On the Theory of Shrink Fits with Application to Waveguide Pressure Seals

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(Manuscript received November 15, 1961)

Waveguide antennas for missiles and space vehicles usually require a window or radome to provide a pressure seal. The technique of shrink fitting is a simple method to seal rectangular waveguides. A theoretical and experimental investigation has been undertaken to study the stresses and displacements that result from shrink fits between rectangular cross sections.

An appropriate differential equation is derived by applying simplifying assumptions to the theory of thermal stresses. The solution of this equation indicates that for certain combinations of materials and sizes of rectangles, there is a critical wall thickness below which a pressure-tight shrink fit cannot be made regardless of the temperature at which the shrink fit process is initiated. A comparison is presented of the analytical solution with experimental results of "Teflon" shrunk fit into precision fabricated X-band waveguide. Finally, tabulated results are presented to indicate the magnitude of the stresses and displacements to be expected from typical materials and rectangles of various sizes.*

I. INTRODUCTION

The derivations and conclusions reported in this paper have been stimulated by the current interest in light-weight microwave antennas for missiles and space vehicles. These antennas must be capable of maintaining a pressure seal in temperature environments from $-80°F$ to $+400°F$ or higher. A typical seal is intended to maintain a pressure of one atmosphere. Such pressures are required to prevent electrical breakdown of the waveguide at nominal operating power.

A common method of achieving such a seal is to provide a radome over the antenna. This scheme is usually effective but requires gaskets

* "Teflon" is a registered trademark of the E. I. du Pont DeNemours and Co. The type referred to in this paper is a polytetrafluorethylene resin.

and screws and, moreover, is an additional structure to be carried during flight.

A design is proposed in which a suitable dielectric plug, or core, necessary to give the required antenna pattern, is inserted into the waveguide opening of the antenna. A pressure seal between the plug and the waveguide is accomplished by first chilling the plug to a temperature well below that of the waveguide piece. The plug is then inserted into the waveguide. As the plug warms up, it expands and exerts sufficient pressure on the walls of the waveguide to cause an effective seal.

The purpose of this study is to determine theoretically as well as experimentally the feasibility of such a shrink fit. The study includes the behavior of thin wall rectangular tubes into which a core is shrunk fit, where the core may or may not be of the same material as the tube. The particular solutions that are sought are the resulting stresses and deformations at the interface of the tube and the core; the stresses to indicate the effectiveness of the seal, and the deformations to indicate the degree of distortion of the tube. While it is by no means obvious that a limit on wall thickness exists, for given materials and shape of rectangles, below which no seal can be obtained, it will be shown that such a limit does indeed exist.

II. FORMULATION OF THE PROBLEM

2.1 *General Formulation*

The analysis is performed for the general case, i.e., the size of the rectangular enclosure, type of materials, and the temperature environment are all arbitrary, as shown in Fig. 1. The core material, E_1, is subjected to a temperature change, ΔT, until it can be inserted into the tube, E_2. Therefore, $\Delta T = T_2 - T_1$, where T_2 is the initial temperature of the tube and the core and T_1 is the temperature to which the core is reduced.

For a core with no variation of temperature or stress through the thickness we may assume a condition of plane stress. Consider, therefore, a thin wall rectangular tube of material, E_2, in which a material, E_1, is shrunk fit, as shown in Fig. 1. A set of coordinate axes, x-y and u-v, is located at the centers of the broad and narrow walls respectively. The deformation of the tube from its original shape is y and v. The strains perpendicular to the interface of length b_o and a_o are ε_y and ε_v respectively.

We assume that the corners of the tube do not move, that the stresses induced in the enclosure are due to bending only, and that the right

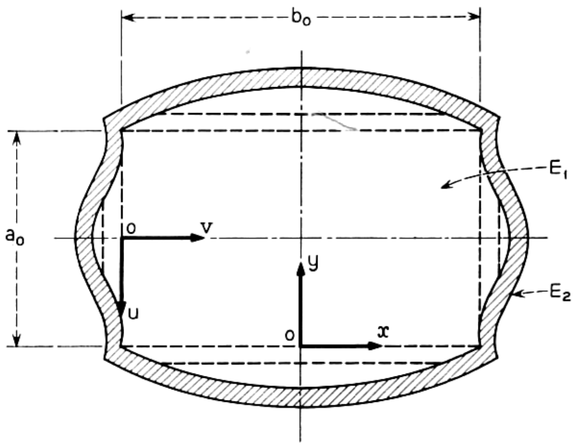

Fig. 1 — Deformation of thin wall rectangular tubing by a shrunk-fit core, general case.

angles formed by both sides remain right angles. The first two assumptions admit a contradiction since the first will result in a tension in the beam. This tension will be neglected because of the following reasons: *i.* the corners actually move and the stress due to tension introduced by the assumption of fixed corners is assumed negligible. To analyze the problem with the admission of the moveable corners would introduce undefined boundary conditions, and *ii.* the enclosure wall thicknesses are such as to consider the enclosure a beam frame where the stresses due to bending are predominant. These simple assumptions are satisfactory approximations in many practical cases where we are concerned only in the elastic range. For instance, in the case of "Teflon" inserted into aluminum or brass the solution obtained on these assumptions is in good agreement with actual measurements.[1] In other cases, the deformations of the sides of the enclosure are so great as to render the displacements of the corners negligible. Also because of symmetry the centers of the interfaces do not move parallel to the interface. We therefore make the further assumption that all strains parallel to the interfaces are zero. We note at this time we are only concerned about the interfaces.

From the theory of elasticity[2] as applied to restricted thermal expansion we have for the stress in the y direction:

$$\sigma_y = \frac{-E_1}{1 - \nu_1^2} (\varepsilon_y + \nu_1 \varepsilon_x) + \frac{\alpha_1 E_1 \Delta T}{1 - \nu_1} \qquad (1)$$

where: ε_y and ε_x are strains anywhere in the region, E_1.
α_1 is the coefficient of linear expansion of E_1.
ΔT is the temperature change experienced by E_1 and is not a function of position or time.
ν_1 is Poisson's ratio for E_1.
E_1 is the modulus of elasticity of material designated E_1.

