Solution of Systems of Linear Ordinary Differential Equations with Periodic Coefficients

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An analysis technique is presented to provide an essentially explicit solution for a system of n simultaneous first-order linear differential equations with periodic coefficients. This representation of a periodic variable-parameter linear system of arbitrary finite order is chosen for its theoretical and practical advantages over the classical nth *order linear differential equation. Emphasis is placed on natural mode solutions of a homogeneous set of equations. The characteristic exponents for these solutions are determined from a polynomial equation the coefficients of which are linear combinations of $n - 1$ convergent infinite-order determinants. Approximate calculation of these determinants is feasible for problems of moderate order.*

I. INTRODUCTION

Systems of linear ordinary differential equations with periodic coefficients are assuming an increasing importance in engineering problems. Two applications of present interest are periodically time-variable networks and multimode waveguide with periodic physical distortions. Such applications have usually been analyzed by methods appropriate to special cases such as the second-order case or by approximate techniques valid for almost constant-parameter systems. However, perturbation techniques for almost stationary systems are inadequate for careful analysis of large-signal behavior of time-variable networks. Similarly, a periodically distorted helix waveguide, for which more than two modes must be considered,[1] should be described by a differential system of order greater than two. These examples illustrate the importance of a technique for obtaining essentially explicit solutions of periodic variable-parameter linear systems. Solutions in terms of characteristic exponents are known to exist for systems of linear differential equations with periodic coefficients.[2] However, the methods usually employed for solving such systems, such as power-series techniques, iterative processes, and incremental numerical solution methods, fail to provide a system response description valid for all values of the independent variable (time, distance, etc.).

The analysis method to be presented below provides an essentially explicit solution for periodic variable-parameter linear systems of arbitrary finite order. The solution describes the system behavior for all values of the independent variable. Emphasis will be placed on obtaining a set of basis functions for a homogeneous system, since the solution in the inhomogeneous case can be obtained from the basis functions. As shown by Darlington,[3] these functions may be regarded as analogues of partial fractions in fixed network theory.

II. FORMULATION OF DIFFERENTIAL SYSTEM

In this discussion the system of equations to be solved will be represented by the vector differential equation

$$F'(t) = B(t)F(t) \tag{1}$$

where $F(t)$ and $B(t)$ are nth-order column and square matrices, respectively, and the prime denotes differentiation with respect to the independent variable t. It is supposed that the elements of $B(t)$ are known functions of t with a common period of unity, i.e.,

$$B(t) = B(t + 1). \tag{2}$$

The formulation of this problem in (1) is chosen not only for its elegance, but also because of its practical advantages. As indicated by Kinariwala[4] these include the ability to write such an equation directly from a time-variable network, the fact that the eigenvalues of $B(t)$ are natural frequencies for stationary networks, and the convenience of (1) in obtaining the quadratic forms representing stored energy and dissipated power in stationary or nonstationary cases. These advantages have their translated versions in other physical problems, including multimode waveguide problems. Moreover, an equation such as (1) is easily obtained from an nth-order linear differential equation, but the transformation from (1) to such an equation can be quite difficult (or analytically inconvenient).[4] Thus, (1) represents a well-founded beginning for the analysis of variable-parameter problems of practical or theoretical interest.

III. FORM OF SOLUTIONS

The form of solutions of (1) is well known;[2] pertinent properties of such solutions will be reviewed here briefly. If $B(t)$ is piecewise continuous (1) has the unique solution

$$F(t) = X(t)F(0) \tag{3}$$

where $X(t)$ is the unique nonsingular square matrix satisfying

$$\begin{aligned} X' &= BX \\ X(0) &= I = \text{diag}\,\{1\}. \end{aligned} \tag{4}$$

