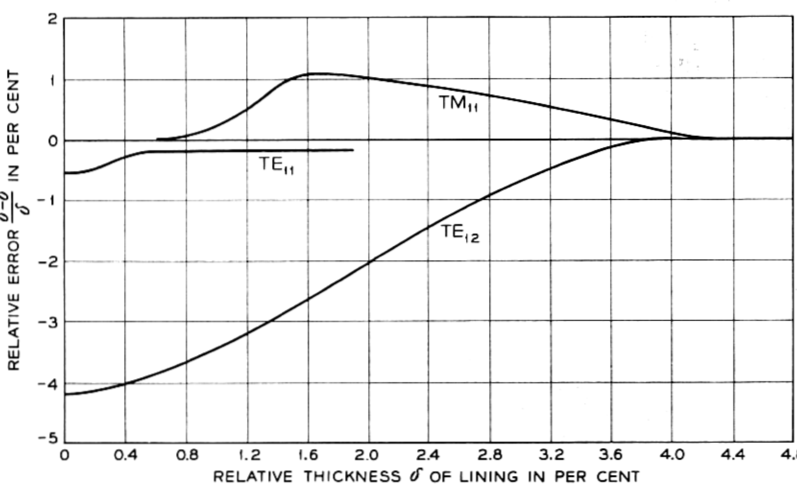

Fig. 5 — For the same phase constant, the thickness of the lining is δ according to (39) and δ' according to (58), $b/\lambda = 4.70$, $\epsilon = 2.5$.

The characteristic equation (58) for the waveguide with a single isotropic lining is not the only form to which the more general expression (57) can be reduced. It can be utilized in much more general cases. As long as we are able to determine the wall impedance Z_w and Z_v, we can use (57) to determine the normal modes of the structure for any lining such as anisotropic lining or heterogeneous linings.

The example of a single isotropic lining has taught us that it is sufficient to use wall impedance values of the corresponding circular symmetric modes. It is relatively easy to find these wall impedances even for quite complicated jacket structures. We will then be able to determine the normal modes characteristics of waveguides with such complicated jacket structures.

To make full use of the wall impedance representation in our analysis of the curved waveguide, some approximations are necessary for the coefficient of curvature coupling (35) and the normalization factor N_n.

To obtain these two quantities, products of field components and other functions had to be integrated over the total cross section. The range of integration included the lining.

In our present representation we do not explicitly determine the field-distributions within the lining, but the effect of the lining is taken into account by only considering the input impedances as seen from the waveguide interior. In this representation we therefore cannot calculate the contributions to the various integrals by extending them over the lining. We will consider the effect of neglecting these contributions.

Under practical circumstances, the area of the lining is always very much smaller than the total cross section. The components of the magnetic field, since they are continuous across a dielectric boundary, are of the same order of magnitude within the lining as in the empty space of the waveguide. Except for a possible change of the order ϵ, the same is true for the components of the electric field.

In summary, then, the integrals of products of field components over the area of the lining are always very much smaller than the corresponding integrals over the total cross section. They consequently might be neglected.

Under these circumstances, (42) reduces to

$$\frac{\pi}{2} N_n^2 k_n^2 J_p^2(k_n) \cdot \left[\left(1 + d_n^2 \frac{h_n^2}{k^2}\right) \cdot \left(1 - \frac{p^2}{k_n^2} + 2 y_n + k_n^2 y_n^2\right) - 2 \left(1 + \frac{h_n^2}{k^2}\right) d_n \frac{p}{k_n^2} \right] = 1 \tag{60}$$

and (43) for circular electric waves to:

$$\pi N_m^2 d_m^2 k_m^2 J_0^2(k_m) \frac{h_m^2}{k^2} (1 + 2y_m + k_m^2 y_m^2) = 1. \tag{61}$$

