Approximate Solutions for the Coupled Line Equations

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The coupled line equations for only two modes, representing the TE_{01} *signal mode and a single spurious mode in circular waveguide, are solved in series form by the method of successive approximations. Bounds are found on the magnitudes of the terms in the series solution. These bounds decrease rapidly only for "short" waveguides; for long guides many terms of the series must be included in the solution.*

The coupled line equations are transformed to a new form, in which one of the unknowns Λ *is given by* $\Lambda = -\ln G_0$, *where* G_0 *is the (complex)* TE_{01} *transfer function of the original coupled line equations. Thus* $\text{Re } \Lambda = -\ln |G_0|$, *the* TE_{01} *loss in nepers,* $\text{Im } \Lambda = -\angle G_0$, *the* TE_{01} *phase in radians. These transformed equations are again solved by successive approximations; the first term is the commonly used solution that has been obtained by physical arguments. Bounds are determined for the magnitudes of the terms in these series solutions; for a suitable restriction on the coupling coefficient that includes many cases of practical interest, these bounds decrease rapidly for long guides.*

In present calculations of the TE_{01} *loss statistics in random guides, only the first term of the series expansion for* Λ *is considered. Unfortunately this approximation has not so far been justified.*

I. INTRODUCTION

Consider the coupled line equations:

$$\begin{aligned} I_0'(z) &= -\Gamma_0 I_0(z) + jc(z) I_1(z), \\ I_1'(z) &= +jc(z) I_0(z) - \Gamma_1 I_1(z). \end{aligned} \tag{1}$$

These equations are of interest in many applications. Our particular interest in them in a companion paper[1] is that they describe the effects of coupling between the TE_{01} signal mode, represented by the complex wave amplitude I_0, and a single spurious mode, represented by the complex wave amplitude I_1, caused by geometric imperfections in circular waveguide. We have, of course, assumed that only a single spurious mode has significant magnitude, so that all other spurious modes may be neglected. For example, we may consider copper waveguide with a rather general straightness deviation; the most important spurious mode under many conditions will be the forward TE_{12}[1] (both polarizations must, of course, be considered unless the straightness deviation is confined to a single plane). However, these equations apply to a variety of other problems which may be described by only two modes with varying degrees of accuracy.[1]

In copper waveguide if the wall losses may be neglected the propagation constants Γ_0 and Γ_1 are pure imaginary and the coupling coefficient $c(z)$ is pure real. In helix guide, where loss is added to the spurious mode, the propagation constant Γ_1 has a significant negative real part; further, as shown by H. G. Unger,[2] the coupling coefficient $c(z)$ also becomes complex.

The case where the geometric imperfection (e.g., straightness deviation) and hence the coupling coefficient is a stationary random process, perhaps Gaussian, is of great interest; here it is desired to compute the statistics of the TE_{01} transmission I_0 in terms of the statistics of the coupling coefficient $c(z)$. Since exact solutions to (1) are easily found in only a few special cases, this has been done by using an approximate solution to these equations that is essentially a second-order perturbation solution, and by studying the statistics of this approximate solution.[1] The present paper will discuss this approximate solution, will give some bounds on the convergence of the approximation, and will indicate a basic gap in our knowledge concerning this problem.

Equation (1) represents a drastic idealization of the real TE_{01} transmission problem, in that it contains only one spurious mode and neglects all other spurious modes. The approximate solution includes all second-order terms; a physical interpretation of this solution states that conversion of TE_{01} to each spurious mode and subsequent reconversion to TE_{01} is considered at all pairs of elementary mode converters, but that higher-order terms involving more than one pair of elementary mode converters are neglected.[1] The exact solution of (1) includes all higher-order terms involving the single spurious mode, but neglects many more higher-order terms involving the many other spurious modes that have been neglected in (1). In view of this it may appear questionable to try to deal with (1) in more exact terms for the TE_{01} mode conversion problem. However, a start has to be made somewhere, and it seems unlikely that the general case involving an infinite number of modes will be understood before the two-mode case of (1) is understood. Even this simple idealized case does not yet have a really satisfactory treatment. Also, (1) does apply more or less rigorously to many other situations than the TE_{01} mode conversion problem.[1]

In dealing with these equations it is convenient to introduce the following change of variables:

$$\begin{aligned} I_0(z) &= e^{-\Gamma_0 z} \cdot G_0(z). \\ I_1(z) &= e^{-\Gamma_1 z} \cdot G_1(z). \end{aligned} \tag{2}$$

Then (1) becomes:

$$\begin{aligned} G_0'(z) &= jc(z)\, e^{\Delta\Gamma z}\, G_1(z), \\ G_1'(z) &= jc(z)\, e^{-\Delta\Gamma z}\, G_0(z). \end{aligned} \tag{3a}$$

$$\begin{aligned} \Delta\Gamma &= \Delta\alpha + j\Delta\beta = \Gamma_0 - \Gamma_1 ; \\ \Delta\alpha &= \alpha_0 - \alpha_1 < 0, \\ \Delta\beta &= \beta_0 - \beta_1 . \end{aligned} \tag{3b}$$