When $B(t)$ satisfies (2), $X(t)$ may be written as

$$X(t) = J(t)\,e^{Kt} \tag{5}$$

where

$$J(t) = J(t+1) \tag{6}$$

and

$$e^{K} = X(1). \tag{7}$$

For convenience it will be assumed here that the eigenvalues of K are distinct, or at least that K can be diagonalized; thus, a constant nonsingular matrix P exists so that

$$K = PMP^{-1} \tag{8}$$

where

$$M = \text{diag}\,\{\mu_i\} \tag{9}$$

and the constants μ_i are the eigenvalues of K. The matrix exponential function in (5) may be similarly diagonalized, so that the solution (3) may be constructed in the form

$$F(t) = J(t)P[\text{diag}\,\{e^{\mu_i t}\}]P^{-1}F(0). \tag{10}$$

By establishing the special initial conditions

$$F_i(0) = P\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \\ \\ \\ \leftarrow i^{\text{th}} \text{ row} \\ \\ \\ \end{matrix} \tag{11}$$

the corresponding unique solution

$$F_i(t) = e^{\mu_i t} J(t) F_i(0), \tag{12}$$

is obtained from (10). Thus, by proper choice of initial conditions a set of n solutions of the form

$$F(t) = e^{\mu t}Q(t) \tag{13}$$

where μ is a scalar constant and $Q(t)$ is a column matrix with period unity, have been shown to exist.

The n solutions in the form (12) or (13) represent natural modes of the periodic system described by (1) and (2). If the n values of μ_i are distinct the corresponding n solutions are certainly independent and form a set of basis solutions of (1). Any other solution of (1) comprises a linear combination of solutions like (12) or (13). Moreover, as Darlington[3] has pointed out, these natural-mode solutions are essentially unique because of their simple form. Hence the natural modes given by (12) represent a complete and essentially unique description of the natural behavior of the periodic system. The eigenvalues μ_i, frequently referred to as characteristic exponents, play a role analogous to response poles or natural frequencies of stationary systems. The strength of each natural mode in the homogeneous case is determined by the initial conditions and the constant matrix P. Moreover, the natural-mode solutions allow a complete solution to be calculated in the inhomogeneous case. Thus, the determination of n corresponding solutions for μ and $Q(t)$ in (13) is central to the problems associated with (1) and (2).

The object of the present treatment is to indicate a technique for determining the characteristic exponents μ, as well as the corresponding matrices $Q(t)$ if desired. Primary attention is given in finding the characteristic exponents μ because of their practical importance and because the solution for $Q(t)$ is not greatly difficult in principle if the appropriate characteristic exponent is known. Solutions for $Q(t)$ are mentioned in Appendices A and B.

The method to be discussed resembles the technique used by Hill[5,6] in solving the second-order equation

$$x''(t) + A(t)x(t) = 0 \tag{14}$$

where $A(t)$ is periodic. It will be shown that the characteristic exponents may be determined from roots of either a transcendental or polynomial equation in which certain infinite-order determinants enter as parameters. A technique similar to Hill's was employed by H. von Koch in the last century to provide an explicit solution in terms of infinite-order determinants for a general nth-order linear differential equation with periodic coefficients. This technique is carefully discussed by Forsyth[7] and Riesz,[8] who also give references to von Koch's original papers. Thus, the method presented here, although developed independently, does not solve an unsolved mathematical problem when applied to a periodic variable-parameter system described by an nth-order linear differential equation. It does, however, solve the stated problem in a way that appears to have several advantages, mostly associated with its formulation as a system of n simultaneous first-order linear equations. These advantages, already mentioned in Section II, seem likely to make the present solution technique more useful in the analysis and synthesis of periodic variable-parameter systems than one based entirely on the classical nth-order linear differential equation.

IV. INTEGRAL FORM FOR THE PERIODIC SYSTEM

The analysis of the periodic system begins by multiplying both members of (1) by e^{-at}, where a is an arbitrary constant, and adding and subtracting aFe^{-at} to yield, whenever F' exists,

$$(Fe^{-at})' + aFe^{-at} = BFe^{-at}. \tag{15}$$

Integration of (15) results in the integral equation

$$Fe^{-at} + a \int Fe^{-at}\, dt = \int BFe^{-at}\, dt + C \tag{16}$$

where C is a constant. Any solution of (15) is also a solution of (16); thus, let F be a solution given by (13) and let

$$a = \mu + j2\pi k \tag{17}$$

where k is an arbitrary integer. Equation (16) becomes

$$Qe^{-j2\pi kt} + (\mu + j2\pi k) \int Qe^{-j2\pi kt}\, dt = \int BQe^{-j2\pi kt}\, dt + C. \tag{18}$$

If (18) is evaluated at $t = 0$ and $t = 1$, and the results subtracted, the first term in (18) makes no contribution, being periodic. Hence, (18) implies

$$(\mu + j2\pi k) \int_0^1 Q\, e^{-j2\pi kt}\, dt = \int_0^1 BQe^{-j2\pi kt}\, dt \tag{19}$$

for all integers k. It will be seen below that this integral equation suffices to determine μ and $Q(t)$, which are essentially eigenvalues and eigenfunctions.