For the coefficient of curvature coupling we get instead of (45)

$$\begin{aligned} c_{mn}^{\pm} = \frac{\pi}{2} \sqrt{h_m h_n} \frac{d_m k_m^2 N_n N_m}{k_m^2 - k_n^2} J_1(k_n) J_0(k_m) \frac{a}{R} \\ \cdot \left\{ \left(1 \pm \frac{h_m}{h_n}\right) \left[1 + d_n + d_n \frac{h_n^2}{k^2} \left(\frac{2k_n^2}{k_n^2 - k_m^2} - \frac{1}{x_n} \right) \right. \right. \\ \left. - \left(1 + d_n \frac{k_n^2 + k_m^2 \left(1 - 2 \frac{\chi_n^2}{k^2}\right)}{k_n^2 - k_m^2} \right) \frac{x_m}{x_n} - d_n k_n^2 x_m \right] \\ \left. + d_n \frac{\chi_n^2}{k^2} \left[\left(k_m^2 \pm \frac{h_m}{h_n} k_n^2 \right) x_m \pm \left(\frac{x_m}{x_n} - 1 \right) \frac{h_m}{h_n} \right] \right\}. \end{aligned} \tag{62}$$

The factor d_n in all these equations can be determined from (55) or (56). For example from (55) we get:

$$d_n = \frac{k_n^2}{p} \frac{k^2}{h_n^2} \left(y_n + \frac{j}{\omega \epsilon_0 a Z_w} \right). \tag{63}$$

After the characteristic equation (57) has been solved for a particular combination of wall impedance values Z_w and Z_v, all the other quantities and eventually the coefficient of curvature coupling $c_{mn}^{\pm}$ can be found by straightforward evaluation of (60) to (63). The wall impedances Z_w and Z_v may of course be determined from the circular symmetric field components.

The approximations which have been made to obtain (60) to (63) have been examined more closely by numerical evaluation. The coefficient of curvature coupling has been calculated using these equations and compared with the plots in Fig. 4. For TM_{11}-TE_{01} and TE_{12}-TE_{01} coupling the differences are small enough not to show in Fig. 4. The wall impedance representation fails only for TE_{01}-TE_{11} coupling and $\delta > 0.8$ per cent. The corresponding coefficient of curvature coupling is shown by the broken line in Fig. 4. Fortunately the coupling is then so small and the phase constants of the waves differ so much that there is no significant interaction between TE_{01} and TE_{11}.

The explanation of why the two methods of calculation result in different values in just this case is as follows:

The TE_{11} wave in round waveguide is most strongly modified by the lining. Even in a thin lining, the TE_{11} field tends to be concentrated within the lining. In the present numerical example, it takes only a relative thickness of $\delta = 0.4$ per cent for the radial propagation constant k_n to become imaginary and consequently the TE_{11} fields to be evanescent towards the center of the guide. When this happens because of very weak TE_{11} fields within the guide, the curvature coupling to TE_{01} will be very weak too.

The wall impedance representation fails for calculating the coupling, because it entirely neglects any field interaction within the lining, which is more and more significant for TE_{11} and a thick lining. This phenomenon is limited to TE_{11} ; coupling to all other modes is accurately described by the wall impedance representation.

VI. WALL IMPEDANCE OF ANISOTROPIC AND HETEROGENEOUS LININGS

It has now been established that the wall impedance representation is a useful method in analyzing wave propagation in straight and curved sections of lined waveguide. To use this method for waveguides with anisotropic or heterogeneous linings we need to know the wall impedances of these linings.

6.1 *Anisotropic Lining*

Flock coating shows promise as a lining for circular electric waveguide. Resistive fibers of the flock are parallel to the electric field of unwanted modes but perpendicular to circular electric fields. A flock coat is anisotropic, and in an (x,y,z) system identified by

$$u = x, \qquad av = y, \qquad w = z \tag{64}$$

it may be described by the permittivity tensor

$$\| \epsilon \| = \begin{Vmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_z & 0 \\ 0 & 0 & \epsilon_z \end{Vmatrix}. \tag{65}$$

Wall impedances of circular symmetric modes are used in the wall impedance representation. In our present system of coordinates, circular symmetry corresponds to $\partial/\partial y = 0$.