Note that we assume $\Delta\alpha < 0$, because in circular waveguide the TE_{01} signal mode will have lower heat loss than any of the spurious modes. We will always take as initial conditions at $z = 0$ a TE_{01} wave of unit magnitude and zero phase, and a spurious mode of zero magnitude:

$$G_0(0) = 1, \qquad G_1(0) = 0. \tag{4}$$

II. SOLUTION OF THE COUPLED LINE EQUATIONS BY SUCCESSIVE APPROXIMATIONS (PICARD'S METHOD)

We summarize the solution of (3) by successive approximations. Let $G_{0(n)}(z)$ and $G_{1(n)}(z)$ be the n^{th} approximation to the solution of (3). Let the initial approximation be given simply by the initial conditions of (4):

$$G_{0(0)}(z) = 1, \qquad G_{1(0)}(z) = 0. \tag{5}$$

Then following Picard's method[3,4] we obtain the successive approximations as follows:

$$G_{0(0)}(z) = 1, \qquad G_{1(0)}(z) = 0$$

$$G_{0(1)}(z) = 1 + j\int_0^z c(s)e^{\Delta\Gamma s}G_{1(0)}(s)\,ds$$
$$= 1$$
$$G_{1(1)}(z) = j\int_0^z c(s)e^{-\Delta\Gamma s}G_{0(0)}(s)\,ds$$
$$= j\int_0^z c(s)e^{-\Delta\Gamma s}\,ds$$

$$G_{0(2)}(z) = 1 + j\int_0^z c(s)e^{\Delta\Gamma s}G_{1(1)}(s)\,ds$$
$$= 1 - \int_0^z c(s)e^{\Delta\Gamma s}\,ds\int_0^s c(t)e^{-\Delta\Gamma t}\,dt \tag{6}$$
$$G_{1(2)}(z) = j\int_0^z c(s)e^{-\Delta\Gamma s}G_{0(1)}(s)\,ds$$
$$= j\int_0^z c(s)e^{-\Delta\Gamma s}\,ds$$

$$\vdots \cdots\cdots\cdots\cdots\cdots\cdots$$

$$G_{0(n)}(z) = 1 + j\int_0^z c(s)e^{\Delta\Gamma s}G_{1(n-1)}(s)\,ds$$
$$G_{1(n)}(z) = j\int_0^z c(s)e^{-\Delta\Gamma s}G_{0(n-1)}(s)\,ds.$$

The n^{th} approximation is obtained by substituting the $(n - 1)^{\text{th}}$ approximation in the right-hand side of (3) and integrating. Writing

$$G_{0(n)}(z) - G_{0(n-1)}(z) = g_{0(n)}(z),$$
$$G_{1(n)}(z) - G_{1(n-1)}(z) = g_{1(n)}(z), \tag{7}$$

we have

$$G_{0(n)}(z) = 1 + \sum_{k=1}^{n} g_{0(k)}(z), \tag{8a}$$

$$G_{1(n)}(z) = \sum_{k=1}^{n} g_{1(k)}(z), \tag{8b}$$

where the g's are given as follows:

$$g_{0(n)}(z) = j \int_0^z c(s) e^{\Delta\Gamma s} g_{1(n-1)}(s)\, ds, \quad n \geqq 1. \tag{9a}$$

$$g_{1(n)}(z) = j \int_0^z c(s) e^{-\Delta\Gamma s} g_{0(n-1)}(s)\, ds, \quad n \geqq 1. \tag{9b}$$

$$g_{0(0)}(z) = 1, \qquad g_{1(0)}(z) = 0. \tag{9c}$$

It is readily seen that

$$\begin{aligned} g_{0(n)}(z) &= 0, \qquad n \text{ odd} \\ g_{1(n)}(z) &= 0, \qquad n \text{ even} \end{aligned} \tag{10}$$

so that only even terms appear in the summation of (8a) and only odd terms in the summation of (8b).

In the standard proof of Picard's method the series of (8) are shown to converge to the unique solution of the coupled line equations, (3), and bounds are given on the magnitudes of the terms in (8). Thus we may write

$$G_0(z) = \sum_{n=0}^{\infty} g_{0(n)}(z), \tag{11a}$$

$$G_1(z) = \sum_{n=0}^{\infty} g_{1(n)}(z), \tag{11b}$$

where the g's are given in (9) and (10). However, better bounds than those given by Picard's general method may be found for the present special case. We show that

$$\left| g_{0(n)}(z) \right| \begin{array}{ll} \leqq \dfrac{\left[\int_0^z |\, c(s)\, |\, ds \right]^n}{n!}, & n \text{ even.} \\ = \quad 0 \quad , & n \text{ odd.} \end{array} \tag{12a}$$

$$\left| g_{1(n)}(z) \right| \begin{array}{ll} = \quad 0 \quad , & n \text{ even.} \\ \leqq \dfrac{\left[\int_0^z |\, c(s)\, |\, ds \right]^n}{n!} e^{-\Delta\alpha z}, & n \text{ odd.} \end{array} \tag{12b}$$

Suppose that (12a) is true for some even value of n. Then from (9b)

$$
\begin{aligned}
| g_{1(n+1)}(z) | &\leqq \int_0^z | c(t) | e^{-\Delta\alpha t} \frac{\left[\int_0^t | c(s) | \, ds \right]^n}{n!} \, dt \\
&\leqq \frac{e^{-\Delta\alpha z}}{n!} \int_0^z \left[\int_0^t | c(s) | \, ds \right]^n d \left[\int_0^t | c(s) | \, ds \right] \qquad (13) \\
&= \frac{\left[\int_0^z | c(s) | \, ds \right]^{n+1}}{(n+1)!} e^{-\Delta\alpha z},
\end{aligned}
$$

where we recall from (3b) that $\Delta\alpha < 0$. Substituting (13) into (9a),

$$
\begin{aligned}
| g_{0(n+2)}(z) | &\leqq \int_0^z | c(t) | e^{\Delta\alpha t} e^{-\Delta\alpha t} \frac{\left[\int_0^t | c(s) | \, ds \right]^{n+1}}{(n+1)!} \, dt \\
&= \frac{1}{(n+1)!} \int_0^z \left[\int_0^t | c(s) | \, ds \right]^{n+1} d \left[\int_0^t | c(s) | \, ds \right] \qquad (14) \\
&= \frac{\left[\int_0^z | c(s) | \, ds \right]^{n+2}}{(n+2)!}.
\end{aligned}
$$

Noting (9c), the results of (12) hold for all n by induction.