V. MATRIX DIFFERENCE EQUATION

To make use of (19) in finding solutions of (1) it will be assumed that the given matrix $B(t)$ and the solution matrix $Q(t)$ may be expanded in the Fourier series

$$B(t) = \sum_{p=-\infty}^{\infty} B_p e^{j2\pi pt} \tag{20}$$

and

$$Q(t) = \sum_{p=-\infty}^{\infty} Q_p e^{j2\pi pt} \tag{21}$$

where matrices B_p are square matrices and Q_p are column matrices. Requirements on the asymptotic behavior of the elements of matrices B_p and Q_p for large values of $| p |$ will be discussed in Appendix A in relation to convergence of certain infinite-order determinants. The Fourier series for the matrix product BQ may be written as

$$BQ = \sum_{p=-\infty}^{\infty} (BQ)_p e^{j2\pi pt} \tag{22}$$

in which the column matrices $(BQ)_p$ are given by the convolution

$$(BQ)_p = \sum_{r=-\infty}^{\infty} B_{p-r}Q_r. \tag{23}$$

Except for a factor of 2π the integrals in (19) express the Fourier coefficients of Q and BQ. Thus, if Q and BQ possess Fourier series (19) is equivalent to the infinite set of linear equations

$$(\mu + j2\pi k)Q_k = (BQ)_k \tag{24}$$

or

$$(\mu + j2\pi k)Q_k = \sum_{r=-\infty}^{\infty} B_{k-r}Q_r, \tag{25}$$

where k assumes all integral values. Equation (25) might be regarded as a matrix difference equation for Q_p; however it is more convenient here to consider (25) as defining an eigenvalue problem for an infinite matrix. In terms of Kronecker's δ, (25) is

$$0 = \sum_{r=-\infty}^{\infty} [B_{k-r} - \delta_{kr}(\mu + j2\pi k)I]Q_r \tag{26}$$

where I is the nth-order unit matrix. The expanded form of (26) is shown in the following infinite-order matrix equation, in which the first matrix is partitioned into $n \times n$ size blocks and the second into $n \times 1$ size blocks. The "origins" of the matrices fall at $(B_0 - \mu I)$ and Q_0.

$$\begin{Vmatrix} \ddots & & & & & \\ \cdot\; B_0-(\mu-j4\pi)I & B_{-1} & B_{-2} & & & \\ \cdot\; B_1 & B_0-(\mu-j2\pi)I & B_{-1} & B_{-2} & & \\ \cdot\; B_2 & B_1 & B_0-\mu I & B_{-1} & B_{-2} & \\ & B_2 & B_1 & B_0-(\mu+j2\pi)I & B_{-1} & \\ & & B_2 & B_1 & B_0-(\mu+j4\pi)I & \cdots \\ & & & & & \ddots \end{Vmatrix} \begin{vmatrix} \vdots \\ Q_{-2} \\ Q_{-1} \\ Q_0 \\ Q_1 \\ Q_2 \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{vmatrix} \tag{27}$$

For convenience it will be assumed that B_0 is a triangular matrix so that its eigenvalues appear explicitly as main diagonal elements. To show that a constant linear transformation of the dependent variable F can always produce this property, let

$$B(t) = B_0 + A(t) \tag{28}$$

where $A(t)$ has a zero mean, and let

$$X = PF \tag{29}$$

where P is a nonsingular matrix of constants. Then (1) is transformed to

$$X' = (PB_0P^{-1} + PAP^{-1})X. \tag{30}$$

This equation has the same form as (1), but the constant term in its coefficient matrix is the matrix PB_0P^{-1} derived from B_0 by a similarity transformation. It is well known that a square matrix is reducible by a similarity transformation to the classical canonical form having eigenvalues on the main diagonal and possibly nonzero constants in some positions of the next higher diagonal.[9] (These constants cannot appear if B_0 has distinct eigenvalues; hence, B_0 can often be assumed to be diagonal.) The matrix B_0 can also be reduced to a triangular form by a similarity transformation in which P is a unitary matrix.[10]. This reduction, which is always possible, may sometimes have advantages in studying energy functions or related quadratic forms. Thus, by either technique B_0 can be reduced to triangular form. It will be assumed that such a transformation has been effected in obtaining (1).