Assuming furthermore a z-dependence of e^{-jhz}, Maxwell's equations may be written in the following form:

$$\begin{aligned} j\,h\,E_y &= j\omega\mu H_x \\ -j\,h\,E_x - \frac{\partial E_z}{\partial x} &= -j\omega\mu H_y \\ \frac{\partial E_y}{\partial x} &= -j\omega\mu H_z \\ j\,h\,H_y &= j\,\omega\,\epsilon_x\,E_x \\ -j\,h\,H_x - \frac{\partial H_z}{\partial x} &= j\,\omega\,\epsilon_z\,E_y \\ \frac{\partial H_y}{\partial x} &= j\,\omega\,\epsilon_z\,E_z\,. \end{aligned} \tag{66}$$

Eliminating the field components E_x, E_y and H_x, H_y from (66) we get:

$$\frac{\partial^2 E_z}{\partial x^2} + (\omega^2\mu\,\epsilon_x - h^2)\,\frac{\epsilon_z}{\epsilon_x}\,E_z = 0 \tag{67}$$

$$\frac{\partial^2 H_z}{\partial x^2} + (\omega^2\,\mu\,\epsilon_z - h^2)\,H_z = 0. \tag{68}$$

The general solution of these equations has the form

$$Ae^{j\chi x} + Be^{-j\chi x}$$

where for (67)

$$\chi = \chi_x\sqrt{\frac{\epsilon_z}{\epsilon_x}} = \sqrt{\frac{\epsilon_z}{\epsilon_x}\,(\omega^2\mu\,\epsilon_x - h^2)}$$

and for (68)

$$\chi = \chi_z = \sqrt{\omega^2\mu\epsilon_z - h^2}$$

are the propagation constants in x- or radial direction.

A wave traveling in positive x-direction or outwardly in the cylindrical system is represented by the second term. For such waves

$$\frac{\partial}{\partial x} = -j\chi$$

and we obtain from Maxwell's equations and (67)

$$j\,\omega\,\epsilon_z\,E_z = -j\,\chi_x \sqrt{\frac{\epsilon_z}{\epsilon_x}}\,H_y$$
$$Z_z = \frac{\chi_x}{\omega\sqrt{\epsilon_x\epsilon_z}} \tag{69}$$

and from (68)

$$-j\omega\mu\,H_z = -j\,\chi_z\,E_y$$
$$Z_y = \frac{\omega\mu}{\chi_z}. \tag{70}$$

Z_z and Z_y are the wave impedances of an anisotropic medium. The wall impedances of the anisotropic lining are input impedances of a radial transmission line of length $(b - a)$ short-circuited at the end. In our present approximation:

$$Z_w = j\,\frac{\chi_x}{\omega\sqrt{\epsilon_x\epsilon_z}}\tan\chi_x\sqrt{\frac{\epsilon_z}{\epsilon_x}}\,(b - a) \tag{71}$$

$$Z_v = j\,\frac{\omega\mu}{\chi_z}\tan\chi_z\,(b - a). \tag{72}$$

To make Z_w and Z_v constants of the waveguide, independent of a particular mode, we consider only modes which are sufficiently far from cutoff to propagate nearly as in free space. Then

$$\chi_x = \frac{2\pi}{\lambda}\sqrt{\frac{\epsilon_x}{\epsilon_0} - 1}$$
$$\chi_z = \frac{2\pi}{\lambda}\sqrt{\frac{\epsilon_z}{\epsilon_0} - 1}. \tag{73}$$

For circular electric waves only Z_v enters the boundary condition. Note that in (72) Z_v is independent of ϵ_x. Resistive components in the flock coat will cause a loss factor only of ϵ_x. Such resistive components leave the circular electric wave loss unaffected.

6.2 *Double Lining*

A base layer of dissipative material and a top layer of low-loss material provide mode filtering for TE_{01} transmission and reduce TE_{01} loss in bends.[3]

Let the base lining have a relative permittivity ϵ_b and thickness $b - a$, the top lining ϵ_t and $a_1 - a$.