We may ask whether the bounds of (12) are the best that can be obtained in general, or if by being sufficiently clever we can do better. It is easy to find examples whose terms are actually as large as those given in (12), so that no improvement in these results is to be expected unless suitable restrictions are placed on the problem. Thus, consider the following special case:

$$\Delta\Gamma = 0. \qquad (15)$$

The coupling coefficient is non-negative but otherwise arbitrary. The general solution to (3a), subject of course to the initial conditions of (4), is[1]

$$G_0(z) = \cos \left[\int_0^z c(s) \, ds \right]. \qquad (16)$$

Expanding the cosine in power series,

$$G_0(z) = 1 - \frac{\left[\int_0^z c(s) \, ds \right]^2}{2!} + \frac{\left[\int_0^z c(s) \, ds \right]^4}{4!} - \cdots. \qquad (17)$$

The successive terms of (17) are simply the $g_{0(n)}$ given in (9) and (10).

It is readily seen that the magnitudes of these terms are equal to the bounds given in (12a) if we require $0 \leqq c(z)$ so that $| c(z) | = c(z)$.

As another similar example, let $\Delta\Gamma \neq 0$ and $c(z)$ be a single δ-function located at z_0,

$$c(z) = C \cdot \delta(z - z_0). \tag{18}$$

In our present case, i.e., straightness deviation, a discrete coupling of the type given in (18) corresponds to a discrete tilt located at $z = z_0$. The solution to (3a), subject again to the initial conditions of (4), is[1]

$$G_0(z) = \cos C, \qquad z > z_0. \tag{19}$$

Expanding the cosine,

$$G_0(z) = 1 - \frac{C^2}{2!} + \frac{C^4}{4!} - \cdots. \tag{20}$$

Again the terms of (20) are the $g_{0(n)}$ of (9) and (10), and their magnitudes are equal to the bounds given in (12a). Of course, this above solution, which mathematically is valid for an arbitrarily large tilt in the present idealized two-mode case, must fail for large tilts in the physical case, the error being caused by neglecting the higher-order spurious modes excited by the tilt. While this serves as a further warning against uncritical application of the results of the two-mode theory to the physical problem, it is still of interest to inquire into the mathematical properties of the solutions to (3).

It is often desirable to express the TE_{01} loss in db rather than as the magnitude of the TE_{01} normalized gain, $| G_0 |$. Define the complex TE_{01} loss Λ as

$$\Lambda = -\ln G_0 = A - j\Theta. \tag{21}$$

Then

$$\begin{aligned} A &= -\ln | G_0 |, \\ \Theta &= \angle\, G_0. \end{aligned} \tag{22}$$

A is the TE_{01} loss in nepers; the TE_{01} loss in db is simply 8.686 A. If we have a number of sections of waveguide separated by ideal mode filters, the over-all TE_{01} gain, G_{OT}, and loss Λ_T, are given by

$$\begin{aligned} G_{OT} &= G_{0,1}\, G_{0,2} \cdots, \\ \Lambda_T &= \Lambda_1 + \Lambda_2 + \cdots, \\ A_T &= A_1 + A_2 + \cdots, \\ \Theta_T &= \Theta_1 + \Theta_2 + \cdots. \end{aligned} \tag{23}$$

The statistics of A and Θ for the composite guide may thus be expressed simply in terms of the statistics of A and Θ for the individual guide sections.

Suppose that the series solution for $G_0(z)$ given in (11a) converges very rapidly, so that only the first two terms need be retained. Then we have from (8–11)

$$G_0(z) \approx G_{0(2)}(z) \tag{24}$$

so that approximately

$$G_0(z) = 1 - \int_0^z c(s)e^{\Delta\Gamma s}\, ds \int_0^s c(t)e^{-\Delta\Gamma t}\, dt. \tag{25}$$

Then, assuming that the second term is small compared to 1, we have approximately:

$$|\, G_0(z) \,| = \text{Re } G_0 = 1 - \text{Re} \iint \tag{26a}$$

$$\Lambda = -\ln G_0 = \iint \tag{26b}$$

$$A = -\ln |\, G_0 \,| = \text{Re} \iint \tag{26c}$$

$$\Theta = \angle\, G_0 = -\text{Im} \iint \tag{26d}$$

where $\iint$ is shorthand for

$$\iint = \int_0^z c(s)e^{\Delta\Gamma s}\, ds \int_0^s c(t)\, e^{-\Delta\Gamma t}\, dt \tag{27a}$$

$$= \int_0^z e^{\Delta\Gamma u}\, du \int_0^{z-u} c(s)c(s+u)\, ds \tag{27b}$$

$$= \frac{1}{2} \int_0^z \int_0^z c(s)c(t)e^{\Delta\Gamma|t-s|}\, ds\, dt. \tag{27c}$$