VI. CONVERGENT LINEAR EQUATIONS AND INFINITE-ORDER DETERMINANT

To produce convergence of the determinant of coefficients of the infinite set of homogeneous equations defining Q, Equations (26) or

(27), divide each elementary row of the coefficient matrix in Equation (27) by its main-diagonal elements. When B_0 is diagonal this process is identical with multiplication of each equation in Equation (26) or each matrix row in the square matrix of Equation (27) by the diagonal matrix $[B_0 - (\mu + j2\pi k)I]^{-1}$. In general, let the matrix Λ be defined by

$$\Lambda = \operatorname{diag}\{\lambda_p\} \tag{31}$$

where λ_p represent the n eigenvalues of B_0. Then the set of equations with convergent determinant may be written as

$$[M_{kr}][Q_r] = 0 \tag{32}$$

where submatrices M_{kr} are given by

$$M_{kk} = [\Lambda - (\mu + j2\pi k)I]^{-1}[B_0 - (\mu + j2\pi k)I] \tag{33}$$

and

$$M_{kr} = [\Lambda - (\mu + j2\pi k)I]^{-1}B_{k-r}, \qquad k \neq r. \tag{34}$$

When B_0 is diagonal, M_{kk} reduces to the unit matrix. Thus, a typical determinant $d[M_{kr}]$ of (32) may be illustrated for $n = 2$ by the following scheme:

$$d(\mu) = \begin{vmatrix} \ddots & & & & & & & \\ \cdots & 1 & 0 & \frac{a_{-1}}{\lambda_1+j2\pi-\mu} & \frac{b_{-1}}{\lambda_1+j2\pi-\mu} & \frac{a_{-2}}{\lambda_1+j2\pi-\mu} & \frac{b_{-2}}{\lambda_1+j2\pi-\mu} & \\ \cdots & 0 & 1 & \frac{c_{-1}}{\lambda_2+j2\pi-\mu} & \frac{d_{-1}}{\lambda_2+j2\pi-\mu} & \frac{c_{-2}}{\lambda_2+j2\pi-\mu} & \frac{d_{-2}}{\lambda_2+j2\pi-\mu} & \\ \cdots & \frac{a_1}{\lambda_1-\mu} & \frac{b_1}{\lambda_1-\mu} & 1 & 0 & \frac{a_{-1}}{\lambda_1-\mu} & \frac{b_{-1}}{\lambda_1-\mu} & \\ & \frac{c_1}{\lambda_2-\mu} & \frac{d_1}{\lambda_2-\mu} & 0 & 1 & \frac{c_{-1}}{\lambda_2-\mu} & \frac{d_{-1}}{\lambda_2-\mu} & \\ & \frac{a_2}{\lambda_1-j2\pi-\mu} & \frac{b_2}{\lambda_1-j2\pi-\mu} & \frac{a_1}{\lambda_1-j2\pi-\mu} & \frac{b_1}{\lambda_1-j2\pi-\mu} & 1 & 0 & \\ & \frac{c_2}{\lambda_2-j2\pi-\mu} & \frac{d_2}{\lambda_2-j2\pi-\mu} & \frac{c_1}{\lambda_2-j2\pi-\mu} & \frac{d_1}{\lambda_2-j2\pi-\mu} & 0 & 1 & \cdots \\ & & & & & & & \ddots \end{vmatrix} \tag{35}$$

The notation used in (35) is

$$B_p = \begin{bmatrix} a_p & b_p \\ b_p & d_p \end{bmatrix}, \qquad p \neq 0 \tag{36}$$

$$B_0 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{37}$$

The determinant of the infinite-order matrix $[M_{kr}]$ of (32), illustrated by (35) for $n = 2$, will be denoted by $d(\mu)$ to show its functional dependence on the argument μ. The function $d(\mu)$ is actually a determinant of infinite order. If this determinant converges it represents a function of μ which must vanish in order to obtain nontrivial solutions for Q_r in (32). Requirements necessary for the convergence of $d(\mu)$ are discussed in Appendix A, where it will be shown that $d(\mu)$ converges for a large class of problems. Hence the basic equation

$$d(\mu) = 0 \tag{38}$$

defines the characteristic exponents of the differential system (1).