Input impedances of the base lining are

$$Z_{w1} = j \frac{\chi_1}{\omega \epsilon_b \epsilon_0} \tan \chi_1(b - a), \qquad Z_{v1} = j \frac{\omega \mu}{\chi_1} \tan \chi_1(b - a)$$

where

$$\chi_1 = k \sqrt{\epsilon_b - 1}.$$

These input impedances are transformed by the second lining according to ordinary transmission line theory

$$Z_w = j \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\sqrt{\epsilon_t - 1}}{\epsilon_t} \frac{\epsilon_t \sqrt{\epsilon_b - 1} \tan \varphi_b + \epsilon_b \sqrt{\epsilon_t - 1} \tan \varphi_t}{\epsilon_b \sqrt{\epsilon_t - 1} - \epsilon_t \sqrt{\epsilon_b - 1} \tan \varphi_b \tan \varphi_t} \tag{74}$$

$$Z_v = j \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{1}{\sqrt{\epsilon_t - 1}} \frac{\sqrt{\epsilon_t - 1} \tan \varphi_b + \sqrt{\epsilon_b - 1} \tan \varphi_t}{\sqrt{\epsilon_b - 1} - \sqrt{\epsilon_t - 1} \tan \varphi_b \tan \varphi_t} \tag{75}$$

where

$$\varphi_b = 2\pi \sqrt{\epsilon_b - 1} \frac{b - a_1}{\lambda}$$

and

$$\varphi_t = 2\pi \sqrt{\epsilon_t - 1} \frac{a_1 - a}{\lambda}.$$

For a thin base layer $\varphi_b \ll 1$ and

$$Z_v = j \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\frac{2\pi}{\lambda}(b - a_1) + \frac{\tan \varphi_t}{\sqrt{\epsilon_t - 1}}}{1 - \frac{2\pi}{\lambda}(b - a_1) \sqrt{\epsilon_t - 1} \tan \varphi_1}. \tag{76}$$

Note that in (76) Z_v is independent of the permittivity in the base layer. Any loss in the base layer will not significantly raise circular electric wave loss.

VII. CONCLUSION

In previous first-order approximations, normal modes of lined waveguide were considered perturbed modes of plain waveguide, and coefficients of curvature couplings were assumed the same as in metallic waveguide. In some respects these approximations hold only for extremely thin linings, thinner for example than 0.05 per cent of the waveguide radius. The present more exact analysis shows that the TE_{11} wave has a phase constant much higher than would be expected from these approximations. Neither the thickness nor the permittivity of the lining can increase the phase difference between TM_{11} and TE_{01} beyond a certain limit. The phase difference eventually is almost independent of δ and ϵ and is small for high frequency.

Curvature coupling between TE_{11} and TE_{01} is substantially smaller in lined waveguide than in plain waveguide, while it is nearly independent of the lining between TE_{12} and TE_{01}. Between TM_{11} and TE_{01}, however, it first increases and then stays constant.

Waveguides with sandwiched or anisotropic linings may be analyzed by using a wall impedance representation. Wall impedances which the lining presents to fields of circular symmetry may be used in this analysis. They may easily be calculated for flock coatings and double linings. The wall impedance representation is found to be accurate for all typical cases.

REFERENCES

1. Unger, H. G., Circular Electric Wave Transmission in a Dielectric-Coated Waveguide, B.S.T.J., **36,** Sept., 1957, p. 1253.
2. Unger, H. G., Round Waveguide with Lossy Lining, Proc. of Symposium on Millimeter Waves, New York, 1959.
3. Unger, H. G., Round Waveguide with Double Lining, B.S.T.J., **39,** Jan., 1961, p. 161.
4. Noda, Ken-ichi, Circular Electric Wave Transmission through Hybrid-Mode Waveguide, Review of the Electrical Communication Lab. (Tokyo), **8,** No. 9–10, Sept.–Oct., 1960.
5. Katsenelenbaum, B. Z., The Effect of a Dielectric Film on the Attenuation of H_{01} Waves in a Straight Nearly Circular Waveguide, Radiotekhnika i Electronika, 1958, p. 38.
6. Kreipe, H. L., and Unger, H. G., Imperfections in Lined Waveguide, to be published.
7. Bressler, A. D., Joshi, G. H., and Marcuvitz, N., Orthogonality Properties for Modes in Passive and Active Uniform Waveguides. J. Appl. Phys., **29,** p. 794–799.
8. Kindermann, H. P., Kruemmungskopplung im Rundhohlleiter mit Dielektrischem Wandbelag. Diplom-Thesis submitted to 'Institut fuer Hoechstfrequenztechnik,' Technische Hochschule Braunschweig.
9. Unger, H. G., Helix Waveguide Theory and Application, B.S.T.J., **37,** Nov., 1958, p. 1599.