If the coupling coefficient $c(z)$ is pure real but the differential propagation constant is complex (possibly not a physical case), (26c) becomes, using (27c),

$$A = -\ln |\, G_0 \,| = \int_0^z e^{\Delta\alpha u} \cos \Delta\beta u\, du \int_0^{L-u} c(s)c(s+u)\, ds. \tag{28}$$

If $c(z)$ is complex, it turns out that for uniform waveguides its phase angle remains constant and only its magnitude varies, so that we may write[2]

$$c(z) = (C_r + j\,C_i)\,\bar{c}(z), \tag{29}$$

where $\bar{c}(z)$ is real. Then (28) becomes

$$A = -\ln|G_0| = (C_r^2 - C_i^2)\int_0^z e^{\Delta\alpha u}\cos\Delta\beta u\,du \cdot\int_0^{L-u}\bar{c}(s)\bar{c}(s+u)\,ds - 2C_rC_i\int_0^z e^{\Delta\alpha u}\sin\Delta\beta u\,du \cdot\int_0^{L-u}\bar{c}(s)\bar{c}(s+u)\,ds. \tag{30}$$

These approximate expressions of (26-30) may be regarded as the first terms of series expansions for the various quantities. The above approximations will be valid when $\left|\iint\right| \ll 1$ and when the higher-order terms may be neglected. From the above analysis it would appear that when $\left|\iint\right| \gg 1$, all of the above approximations would fail, since, in particular, (26a) obviously fails. In spite of this fact, (26b-d), (28) and (30) may remain valid for a wide class of long guides of practical interest; roughly speaking, the required conditions are that the differential loss $|\Delta\alpha|$ be large enough and that the coupling coefficient $c(z)$ be sufficiently small and uniformly distributed in an appropriate sense. This result has been suggested by simple physical arguments;[1] a formal mathematical derivation starting with the appropriate restriction on $c(z)$ and $\Delta\alpha$ is given in the following section. These results are of importance because in a random guide the expected value of the $\iint$ term increases linearly with distance z;[1] while the approximations of (25) and (26a) fail, the results of (26b-d), (28) and (30) may remain valid, and so provide us with a theory for long guides.

It is apparent that further restrictions are required to obtain these additional results, by considering the example of (18-20). Thus, let the magnitude of the δ-function coupling coefficient be $\pi/2$, so that we have in (18)

$$C = \frac{\pi}{2}, \qquad c(z) = \frac{\pi}{2}\cdot\delta(z - z_0). \tag{31}$$

Then from (19)

$$G_0(z) = 0, \qquad z > z_0 \tag{32}$$

so that

$$\Lambda = -\ln G_0 = \infty. \tag{33}$$

However, the approximate result of (26b) yields

$$\Lambda = -\ln G_0 \approx \frac{C^2}{2} = \frac{\pi^2}{8}. \tag{34}$$

The approximation of (34) is obviously invalid; since these relations are independent of $\Delta\alpha$, this approximation remains invalid no matter how high $|\Delta\alpha|$ becomes. Cases such as this are ruled out by the additional restrictions that require the coupling coefficient to be more or less uniformly distributed with z in a certain sense, described in the following section.

III. TRANSFORMATION OF THE COUPLED LINE EQUATIONS TO LOGARITHMIC FORM, AND SOLUTION BY SUCCESSIVE APPROXIMATIONS

We repeat for convenience the coupled line equations, given in (3), together with the desired initial conditions, (4).

$$\begin{aligned} G_0'(z) &= j\, c(z)\, e^{\Delta\Gamma z}\, G_1(z), \\ G_1'(z) &= j\, c(z)\, e^{-\Delta\Gamma z}\, G_0(z). \end{aligned} \tag{35}$$

$$G_0(0) = 1, \qquad G_1(0) = 0. \tag{36}$$

Next, the following transformation of variables is made:

$$G_0(z) = e^{-\Lambda(z)}. \tag{37a}$$

$$G_1(z) = e^{-\Lambda(z)} \cdot H(z). \tag{37b}$$

The transformation of (37a) is dictated by the desire to obtain a series solution for Λ, defined in (21). That of (37b) was obtained partly by trial and error and partly by intuitive means. Substituting (37) into (35), we obtain:

$$\Lambda'(z) = -j\, c(z)\, e^{\Delta\Gamma z}\, H(z) \tag{38a}$$

$$H'(z) = j\, c(z)\, e^{\Delta\Gamma z} + \Lambda'(z)\, H(z). \tag{38b}$$

By substituting (38a) into the second term on the right-hand side of (38b), we have:

$$\Lambda'(z) = -j\, c(z)\, e^{\Delta\Gamma z}\, H(z) \tag{39a}$$

$$H'(z) = j\, c(z)\, e^{-\Delta\Gamma z} - j\, c(z)\, e^{\Delta\Gamma z}\, H^2(z). \tag{39b}$$

The initial conditions of (36) transform via (37) to

$$\Lambda(0) = 0, \qquad H(0) = 0. \tag{40}$$

The method of successive approximations may now be applied to (39) (or 38), subject to the initial conditions of (40). We note that $\Lambda(z)$ is absent from (39b), so that this equation contains only a single dependent variable, $H(z)$. Thus the successive approximations to $H(z)$ may be found without reference to (39a) or to $\Lambda(z)$; the corresponding approximations for $\Lambda(z)$ are then found by a simple integration of (39a). We note further that (39b) for $H(z)$ is a Riccati equation.[3]