Since this stress exists throughout the core, it exists at its boundary and becomes the loading on the enclosure. In our problem for the interface of length b_o:

$$\varepsilon_x = 0 \therefore \sigma_y = -\frac{E_1}{1 - \nu_1^2}\varepsilon_y + \frac{\alpha_1 E_1 \Delta T}{1 - \nu_1}$$

We make the further simplified assumption that the strain ε_y at the interface is simply the ratio of the change in length to the original length. The original length is the initial (cold) state length: $a_o/2$, therefore $\varepsilon_y = \Delta L/L_o = -(2y/a_o)$.

$$\therefore \sigma_y = \frac{2E_1}{a_o(1 - \nu_1^2)}y + \frac{\alpha_1 E_1 \Delta T}{1 - \nu_1}. \tag{2}$$

Now this normal stress (2), being continuous across the boundary between E_1 and E_2, provides an outward load on the tube. Considering E_2 of length b_o as a uniform simple beam of unit depth this stress is merely the load per unit length. Using the differential equation for the deflection of beams:[3]

$$E_2 I_2 \frac{d^4 y}{dx^4} = -q \tag{3}$$

where q is the intensity of the load per unit length and is considered positive acting in the negative y direction. Substituting (2) into (3):

$$\frac{d^4 y}{dx^4} + \left(\frac{E_1}{E_2 I_2}\right)\frac{2}{a_o(1 - \nu_1^2)}y = -\left(\frac{E_1}{E_2 I_2}\right)\frac{\alpha_1 \Delta T}{1 - \nu_1}$$

and I_2 is the moment of inertia of the beam. Let

$$P = \frac{E_1}{E_2 I_2}; \qquad s^4 = \frac{2P}{a_o(1 - \nu_1^2)}; \qquad M = \frac{P\,\alpha_1 \Delta T}{1 - \nu_1}; \qquad \alpha^4 = \frac{s^4}{4}$$

$$\therefore \frac{d^4 y}{dx^4} + 4\alpha^4 y = -M. \tag{4}$$

Similar reasoning for the interface of length a_o leads to an analogous equation:

$$\frac{d^4 v}{du^4} + 4\beta^4 v = -M \tag{5}$$

where $t^4 = 2P/b_o(1 - \nu_1^2)$, $\beta^4 = t^4/4$ and all other terms are as previously defined.

Equations (4) and (5) are the governing differential equations which describe the general behavior of the rectangular shrink fit.

Note that both (4) and (5) have the same form as equations for beams on elastic foundations. This is understandable if we consider that the same condition exists whether the beam is embedded in a foundation that helps support the load as the beam deflects or that the intensity of the load decreases as the deflection increases, as is the case in this analysis.

2.2 *A Special Case — the square shrink fit*

The solutions and results of this case are carried out in Section A.2. It is determined that there is a definite wall thickness limit, determined by the choice of materials and size, below which no square enclosure can effectively be sealed by a shrink fit process.

III. RESULTS OF ANALYSIS

3.1 *General*

The final equations were programmed on the IBM 704 computer to include as wide a variety of combinations of materials and different size enclosures, as is practicable with materials that obey, at least in part, Hooke's Law of stress and strain.

The parameter chosen as a convenient variable to describe the merit of the shrink fit is called, here, the "shrink fit resistance," i.e., the higher the value of this parameter or "resistance" the lower the resulting compressive stresses. This parameter contains such constants of the configuration as Modulus of Elasticity of both the core and the enclosure, Poisson's ratio of the core, dimension of the enclosure, and thickness of the enclosure wall. This resistance is described by:

$$\frac{\alpha b_o}{2}$$

The initial run through the computer was for values of $\alpha b_o/2$ from 0 to 300. This range covered all possible combinations of interest. This run indicated that for values of resistance greater than 30 the expansion of the core was practically free-expansion, and the stresses between the core and the enclosure were zero at least to four decimal places. As an example, one can visualize trying to obtain a shrink fit of a block of steel or titanium in a thin shell of cork or plastic. The block would expand as if the shell were nonexistent.

In a second run of the computer for values expanded between 0 and 30, it was determined that a seal could exist for a value of resistance lower than 2.10 for all sizes of rectangles. Above this value, separations would occur in a definite pattern for particular sizes. A "definite pattern" implies that at low values we obtain few but large separations and at progressively higher values we obtain many more but smaller separations until we approach (around 30) a "just-touching" situation of zero pressure. The results are approximately as shown in Fig. 2.

Relatively few rectangles can be sealed that possess values of resistance greater than 2.10, and less than 2.40. As was mentioned previously, these become fewer and fewer as the resistance becomes greater. Rectangles with resistance greater than 2.40 cannot be sealed.

An interesting phenomenon is that if the value of the resistance is great enough to prevent a seal, then this situation exists no matter at what temperature the shrink fit takes place. This is because the condition for a seal is independent of the temperature, while the intensity or "tightness" of the fit is directly proportional to the temperature.

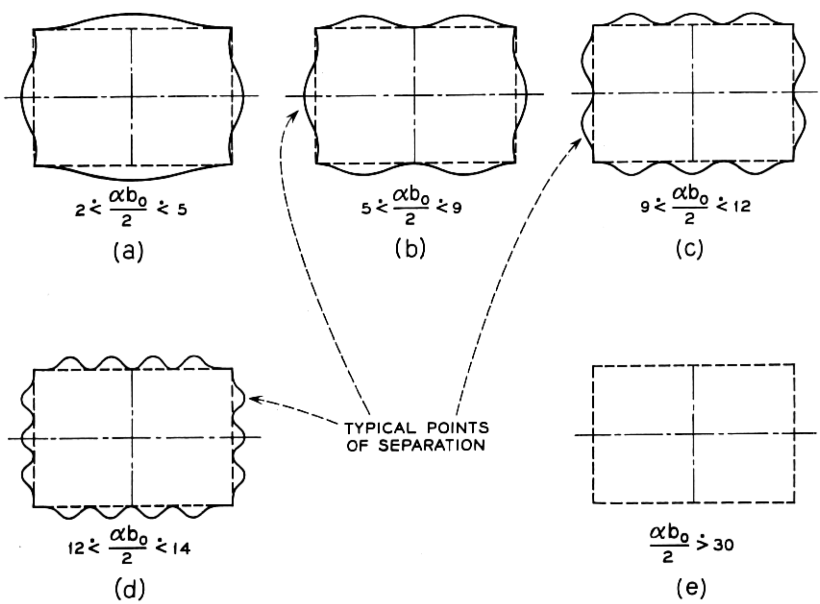

Fig. 2 — Typical separation patterns for values of $\alpha b_0/2$ between 2 and 30.

3.2 *Detailed Results*

An examination of Fig. 2 indicates typical modes of separation or leakage points as the resistance increases. It can be seen that separations occur in sequence. If $m = 0, 1, 2, 3 \cdots$, modes of separation in any one narrow wall then there are $n = 1, 2, 3, 4 \cdots$, corresponding separations in any one broad wall. i.e., $m = n - 1$ and the total number of separations in any mode is $4n - 2$.

A comparison of Fig. 3, which is a plot of stresses and displacements at the wall centers, with Fig. 2 clearly demonstrates how the wall centers have alternately positive and negative stresses. Fig. 3 also indicates the extreme reduction in the magnitude of the stresses beyond a resistance value of approximately 2.10. Therefore, although there are a few rectangles that can be sealed beyond $\alpha b_o/2 = 2.10$, the intensity of the fit is very low. A good rule of thumb is to design a shrink fit to have a resistance less than 1.0. In this manner the lowest compressive stress is more than 75 per cent of the stress in an infinitely restricted expanded core.

For example, the resistance of teflon into small X-band waveguide is approximately 2.166. Results from the computer indicate a minimum compressive stress of approximately 10 pounds per square inch while the maximum compressive stress is over 1200 pounds per square inch.