Thus, let $\Lambda_n(z)$ and $H_n(z)$ be the n^{th} approximation to the solution of (39), subject to the initial conditions of (40). Then:

$$\begin{aligned} H_0(z) &= 0 \\ H_1(z) &= j\int_0^z c(s)e^{-\Delta\Gamma s}\,ds \\ H_2(z) &= j\int_0^z c(s)e^{-\Delta\Gamma s}\,ds - j\int_0^z c(s)e^{\Delta\Gamma s}H_1^2(s)\,ds \\ &= j\int_0^z c(s)e^{-\Delta\Gamma s}\,ds \\ &\quad + j\int_0^z c(s)e^{\Delta\Gamma s}\,ds\int_0^s\int_0^s c(t)c(u)e^{-\Delta\Gamma(t+u)}\,dt\,du \\ &\vdots \\ H_n(z) &= j\int_0^z c(s)e^{-\Delta\Gamma s}\,ds - j\int_0^z c(s)e^{\Delta\Gamma s}H_{n-1}^2(s)\,ds \end{aligned} \tag{41a}$$

$$\begin{aligned} \Lambda_0(z) &= 0 \\ \Lambda_1(z) &= -j\int_0^z c(s)e^{\Delta\Gamma s}H_1(s)\,ds \\ &= \int_0^z c(s)e^{\Delta\Gamma s}\,ds\int_0^s c(t)e^{-\Delta\Gamma t}\,dt \\ \Lambda_2(z) &= -j\int_0^z c(s)e^{\Delta\Gamma s}H_2(s)\,ds \\ &= \int_0^z c(s)e^{\Delta\Gamma s}\,ds\int_0^s c(t)e^{-\Delta\Gamma t}\,dt \\ &\quad + \int_0^z c(s)e^{\Delta\Gamma s}\,ds\int_0^s c(t)e^{\Delta\Gamma t}\,dt\int_0^t\int_0^t c(u)c(v)e^{-\Delta\Gamma(u+v)}\,du\,dv \\ &\vdots \\ \Lambda_n(z) &= -j\int_0^z c(s)e^{\Delta\Gamma s}H_n(s)\,ds. \end{aligned} \tag{41b}$$

Note that Λ_1 is identical to the approximation of (26b). Writing as before

$$\begin{aligned} \Lambda_n(z) - \Lambda_{n-1}(z) &= \lambda_n(z), \\ H_n(z) - H_{n-1}(z) &= h_n(z), \end{aligned} \tag{42}$$

we have

$$\begin{aligned} \Lambda_n(z) &= \sum_{k=1}^{n} \lambda_k(z), \\ H_n(z) &= \sum_{k=1}^{n} h_k(z). \end{aligned} \tag{43}$$

The quantities $\lambda_n(z)$ and $h_n(z)$ are given as follows:

$$\lambda_n(z) = -j \int_0^z c(s) e^{\Delta\Gamma s} h_n(s)\, ds,\ n \geqq 1. \tag{44a}$$

$$\begin{aligned} h_n(z) &= -j \int_0^z c(s) e^{\Delta\Gamma s} [H_{n-1}^2(s) - H_{n-2}^2(s)]\, ds \\ &= -j \int_0^z c(s) e^{\Delta\Gamma s} h_{n-1}(s) [H_{n-1}(s) + H_{n-2}(s)]\, ds,\ n \geqq 2. \end{aligned} \tag{44b}$$

$$h_1(z) = H_1(z) = j \int_0^z c(s) e^{-\Delta\Gamma s}\, ds. \tag{44c}$$

Then under certain conditions described below,

$$\Lambda(z) = \sum_{n=1}^{\infty} \lambda_n(z), \tag{45a}$$

$$H(z) = \sum_{n=1}^{\infty} h_n(z). \tag{45b}$$

We next obtain bounds on $| \lambda_n(z) |$ and $| h_n(z) |$. As stated in the last section, it is first necessary to impose additional restrictions on the problem. We assume that the coupling coefficient $c(z)$ and the differential loss $\Delta\alpha$ are such that a number K exists satisfying the following relation:

$$\int_0^z | c(s) | e^{\Delta\alpha(z-s)}\, ds \leqq K \quad \text{for every } z \geqq 0. \tag{46}$$

We recall from (3) that $\Delta\alpha < 0$. It will subsequently appear that convergence of the approximate solution can be guaranteed in general only for $K \leqq 0.455$; further, the smaller K the more rapid the bounds on $|\lambda_n(z)|$ and $|h_n(z)|$ decrease as n increases. The restriction of (46) was again obtained partly by physical reasoning and partly by trial and error. Roughly speaking, for a given K it guarantees that the coupling coefficient $c(z)$ is more or less uniformly distributed along z; we thus rule out cases where the coupling coefficient is zero over most of the guide and large over a very short section (e.g., the example of (31)-(34), where $c(z)$ is a single δ-function). Physically, such a condition says that for small K the spurious mode is dissipated much faster than it is coupled from the signal mode; the larger $c(z)$ the larger must be $|\Delta\alpha|$ in order to satisfy (46) for a given value of K. This will normally be the only case of practical interest in long random guides.

Bounds on the first few $h_n(z)$ are readily obtained. From (44c) and (46),

$$\begin{aligned} |h_1(z)| &\leqq \int_0^z |c(s)|\, e^{-\Delta\alpha s}\, ds = e^{-\Delta\alpha z} \int_0^z |c(s)|\, e^{\Delta\alpha(z-s)}\, ds \\ &\leqq e^{-\Delta\alpha z}\cdot K. \end{aligned} \tag{47}$$

Next, from (44b), (43) and (41a),

$$h_n(z) = -j2 \int_0^z c(s) e^{\Delta\Gamma s} h_{n-1}(s) \left[\sum_{k=1}^{n-2} h_k(s) + \frac{h_{n-1}(s)}{2} \right] ds. \tag{48}$$

Thus,

$$\begin{aligned} |h_n(z)| \leqq 2 \int_0^z |c(s)|\, e^{\Delta\alpha s}\, |h_{n-1}(s)| \left[\sum_{k=1}^{n-2} |h_k(s)| \right. \\ \left. + \frac{|h_{n-1}(s)|}{2} \right] ds. \end{aligned} \tag{49}$$

Equation (49) yields for the first few $h_n(z)$:

$$\begin{aligned} |h_2(z)| &\leqq 2 \int_0^z |c(s)|\, e^{\Delta\alpha s} e^{-2\Delta\alpha s} \frac{K^2}{2}\, ds = e^{-\Delta\alpha z} K^2 \\ &\quad \cdot \int_0^z |c(s)|\, e^{\Delta\alpha(z-s)}\, ds \leqq e^{-\Delta\alpha z}\cdot K^3 \end{aligned} \tag{50}$$

$$\begin{aligned} |h_3(z)| &\leqq 2 \int_0^z |c(s)|\, e^{\Delta\alpha s}\, e^{-\Delta\alpha s} K^3\, e^{-\Delta\alpha s} \left[K + \frac{K^3}{2} \right] ds \\ &= e^{-\Delta\alpha z} 2K^4 \left[1 + \frac{K^2}{2} \right] \int_0^z |c(s)|\, e^{\Delta\alpha(z-s)}\, ds \\ &\leqq e^{-\Delta\alpha z}\cdot 2K^5 \left[1 + \frac{K^2}{2} \right]. \end{aligned} \tag{51}$$

By an exactly similar process:

$$| h_4(z) | \leqq e^{-\Delta\alpha z} \cdot 2^2 K^7 \left[1 + \frac{K^2}{2}\right]\left[1 + K^2 + K^4 + \frac{K^6}{2}\right] \tag{52}$$

$$\begin{aligned} | h_5(z) | \leqq e^{-\Delta\alpha z} \cdot 2^3 K^9 \left[1 + \frac{K^2}{2}\right]\left[1 + K^2 + K^4 + \frac{K^6}{2}\right] \\ \cdot\left[1 + K^2 + 2K^4 + 3K^6 + 3K^8 + 3K^{10} + 2K^{12} + \frac{K^{14}}{2}\right]. \end{aligned} \tag{53}$$

It is difficult to continue the above process and write out explicitly the n^{th} term. However, by accepting a slightly poorer bound the analysis may be greatly simplified. We show that

$$| h_n(z) | \leqq e^{-\Delta\alpha z} M^{n-2} K^{n+1}; \qquad n \geqq 2. \tag{54}$$

M is a constant to be determined, as a function of K. Assume that (54) is true for some value of n. Then, from (49)

$$\begin{aligned} | h_{n+1}(z) | &\leqq 2 \int_0^z | c(s) | M^{n-2} K^{n+1} e^{-\Delta\alpha s} \left[K + \sum_{k=2}^{n-1} M^{k-2} K^{k+1} + \frac{M^{n-2}K^{n+1}}{2}\right] ds \\ &= e^{-\Delta\alpha z} \cdot 2M^{n-2} K^{n+2} \left[1 + K^2 \sum_{k=0}^{n-3} M^k K^k + \frac{M^{n-2}K^n}{2}\right] \cdot \int_0^z | c(s) | e^{\Delta\alpha(z-s)} ds \\ &\leqq e^{-\Delta\alpha z} \cdot 2M^{n-2} K^{n+3} \left[1 + K^2 \sum_{k=0}^{n-3} M^k K^k + \frac{M^{n-2}K^n}{2}\right]. \end{aligned} \tag{55}$$

If (54) is to remain true for $n \to n + 1$, we have from (55)

$$\begin{aligned} 2M^{n-2}K^{n+3}\left[1 + K^2 \sum_{k=0}^{n-3} M^k K^k + \frac{M^{n-2}K^n}{2}\right] \leqq M^{n-1}K^{n+2}, \\ K\left[1 + K^2 \sum_{k=0}^{n-3} M^k K^k + \frac{M^{n-2}K^n}{2}\right] \leqq \frac{M}{2}. \end{aligned} \tag{56}$$

But since the left-hand side of (56) is increased by allowing $n \to \infty$ and dropping the final term inside the brackets, the inequality of (56) will be satisfied if