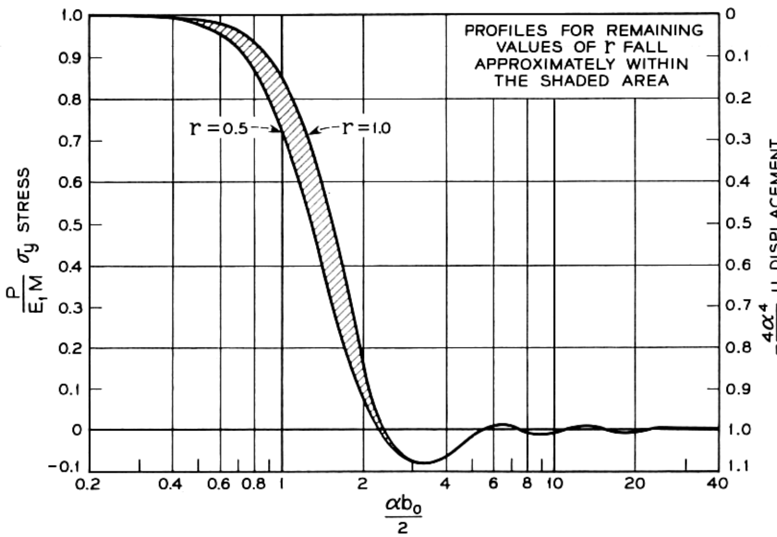

Fig. 3 — Stresses and displacements at wall centers for a square enclosure ($r = 1$) and a rectangular enclosure ($r = 0.5$).

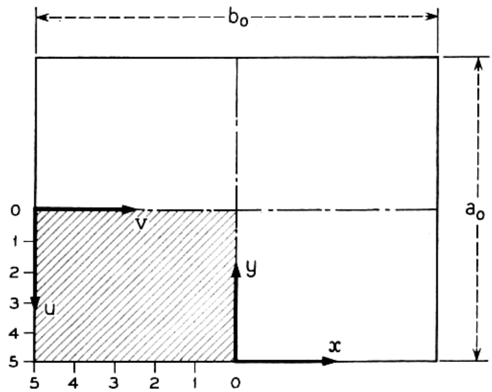

TABLE I — STRESSES AND DISPLACEMENTS FOR $\frac{1}{4}$-SECTION OF RECTANGLE

$\frac{\alpha b_o}{2}$	$r = \frac{a_o}{b_o}$	Point	$\frac{4\alpha^4}{M}\, y$	$\frac{4\alpha^4}{M}\, v$	$\frac{P}{E_1 M}\, \sigma_y$	$\frac{P}{E_1 M}\, \sigma_v$
2.40	0.60	0	−1.00626	−0.549488	−0.00625819	0.670307
		1	−0.945438	−0.507973	0.0545624	0.695216
		2	−0.764464	−0.392764	0.235536	0.764342
		3	−0.480664	−0.232534	0.519336	0.860479
		4	−0.165034	−0.0771141	0.834966	0.953732
		5	0.000000	0.000000	1.000000	1.000000
2.40	0.70	0	−0.999453	−0.804216	0.000547044	0.437049
		1	−0.936954	−0.746459	0.0630464	0.477479
		2	−0.751725	−0.584672	0.248275	0.590730
		3	−0.463771	−0.355304	0.536229	0.751288
		4	−0.149463	−0.125321	0.850537	0.912275
		5	0.000000	0.000000	1.000000	1.000000
2.40	0.80	0	−1.00026	−0.957217	−0.000261307	0.234226
		1	−0.937961	−0.891096	0.0620387	0.287123
		2	−0.753238	−0.703685	0.246762	0.437052
		3	−0.465777	−0.432318	0.534223	0.654145
		4	−0.151312	−0.153320	0.848688	0.877344
		5	0.000000	0.000000	1.000000	1.000000

NOTE: Any negative sign appearing in the last two columns indicates a separation. All other values of r with $\alpha b_o/2 = 2.40$ cannot be sealed.

This is not considered an effective seal for the intended environment of one atmosphere.

Table I is an abstract from the computer runs. It indicates that there are a few rectangles with a resistance value greater than 2.10 that can be sealed — although not too effectively.

Therefore, although the resistance value is 2.40 and most of the sizes of rectangles with this resistance value cannot be sealed, a rectangle with dimensions $a_o/b_o = 0.70$ can be sealed. If this intended seal is to withstand low pressures this combination may well be an effective seal. On the other hand, when $a_o/b_o = 0.6$ or 0.8 a seal cannot be made regardless of the temperature of the shrunk fit, since the quantity $(P/E_1M)\sigma_y$ is negative. In fact the larger ΔT (and therefore M) becomes, the greater the separation that results.

IV. RESULTS OF EXPERIMENT

In order to verify the preceding analytical interpretation it was necessary to experimentally perform the shrink fitting operation, and measure the subsequent deformation of the waveguide. Precision brass waveguide was selected with internal dimensions of 0.4000″ x 0.9000″, a wall thickness of 0.0500″, a Modulus of Elasticity of approximately 17×10^6 p.s.i. Type 1 "Teflon" was selected for the inserts, with a Modulus of Elasticity of approximately 60,000 p.s.i. and a value of Poisson's Ratio of 0.46.

A total of 5 waveguide sections were used in the analysis. The guide was machined to 3″ lengths while the "Teflon" inserts were cut in 2″ lengths. The five "Teflon" inserts were machined oversize for the shrink fitting operation on the basis of the coefficients of expansion at three different temperatures. The temperatures selected were −60°F, −90°F and −120°F. This range was chosen since the proposed application of this analysis is for equipment which must operate satisfactorily over a temperature environment of −60°F to +225°F. Since "Teflon" has a larger coefficient of expansion than brass the high temperature environ-

TABLE II

Block	Temp. °F	Coefficient Expansion—%	Calculated Size—inches	Actual Size—inches
1	−60	−1.00	0.4040 x 0.9090	0.4040 x 0.9080
2	−90	−1.18	0.4047 x 0.9106	0.4045 x 0.9098
3	−90	−1.18	0.4047 x 0.9106	0.4049 x 0.9098
4	−120	−1.28	0.4051 x 0.9115	0.4049 x 0.9107
5	−120	−1.28	0.4051 x 0.9115	0.4049 x 0.9103

Average Coefficients of Thermal Expansion[4] for Type I "Teflon"

Temperature Range °C	Coefficient of Expansion per degree C
+25 to −50	135×10^{-6}
+25 to −100	112×10^{-6}
+25 to −150	96×10^{-6}

ment does not offer a problem. To effectively maintain a seal at the low end however, the "Teflon" blocks must be machined oversize for a temperature lower than −60°F. Each block is listed in Table II along with its associated temperature, coefficient of expansion and actual block dimensions.

To accurately measure deflections of the waveguide due to the expanding "Teflon" a recording system[5] was devised which could provide a maximum magnification factor of 31,000 to 1. Basically it consists of a strain-gage activated transducer whose output is fed to an amplifying and recording system. The transducer itself (see Fig. 4) is essentially a cantilever beam which is deflected as the waveguide section is translated on a reference platform beneath the point probe. These deflections are picked off in strain gage outputs and fed to an amplifying system which controls a recorder chart. A deflectometer was used to insure synchronization of the chart drive and the trace of the transducer probe across the waveguide wall. The result is a magnified profile of the waveguide walls.

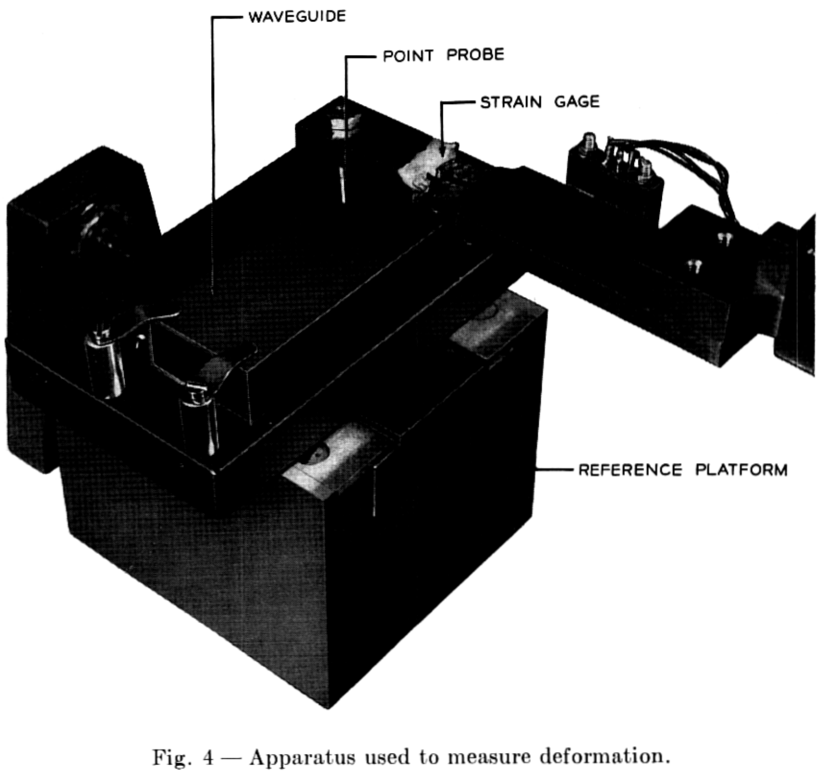

Fig. 4 — Apparatus used to measure deformation.

Prior to a discussion of the results a brief description of the shrink fit operation follows. Although the "Teflon" inserts as noted in Table I were machined oversize for three different temperatures, they were all inserted at a temperature of approximately −300°F. The coolant used in this operation was liquid nitrogen. The blocks were submerged in the liquid nitrogen for several minutes and then inserted into the waveguide allowing $\frac{1}{2}''$ opening on either end. The entire operation was performed in a controlled atmosphere of dry nitrogen to prevent the formation of frost on the "Teflon" blocks. By utilizing this extreme temperature coolant a tolerance was achieved so that upon removal of the blocks from the bath and prior to their immediate insertion into the guide, the resultant expansion was not sufficient to interfere with the placement of the cores. The specimens were allowed to stabilize for 30 minutes at the end of which time their profiles and resultant deflections were recorded. A 3-point suspension was used in mounting the waveguide sections to the reference platform so that accurate measurement of opposite sides of waveguide could be recorded after deflection had taken place. The profiles were monitored before and after insertion of the plug at various stations ($\frac{1}{2}$ inch) along the length of the waveguide, including

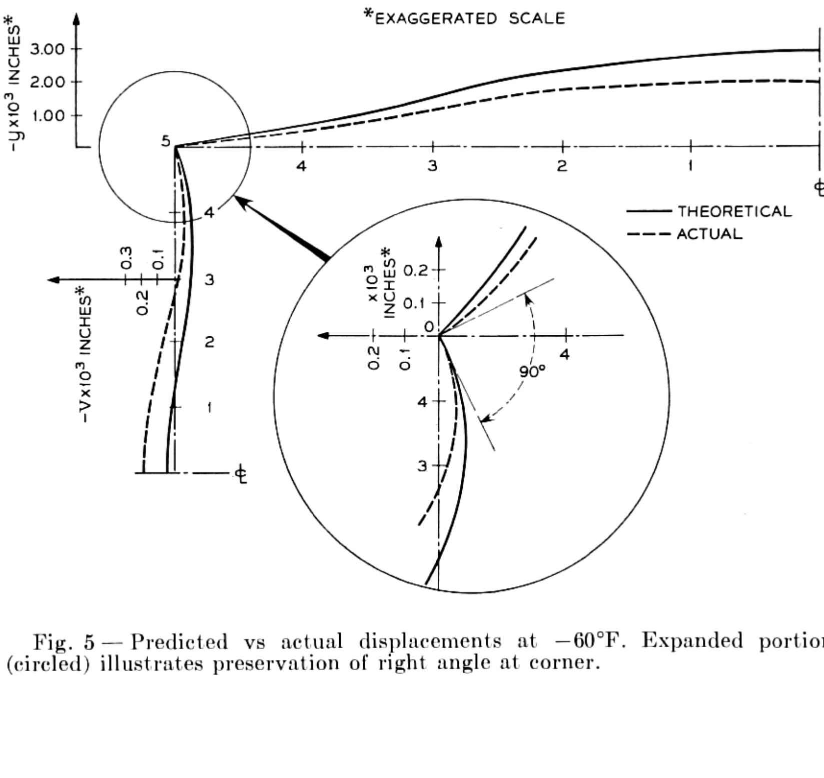

Fig. 5 — Predicted vs actual displacements at −60°F. Expanded portion (circled) illustrates preservation of right angle at corner.

the middle of the 2″ "Teflon." The profiles at each station followed the same general pattern with the maximum expansion occurring at the center of the 2″ "Teflon."

The curves (Figs. 5, 6, 7) were plotted for a quarter section of waveguide to illustrate the resultant expansion produced by the shrink fit operation. The predicted analytical expansion was plotted on the same scale, so a comparison could be made between the two. The analytical analysis assumed that the waveguide corners remained at right angles to each other during the expansion process. This initial assumption was borne out in the experimental data as a valid one. In order to insure the validity of this phenomenon additional profile measurements were performed on an expanded scale directly in the corner region. These measurements definitely indicated a negative or inward expansion of the narrow wall, preserving the right angle corner of the waveguide.

Although predicted and experimental expansions are not in exact agreement, the profiles of each follow the same pattern. The differences in the magnitudes of the total expansions can be attributed to several factors. Machining tolerances in both the brass waveguide and "Teflon" inserts could not be held much closer than 0.0005 inch. In addition, cold flow of the "Teflon" inserts occurring immediately after their insertion

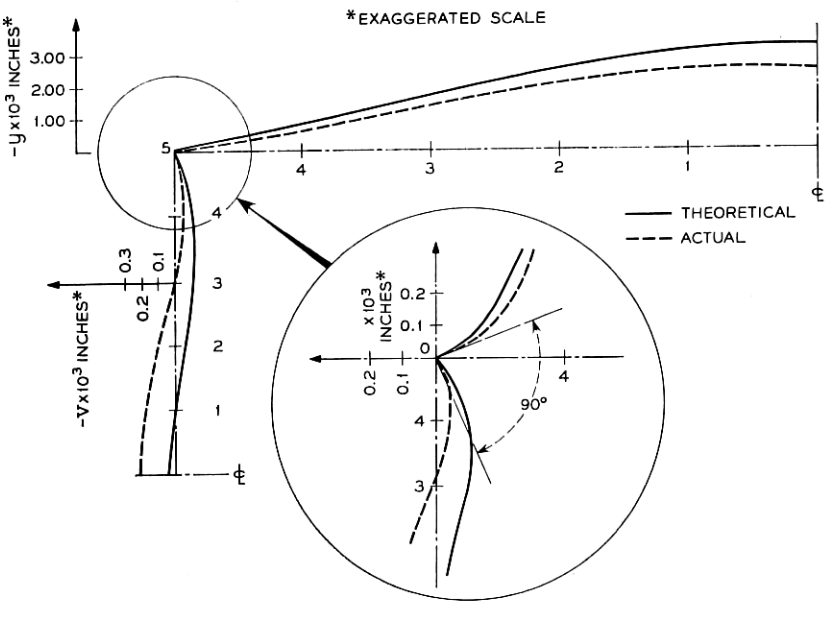

Fig. 6 — Predicted vs actual displacements at −90°F. Expanded portion (circled) illustrates preservation of right angle at corner.