$$K\left[1 + \frac{K^2}{1 - MK}\right] \leqq \frac{M}{2}. \tag{57}$$

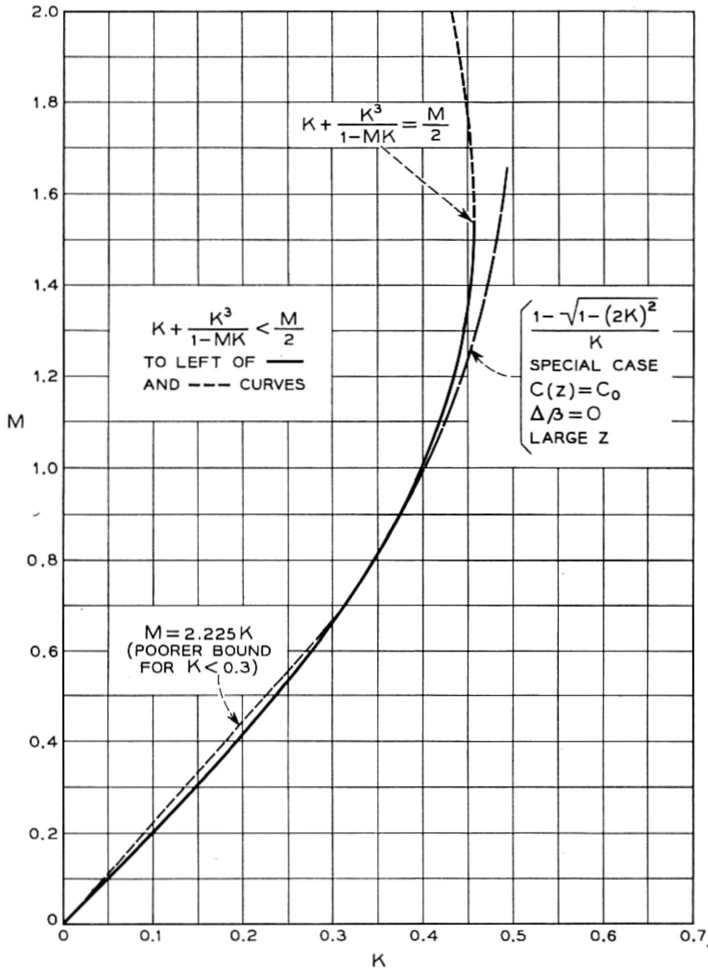

Fig. 1 — M vs K

A plot of M vs K taking (57) as an equality is shown in Fig. 1; the inequality of (57) is satisfied to the left of this curve. If for a given K we have chosen M to satisfy the inequality of (57), then since (54) holds true for $n = 2$, (50), it is valid for all n by induction. For a given K we should choose the smallest value of M satisfying (57) in order to obtain the best bound. This smallest value of M is given by the solid curve of Fig. 1 (i.e., $M < 1.554$); the other branch, indicated by the dashed curve (i.e., $M > 1.554$), thus has no significance for our problem. We note that convergence of the series solution of (45), and hence of the successive approximations of (41), is guaranteed only for $0 \leqq K \leqq 0.455$; for greater values of K the present analysis cannot guarantee convergence.

Summarizing the above results:

$$\begin{aligned} |h_1(z)| &\leqq e^{-\Delta\alpha z}\cdot K \\ |h_n(z)| &\leqq e^{-\Delta\alpha z}\cdot M^{n-2} K^{n+1}; \qquad n \geqq 2. \\ & 0 \leqq K \leqq 0.455. \end{aligned} \tag{58}$$

M is given as a function of K by the solid curve of Fig. 1. If K is further restricted and if we are willing to degrade the bounds slightly, their form becomes simpler still. For example, if $0 \leqq K \leqq 0.3$ we may replace the bound of the solid curve on Fig. 1 by the slightly poorer dotted chord drawn from the origin. For this chord $M = 2.225\ K$ and the results of (58) become:

$$\begin{aligned} |h_1(z)| &\leqq e^{-\Delta\alpha z}\cdot K \\ |h_n(z)| &\leqq e^{-\Delta\alpha z}\cdot K^3\ (2.225\ K^2)^{n-2}; \qquad n \geqq 2. \\ & 0 \leqq K \leqq 0.3. \end{aligned} \tag{59}$$

Finally, by (45), (57) and (58)

$$\begin{aligned} |H(z)| &\leqq e^{-\Delta\alpha z}\left[K + \sum_{n=2}^{\infty} M^{n-2}K^{n+1}\right] = e^{-\Delta\alpha z}\left[K + \frac{K^3}{1 - MK}\right], \\ |H(z)| &\leqq e^{-\Delta\alpha z}\cdot\frac{M}{2}, \end{aligned} \tag{60}$$

where M is again given as a function of K by the solid curve of Fig. 1.

Having found bounds on $h_n(z)$, we may now find bounds on $\lambda_n(z)$, our original objective. From (44a),

$$|\lambda_n(z)| \leqq \int_0^z |c(s)|\, e^{\Delta\alpha s}\, |h_n(s)|\, ds; \qquad n \geqq 1. \tag{61}$$

From (58):

$$\begin{aligned} |\lambda_1(z)| &\leqq K \int_0^z |c(s)|\, ds. \\ |\lambda_n(z)| &\leqq M^{n-2}K^{n+1} \int_0^z |c(s)|\, ds; \qquad n \geqq 2. \\ |\Lambda(z)| &\leqq \frac{M}{2} \int_0^z |c(s)|\, ds. \\ & 0 \leqq K \leqq 0.455. \end{aligned} \tag{62}$$

M is again given by Fig. 1. Again, if K is further restricted, simpler but slightly poorer results are obtained. For example:

$$
\begin{aligned}
|\lambda_1(z)| &\leqq K \int_0^z |c(s)|\, ds. \\
|\lambda_n(z)| &\leqq K^3(2.225K^2)^{n-2} \int_0^z |c(s)|\, ds; \qquad n \geqq 2. \\
& 0 \leqq K \leqq 0.3.
\end{aligned} \tag{63}
$$

Finally, the slightly better bounds of (51-53) may be used for the smaller values of n.