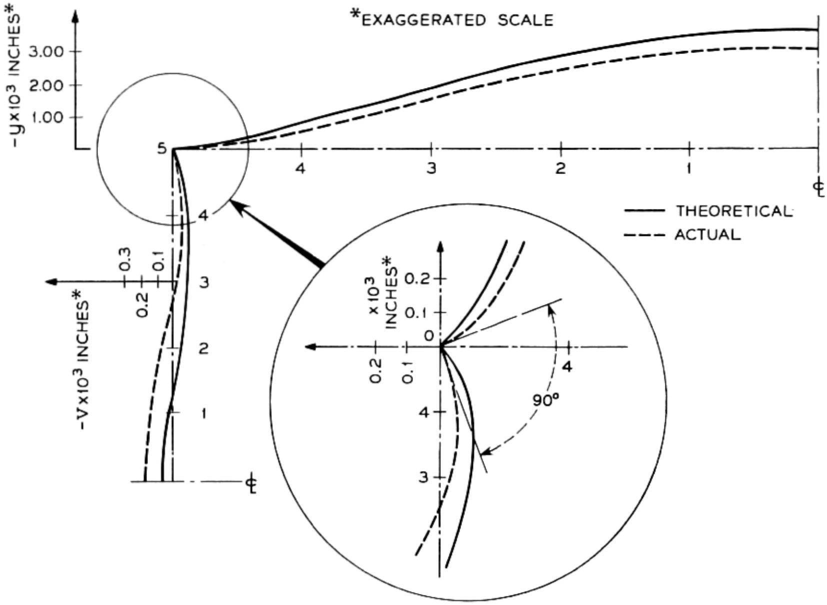

Fig. 7 — Predicted vs actual displacements at −120°F. Expanded portion (circled) illustrates preservation of right angle at corner.

and prior to remeasurement of the waveguide sections introduced an additional discrepancy.

It is apparent from the curves that because of this cold flow the loading intensity of the shrink-fit was decreased, resulting in smaller deflections in both the broad and narrow walls. Nevertheless, the predicted and experimental profiles are in relative agreement with respect to the basic assumptions.

V. GENERAL INFORMATION

Table III indicates the magnitudes of the stresses and displacements one might expect from typical materials and various sizes of rectangles.

VI. SUMMARY AND CONCLUSIONS

The stresses and deformations resulting from rectangular shrink fits can be described by general differential equations of the form:

$$\frac{d^4y}{dx^4} + Ay = B; \qquad \sigma_y = Cy + D$$

TABLE III — STRESSES AND DISPLACEMENTS AT WALL CENTERS

$\frac{\alpha b_o}{2}$	$r = \frac{a_o}{b_o}$	$\frac{4\alpha^4}{M} y$	$\frac{4\alpha^4}{M} v \times 10^2$	$\frac{P}{E_1 M} \sigma_y$	$\frac{P}{E_1 M} \sigma_v$
0.5	0.1	−0.0140475	0.0369430	0.985952	1.00004
	0.2	−0.0168590	0.129932	0.983141	1.00026
	0.3	−0.0188721	0.250018	0.981128	1.00075
	0.4	−0.0200661	0.366116	0.979934	1.00146
	0.5	−0.0204413	0.444737	0.979559	1.00222
	0.6	−0.0200013	0.450008	0.979999	1.00270
	0.7	−0.0187518	0.343972	0.981248	1.00241
	0.8	−0.0167051	0.0873193	0.983295	1.00070
	0.9	−0.0138850	−0.359327	0.986115	0.996766
	1.0	−0.0103330	−1.03330	0.989667	0.989667
1.0	0.1	−0.192576	0.514591	0.807424	1.00051
	0.2	−0.224629	1.74104	0.775371	1.00348
	0.3	−0.245227	3.22422	0.754773	1.00967
	0.4	−0.255226	4.50577	0.744774	1.01802
	0.5	−0.255174	5.10577	0.744826	1.02553
	0.6	−0.245660	4.53701	0.754340	1.02722
	0.7	−0.227738	2.36752	0.772262	1.01657
	0.8	−0.203330	−1.66401	0.796670	0.986688
	0.9	−0.175387	−7.51309	0.824613	0.932382
	1.0	−0.147551	−14.7551	0.852449	0.852449
1.5	0.1	−0.603086	1.68992	0.396914	1.00169
	0.2	−0.657292	5.20538	0.342708	1.01041
	0.3	−0.682426	8.66591	0.317574	1.02600
	0.4	−0.683702	10.3389	0.316298	1.04136
	0.5	−0.664697	8.53652	0.335303	1.04268
	0.6	−0.630410	2.01635	0.369590	1.01210
	0.7	−0.588972	−9.25054	0.411028	0.935246
	0.8	−0.550402	−23.6036	0.449598	0.811171
	0.9	−0.522637	−38.2895	0.477363	0.655394
	1.0	−0.508389	−50.8389	0.491611	0.491611
2.0	0.1	−0.928988	2.91315	0.0710119	1.00291
	0.2	−0.960541	7.98587	0.0394589	1.01597
	0.3	−0.965532	11.0070	0.0344678	1.03302
	0.4	−0.950164	8.13079	0.0498358	1.03252
	0.5	−0.920893	−3.46879	0.0791066	0.982656
	0.6	−0.888056	−23.2032	0.111944	0.860781
	0.7	−0.862943	−46.0441	0.137057	0.677691
	0.8	−0.851041	−65.7452	0.148959	0.474039
	0.9	−0.850384	−78.9297	0.149616	0.289633
	1.0	−0.855904	−85.5904	0.144096	0.144096
2.5	0.1	−1.06440	4.01305	−0.0644005	1.00401
	0.2	−1.07415	9.56188	−0.0741458	1.01912
	0.3	−1.06987	8.90597	−0.0698685	1.02672
	0.4	−1.05587	−4.50505	−0.0558715	0.981980
	0.5	−1.03877	−31.5782	−0.0387744	0.842109
	0.6	−1.02657	−63.3918	−0.0265694	0.619649
	0.7	−1.02256	−88.2925	−0.0225558	0.381953
	0.8	−1.02433	−101.705	−0.0243283	0.186362
	0.9	−1.02825	−105.513	−0.0282519	0.0503840
	1.0	−1.03214	−103.214	−0.0321412	−0.0321412

TABLE III (*Cont'd*)

$\frac{\alpha b_o}{2}$	$r = \frac{a_o}{b_o}$	$\frac{4\alpha^4}{M} y$	$\frac{4\alpha^4}{M} v \times 10^2$	$\frac{P}{E_1 M} \sigma_y$	$\frac{P}{E_1 M} \sigma_v$
3.0	0.1	−1.08931	5.15386	−0.0893139	1.05154
	0.2	−1.09015	10.1753	−0.0901539	1.10175
	0.3	−1.08848	2.16564	−0.0884799	1.02166
	0.4	−1.08547	−26.9990	−0.0854682	0.730010
	0.5	−1.08295	−68.9590	−0.0829545	0.310410
	0.6	−1.08209	−103.274	−0.0820925	−0.0327402
	0.7	−1.08246	−120.106	−0.0824568	−0.201057
	0.8	−1.08320	−122.784	−0.0831998	−0.227840
	0.9	−1.08387	−117.430	−0.0838700	−0.174300
	1.0	−1.08436	−108.436	−0.0843641	−0.0843641
4.0	0.1	−1.04066	7.49643	−0.0406582	1.07496
	0.2	−1.04197	7.03350	−0.0419736	1.07033
	0.3	−1.04981	−28.3538	−0.0498060	0.716462
	0.4	−1.05711	−92.8190	−0.0571056	0.0718099
	0.5	−1.05871	−141.325	−0.0587114	−0.413247
	0.6	−1.05692	−156.244	−0.0569194	−0.562436
	0.7	−1.05484	−150.169	−0.0548359	−0.501691
	0.8	−1.05336	−135.702	−0.0533627	−0.357021
	0.9	−1.05238	−119.646	−0.0523759	−0.196463
	1.0	−1.05163	−105.163	−0.0516336	−0.0516336
5	0.5	−1.01252	−187.848	−0.0125207	−0.878477
	1.0	−1.00910	−100.910	−0.00910099	−0.00910099
7	0.5	−0.997154	−217.039	0.00284579	−1.17039
	1.0	−0.997427	−99.7427	0.00257313	0.00257313
10	0.5	−1.00014	−207.423	−0.000136673	−1.07423
	1.0	−1.00013	−100.013	−0.000125572	−0.000125572
15	0.5	−1.000000	−199.256	0.223517 $\times 10^{-7}$	−0.992557
	1.0	−1.000000	−100.000	−0.596046 $\times 10^{-7}$	−0.596046 $\times 10^{-7}$
20	0.5	−1.000000	−199.981	−0.	−0.999813
	1.0	−1.000000	−100.000	−0.	−0.
30	0.5	−1.000000	−199.999	−0.	−0.999986

where y is the deflection at any point x along the interface, σ_y is the stress at that point, and A, B, C and D are constants.

For any rectangular enclosure and core there exists a practical limit above which no effective seal can be obtained. This limit is defined as the shrink fit resistance and is a function of dimensions and materials. The limiting value of resistance is 2.40. All rectangles with resistance values less than 2.10 can be sealed. Rectangles with resistances between 2.10 and 2.40 may or may not be capable of being sealed, depending on the details of configuration and materials.

Pressure sealing of waveguide is a typical application of shrink fits between rectangular connections. Usually the designer may choose the material and thickness of the waveguide. The choice should be made to limit the resistance to a value of 2.0 or less, regardless of the size of the enclosure or the temperature at which the shrink fit takes place. Therefore, for given materials and size of rectangle a pressure seal will exist if the wall thickness is

$$t \geqq \frac{1}{3.5}\left[\left(\frac{E_1}{E_2}\right)\frac{b_o^4}{a_o(1-\nu_1^2)}\right]^{\frac{1}{3}}$$

where E_1, E_2 are the modulus of elasticity of the core and enclosure, respectively
ν_1 is Poisson's ratio for the core
a_o, b_o are the narrow and broad wall dimension, respectively.

VII. ACKNOWLEDGMENT

The author wishes to acknowledge the helpful and stimulating discussions held with G. J. Bareille. The experimental analysis was performed under his direction.

APPENDIX

A.1 *The General Solution*

The general solution to (4) is:

$$y = A \sin \alpha x \sinh \alpha x + B \sin \alpha x \cosh \alpha x + C \cos \alpha x \sinh \alpha x + D \cos \alpha x \cosh \alpha x - (M/4\alpha^4) \tag{6}$$

where

$$4\alpha^4 = s^4.$$

Taking the origin of the coordinates in the middle of beam b_o as in Fig. 1, we conclude from symmetry that B=C=O.

$$\therefore\ y = A \sin \alpha x \sinh \alpha x + D \cos \alpha x \cosh \alpha x - (M/4\alpha^4). \tag{7}$$

From the boundary conditions at the ends of the beam: $y = 0$ at $x = b_o/2$

$$\therefore\ A = \frac{\dfrac{M}{4\alpha^4} - D \cos \alpha \dfrac{b_o}{2} \cosh \alpha \dfrac{b_o}{2}}{\sin \dfrac{\alpha b_o}{2} \sinh \dfrac{\alpha b_o}{2}}. \tag{8}$$

Similar reasoning leads to the deflection equation for beam a_o with coordinate axes u and v in the middle of beam a_o :

$$v = E \sin \beta u \sinh \beta u + H \cos \beta u \cosh \beta u - (M/4\beta^4). \tag{9}$$

where

$$\beta^4 = \frac{t^4}{4} \quad \text{and} \quad t^4 = \frac{2P}{b_o(1 - \nu_1^2)}$$

$$\therefore E = \frac{\frac{M}{4\beta^4} - H \cos \frac{\beta a_o}{2} \cosh \frac{\beta a_o}{2}}{\sin \frac{\beta a_o}{2} \sinh \frac{\beta a_o}{2}} \tag{10}$$

We assume that the corners of the tube, E_2 , are much stiffer than the "beams," and that the right angles formed by both sides remain right angles. We also assume that any deformations to cause the corners to open would exceed the elastic limit of the tube and would result in permanent deformation. We are not concerned with this situation.

In view of the above, the slopes at the corners are equal:

$$y'_{x=-(b_o/2)} = \dot{v}_{u=a_o/2} \text{ as indicated in Fig. 8} \tag{11}*$$

The last boundary condition necessary is that the moments at the ends of the beams are equal:

$$y''_{x=-(b_o/2)} = \ddot{v}_{u=a_o/2} \tag{12}$$

Now

$$y' = \alpha A \left[\sin \alpha x \cosh \alpha x + \sinh \alpha x \cos \alpha x\right] + \alpha D \left[\cos \alpha x \sinh \alpha x - \sin \alpha x \cosh \alpha x\right]$$

$\therefore$ substituting (8) and evaluating y' at $x = -(b_o/2)$

$$y'_{x=-(b_o/2)} = \alpha D \left[\frac{\sin \alpha b_o + \sinh \alpha b_o}{2 \sin \frac{\alpha b_o}{2} \sinh \frac{\alpha b_o}{2}}\right] - \frac{M}{4\alpha^3}\left(\cot \frac{\alpha b_o}{2} + \coth \frac{\alpha b_0}{2}\right)$$

Likewise for beam a_o : $\dot{v}$ at $u = a_o/2$

$$\dot{v}_{u=a_o/2} = -\beta H \left[\frac{\sin \beta a_o + \sinh \beta a_o}{2 \sin \frac{\beta a_o}{2} \sinh \frac{\beta a_o}{2}}\right] + \frac{M}{4\beta^3}\left(\cot \frac{\beta a_o}{2} + \coth \frac{\beta a_o}{2}\right)$$

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* Note: Primes indicate derivatives with respect to x and dots indicate derivatives with respect to u.