We may again ask whether these bounds are the best that can be obtained. The answer is that we might be able to do a little better, but not much. Thus, consider the following special case:

$$
\begin{aligned}
\Delta\beta &= 0, \\
c(z) &= c_0 = \text{pure real}.
\end{aligned} \tag{64}
$$

From (46) we have

$$K = \frac{c_0}{-\Delta\alpha}. \tag{65}$$

The solution to the coupled line equations, (35), subject to the initial conditions of (36), for this case may be written in the following form.[1]

$$
\begin{aligned}
G_0(z) = &-\frac{1 - \sqrt{1 - (2K)^2}}{2\sqrt{1 - (2K)^2}} \exp \frac{\Delta\alpha}{2} [1 + \sqrt{1 - (2K)^2}]\, z \\
&+ \frac{1 + \sqrt{1 - (2K)^2}}{2\sqrt{1 - (2K)^2}} \exp \frac{\Delta\alpha}{2} [1 - \sqrt{1 - (2K)^2}] z
\end{aligned} \tag{66}
$$

For $K < 0.5$, all of the radicals in (66) are pure real. Under these conditions the first term of (66) has a smaller coefficient and a more rapidly decaying exponential factor than the second term. Therefore, for a large enough value of z the second term dominates, and we may write

$$G_0(z) \approx \frac{1 + \sqrt{1 - (2K)^2}}{2\sqrt{1 - (2K)^2}} \exp \frac{\Delta\alpha}{2} [1 - \sqrt{1 - (2K)^2}] z; \tag{67a}$$

$$
\begin{aligned}
\Lambda = -\ln G_0(z) \approx &-\ln \frac{1 + \sqrt{1 - (2K)^2}}{2\sqrt{1 - (2K)^2}} \\
&- \frac{\Delta\alpha}{2} [1 - \sqrt{1 - (2K)^2}] z;
\end{aligned} \tag{67b}
$$

$$K < 0.5, \qquad -\Delta\alpha z \gg \frac{1}{\sqrt{1 - (2K)^2}}. \tag{67c}$$

The first term of (67b) is constant and the second increases linearly with z, so that we may write for large z

$$\Lambda \approx -\frac{\Delta\alpha}{2}[1 - \sqrt{1 - (2K)^2}]\, z = \frac{1 - \sqrt{1 - (2K)^2}}{2K} \cdot c_0 z, \quad \text{large } z. \tag{68}$$

The bound of (62) becomes simply

$$|\Lambda| \leqq \frac{M}{2} c_0 z. \tag{69}$$

Therefore, a comparison between the bound of the present analysis and the exact results for the special case of (64) for large z is obtained by plotting

$$\frac{1 - \sqrt{1 - (2K)^2}}{K}$$

on Fig. 1 and comparing this quantity with M.

We see that for $K < 0.36$, the exact solution is indistinguishable from the bound of (69) on the plot of Fig. 1; consequently, little improvement may be obtained in these bounds without further restricting the problem. We also note that for $K > 0.5$, the above approximations made in the exact solution of (66) no longer apply. For $K > 0.5$ the character of the solution changes from monotonic to oscillatory; $G_0(z)$ now has periodic zeros, at which $\ln G_0(z)$ must approach infinity. Consequently, the series expansion for Λ in this case will diverge for $K > 0.5$. The present analysis guarantees convergence only for $K < 0.455$; while this might be a little smaller than necessary, the series solution may diverge for values of K not much larger.

IV. DISCUSSION

If K of (46) is very small compared to 1, $K \ll 1$, the bounds of (62-63) on $|\lambda_n(z)|$ converge very rapidly. Under these conditions it is tempting to assume that $\Lambda(z)$ is satisfactorily approximated by the first term of the summation of (45); i.e., from (41b),

$$\Lambda \approx \Lambda_1(z) = \int_0^z c(s) e^{\Delta\Gamma s}\, ds \int_0^s c(t) e^{-\Delta\Gamma t}\, dt, \tag{70}$$

or one of the alternative forms given in (27). [Alternately we might wish to make a similar statement for $G_0(z)$, as in Equation (24), when $\int_0^z |c(s)|\, ds \ll 1$.] This assumption has been made in all calculations of transmission statistics that have so far been made.[1]

Unfortunately there is, at present, no satisfactory justification for this assumption. If $| \lambda_1(z) |$ turns out to be equal to its bound, as given in (62), and if $K \ll 1$, then, of course, we are guaranteed that the higher terms will have much smaller magnitudes than the principal term $\lambda_1(z)$. However, this situation is quite improbable, and occurs only in very specially selected cases. Thus if $| \lambda_1(z) |$ is much smaller than its bound, as will be the usual case, we have no assurance that the magnitude of the next term $| \lambda_2(z) |$ or higher terms may not be much greater than $| \lambda_1(z) |$. However, no instance is known in which $| \lambda_2(z) |$ is *not* small compared to $| \lambda_1(z) |$, for $K \ll 1$.

We do not know whether or not the perturbation solution of (70) provides a useful approximation for *all* cases of interest (i.e., for all cases where $c(z)$ satisfies (46) for some small value of K, e.g., $K = 0.1$). Even if this approximation fails in some cases, we may still hope that it holds true in most cases, so that (70) will yield the correct statistical properties of the loss and phase when the coupling coefficient is a stationary random process, perhaps Gaussian, with a sufficiently small rms value, at least for the simpler statistics of interest. Although this is believed to be true by a number of people, there is nothing in the present paper that bears on this question (and no other information known to the author). It would be most desirable to obtain further information on the way in which $\Lambda_1(z)$ of (70) approximates the true solution $\Lambda(z)$; e.g., does $\Lambda_1(z)$ approximate the fine structure of $\Lambda(z)$ as well as its average value as $\Delta\beta$ (which varies with the frequency of the applied wave) varies.

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