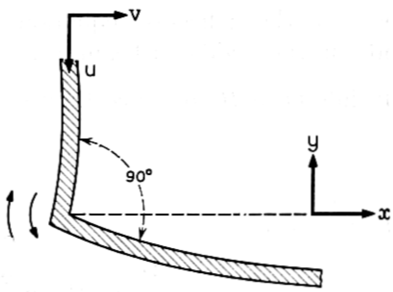

Fig. 8 — Conditions at corner of tube.

From boundary condition (11) we can solve for D:

$$D = \frac{1}{\alpha E}\left[\frac{M}{4}\left(\frac{K}{\beta^3} + \frac{F}{\alpha^3}\right) - H\beta G\right] \tag{13}$$

where

$$E = \frac{\sin \alpha b_o + \sinh \alpha b_o}{2 \sin \frac{\alpha b_o}{2} \sin \frac{\alpha b_0}{2}}$$

$$G = \frac{\sin \beta a_o + \sinh \beta a_o}{2 \sin \frac{\beta a_o}{2} \sinh \frac{\beta a_o}{2}}$$

$$F = \cot \frac{\alpha b_o}{2} + \coth \frac{\alpha b_o}{2}$$

$$K = \cot \frac{\beta a_o}{2} + \coth \frac{\beta a_o}{2} .$$

Now $y'' = 2\alpha^2 \left[A \cos \alpha x \cosh \alpha x - D \sin \alpha x \sinh \alpha x\right]$

$$\therefore\ y''_{x=-b_o/2} = \frac{-2\alpha^2 D\left(\cos^2 \frac{\alpha b_o}{2} + \sinh^2 \frac{\alpha b_o}{2}\right) + \frac{M}{2\alpha^2}\left(\cos \frac{\alpha b_o}{2} \cosh \frac{\alpha b_o}{2}\right)}{\sin \frac{\alpha b_o}{2} \sinh \frac{\alpha b_o}{2}}$$

Likewise for beam a_o :

$$\ddot{v}_{u=a_o/2} = \frac{-2\beta^2 H\left(\cos^2 \frac{\beta a_o}{2} + \sinh^2 \frac{\beta a_o}{2}\right) + \frac{M}{2\beta^2}\left(\cos \frac{\beta a_o}{2} \cosh \frac{\beta a_o}{2}\right)}{\sin \frac{\beta a_o}{2} \sinh \frac{\beta a_o}{2}}$$

∴ from boundary condition (12) we again solve for D:

$$D = \frac{\dfrac{\dfrac{\beta^2 H}{\alpha^2}\left(\cos^2\dfrac{\beta a_o}{2} + \sinh^2\dfrac{\beta a_o}{2}\right) - \dfrac{M}{4\beta^2\alpha^2}\left(\cos\dfrac{\beta a_o}{2}\cosh\dfrac{\beta a_o}{2}\right)}{\sin\dfrac{\beta a_o}{2} + \sinh\dfrac{\beta a_o}{2}}}{\left(\dfrac{\cos^2\dfrac{\alpha b_o}{2} + \sinh^2\dfrac{\alpha b_o}{2}}{\sin\dfrac{\alpha b_o}{2}\sinh\dfrac{\alpha b_o}{2}}\right)} + \frac{\dfrac{M}{4\alpha^4}\left(\cot\dfrac{\alpha b_o}{2}\coth\dfrac{\alpha b_o}{2}\right)}{\left(\dfrac{\cos^2\dfrac{\alpha b_o}{2} + \sinh^2\dfrac{\alpha b_o}{2}}{\sin\dfrac{\alpha b_o}{2}\sinh\dfrac{\alpha b_o}{2}}\right)} \tag{14}$$

∴ to evaluate H, we equate Equations (13) and (14); and let

$$\frac{\beta^4}{\alpha^4} = \frac{a_o}{b_o} = r, \qquad \text{then} \quad \frac{4\alpha^4 H}{M} = \frac{B'}{A'}$$

where,

$$B' = 2\left[\frac{\left(\cot\dfrac{\alpha b_o}{2} + \coth\dfrac{\alpha b_o}{2}\right) + r^{-\frac{3}{4}}\left(\cot\dfrac{\alpha b_o}{2}r^{\frac{5}{4}} + \coth\dfrac{\alpha b_o}{2}r^{\frac{5}{4}}\right)}{\sin\alpha b_o + \sinh\alpha b_o}\right] + \left[\frac{1}{\sqrt{r}}\,\frac{\left(\cot\dfrac{\alpha b_o}{2}r^{\frac{5}{4}}\coth\dfrac{\alpha b_o}{2}r^{\frac{5}{4}}\right) - \left(\cot\dfrac{\alpha b_o}{2}\coth\dfrac{\alpha b_o}{2}\right)}{\cos^2\dfrac{\alpha b_o}{2} + \sinh^2\dfrac{\alpha b_o}{2}}\right]$$

and

$$A' = r^{\frac{1}{4}}\left[\frac{\sin\alpha b_o r^{\frac{5}{4}} + \sinh\alpha b_o r^{\frac{5}{4}}}{\left(\sin\dfrac{\alpha b_o}{2}r^{\frac{5}{4}}\sinh\dfrac{\alpha b_o}{2}r^{\frac{5}{4}}\right)(\sin\alpha b_o + \sinh\alpha b_o)}\right] + \sqrt{r}\left[\frac{\cos^2\dfrac{\alpha b_o}{2}r^{\frac{5}{4}} + \sinh^2\dfrac{\alpha b_o}{2}r^{\frac{5}{4}}}{\left(\sin\dfrac{\alpha b_o}{2}r^{\frac{5}{4}}\sinh\dfrac{\alpha b_o}{2}r^{\frac{5}{4}}\right)\left(\cos^2\dfrac{\alpha b_o}{2} + \sinh^2\dfrac{\alpha b_o}{2}\right)}\right].$$

With this value of H, D can be determined from either (13) or (14). Finally:

$$\frac{4\alpha^4}{M}\,y = \frac{4\alpha^4 D}{M}\left[\cos\alpha x\cosh\alpha x - \left(\cot\frac{\alpha b_o}{2}\coth\frac{\alpha b_o}{2}\right)\cdot\sin\alpha x\sinh\alpha x\right] - \left(1 - \frac{\sin\alpha x\sinh\alpha x}{\sin\frac{\alpha b_o}{2}\sin\frac{\alpha b_o}{2}}\right) \tag{16}$$

where

$$-\frac{b_o}{2} \leqq x \leqq \frac{b_o}{2}$$

and

$$\frac{4\alpha^4}{M}\,v = \frac{4\alpha^4 H}{M}\left[\cos\alpha r^{\frac{1}{4}}u\cosh\alpha r^{\frac{1}{4}}u - \left(\cot\frac{\alpha b_o}{2}r^{\frac{5}{4}}\coth\frac{\alpha b_o}{2}r^{\frac{5}{4}}\right)\sin\alpha r^{\frac{1}{4}}u\sinh\alpha r^{\frac{1}{4}}u\right] - \frac{1}{r}\left(1 - \frac{\sin\alpha r^{\frac{1}{4}}u\sinh\alpha r^{\frac{1}{4}}u}{\sin\frac{\alpha b_o}{2}r^{\frac{5}{4}}\sinh\frac{\alpha b_o}{2}r^{\frac{5}{4}}}\right) \tag{17}$$

where

$$-\frac{rb_o}{2} \leqq u \leqq \frac{rb_o}{2}$$

or

$$-\frac{a_o}{2} \leqq u \leqq \frac{a_o}{2}$$

and the stresses for interfaces b_o and a_o are respectively:

$$\frac{P}{E_1 M}\,\sigma_y = \frac{4\alpha^4}{M}\,y + 1 \qquad \text{and} \qquad \frac{P}{E_1 M}\,\sigma_v = \frac{4\alpha^4 r}{M}\,v + 1\,.$$

A.2 *A Special Case: The Square Shrink Fit*

As a point of interest: if $a_o = b_o$ (square enclosure); $r = 1$

$$H = D = \frac{M}{2\alpha^4}\left(\frac{\cot\frac{\alpha b_o}{2} + \coth\frac{\alpha b_o}{2}}{\sin\alpha b_o + \sinh\alpha b_o}\right)\sin\frac{\alpha b_o}{2}\sinh\frac{\alpha b_o}{2} \tag{18}$$

and

$$y = v = \frac{M}{2\alpha^4}\left[\left(\frac{\cot\alpha\frac{b_o}{2} + \coth\frac{\alpha b_o}{2}}{\sin\alpha b_o + \sinh\alpha b_o}\right)\left(\sin\frac{\alpha b_o}{2}\sinh\frac{\alpha b_o}{2}\cos\alpha x\right.\right.$$
$$\left.\cdot\cosh\alpha x - \cos\frac{\alpha b_o}{2}\cosh\frac{\alpha b_o}{2}\sin\alpha x\sinh\alpha x\right) \tag{19}$$
$$\left. - \frac{1}{2}\left(1 - \frac{\sin\alpha x\sinh\alpha x}{\sin\frac{\alpha b_o}{2}\sinh\frac{\alpha b_o}{2}}\right)\right]$$

The maximum deflection is at $x = 0$:

$$\therefore\ y = v\,|_{\max} = \frac{M}{2\alpha^4}\left[\frac{\left(\cos\frac{\alpha b_o}{2} - \cosh\frac{\alpha b_o}{2}\right)\left(\sinh\alpha\frac{b_o}{2} - \sin\alpha\frac{b_o}{2}\right)}{\sin\alpha b_o + \sinh\alpha b_o}\right] \tag{20}$$

It can also be shown that the slopes are zero at the end of the interfaces and at the four midpoints, when $r = 1$.

The condition for zero stress at $x = u = 0$ is obtained from (2) as

$$y \leqq -\frac{\alpha\Delta T a_o(1 + \nu_1)}{2} = -\frac{M}{4\alpha^4}. \tag{21}$$

The solution breaks down wherever a tension at the interface is indicated, since the load becomes zero over the separated portion of the beam. The calculated tension is that which would be required to prevent separation of E_1 from the computed deflection curve. Substituting (21) into (16) when $x = 0$ gives $D = 0$; then from (18) we have the necessary condition that

$$\cot\frac{\alpha b_o}{2} = -\coth\frac{\alpha b_o}{2}$$

therefore from Fig. 3 the eigenvalues of this equation are determined. (Although these values are for a square and indicate resistance values which result in zero stress at the wall centers, they are approximately the boundaries for the ranges of the modes of separation for any rectangle.) (Compare Fig. 9 with Fig. 3.)

$$\therefore\ \text{eigenvalues of } \frac{\alpha b_o}{2} = 2.365;\ 5.498;\ 8.639;\ 11.781\ \cdots$$

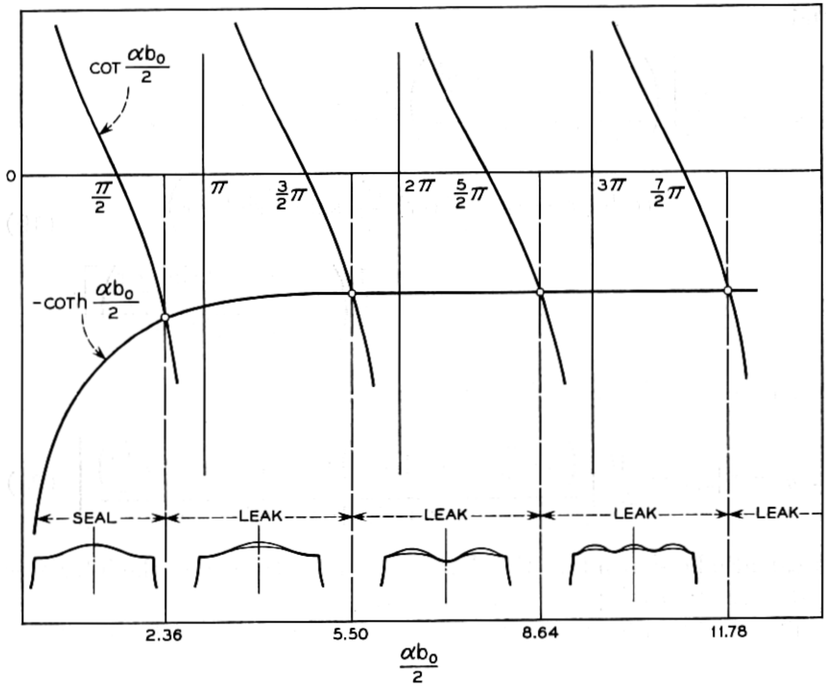

Fig. 9 — Square shrink fit: boundaries for modes of separation. The values shown are also approximately valid for any rectangle.

These eigenvalues after the first (2.365) are for all practical purposes separated by π.

To extend this to the more general case of rectangular enclosures we can determine from (16), (17), (2) and the equivalent of (2) the following at $x = 0$ and $u = 0$ respectively (midpoints of the enclosure walls);

$$\sigma_y = \frac{E_1}{P} 4\alpha^4 D \qquad \text{and} \qquad \sigma_v = \frac{E_1}{P} 4\alpha^4 rH.$$

$\therefore$ σ_y and σ_v have the same sign ($\pm$) as the constants D and H respectively. When these constants are negative the stress on the insert is a pseudo-tension or in reality a separation of the insert and the enclosure at the midpoint of the enclosure walls. Care must be exercised in determining whether a seal exists or not. If the sign of either constant D or H is negative or if either constant is zero then a seal is not possible. A seal will exist only if both constants are positive.

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