VII. FUNCTIONAL EXPRESSIONS FOR THE CHARACTERISTIC DETERMINANT

Equation (38) taken alone is rather unwieldy, involving as it does the equation to zero of an infinite-order determinant whose elements are functions of μ. However, it will now be shown that expressions for $d(\mu)$ in terms of elementary functions may be written to allow a simple solution of (38).

The determinant $d(\mu)$ is shown in Appendix A to converge for all values of μ except those for which the denominators of rows of $d(\mu)$ vanish. Multiplication of one row of an infinite-order determinant by any scalar is equivalent to multiplication of the determinant by the same scalar. Similarly multiplication of any row of $d(\mu)$ by its corresponding denominator $\lambda_p - (\mu + j2\pi k)$ produces a determinant convergent at $\lambda_p = \mu + j2\pi k$, so that each row of $d(\mu)$ introduces exactly one pole in $d(\mu)$. Moreover, $d(\mu)$ is periodic in μ with period $j2\pi$, since replacing μ by $\mu + j2\pi$ only shifts the origin of the infinite-order determinant. Evidently $d(\mu)$ has simple poles at

$$\mu = \lambda_p + j2\pi q, \qquad p = 1,2, \cdots, n \qquad q \text{ integral.} \tag{39}$$

It will be assumed for the moment that these poles are distinct; this restriction may be relaxed slightly, as shown in Appendix B. Finally, as μ approaches infinity along any radial line in the complex μ plane except a vertical line, the off-diagonal elements in $d(\mu)$ tend toward zero, or briefly

$$d(\infty) = 1. \tag{40}$$

The periodicity of $d(\mu)$ implies that the residue of $d(\mu)$ at any of the poles in (39) is independent of the particular integer q. Thus, a formal expansion of $d(\mu)$ in partial fractions is

$$d(\mu) = K_\infty + \sum_{p=1}^{n} \sum_{q=-\infty}^{\infty} \frac{K_p}{\mu - \lambda_p - j2\pi q}. \tag{41}$$

According to the Mittag-Leffler theorem[11] this expansion defines the function

$$d(\mu) = K_\infty + \frac{1}{2} \sum_{p=1}^{n} K_p \coth\left(\frac{\mu - \lambda_p}{2}\right). \tag{42}$$

A relation fixing K_∞ may be derived from (40) by noting that as μ approaches infinity along any nonvertical radial line

$$\lim_{\mu \to \infty} \coth\left(\frac{\mu - \lambda_p}{2}\right) = 1 \tag{43}$$

so that

$$K_\infty = 1 - \frac{1}{2} \sum_{p=1}^{n} K_p . \tag{44}$$

To compute the residues K_p the well-known rule

$$K_p = \lim_{\mu \to \lambda_p} (u - \lambda_p) d(\mu) = [(\mu - \lambda_p) d(\mu)]_{\mu = \lambda_p} \tag{45}$$

is employed. The procedure is simply to multiply every element in the row of $d(\mu)$ containing $\lambda_p - \mu$ (in the denominators) by the factor $(\mu - \lambda_p)$ and to evaluate the resulting determinant. For example, in the case of $n = 2$ used above for illustration, the row of $d(\mu)$ containing $\lambda_1 - \mu$ in the denominators is replaced by

$$\cdots \; -a_2 \; -b_2 \; -a_1 \; -b_1 \; 0 \quad 0 \; -a_{-1} \; -b_{-1} \; -a_{-2} \; -b_{-2} \; \cdots \tag{46}$$

and the resulting determinant evaluated at $\mu = \lambda_1$. Reasonably accurate and efficient methods for computing K_p from such a determinant can be programmed just as for Hill's determinant in the second-order case. Such a technique is discussed briefly in Appendix C.

It is well known that the solution of Hill's equation generally requires the evaluation of only one infinite-order determinant, while the solution of a second-order problem using (38) and (42) appears to require the evaluation of two determinants. Actually it will be shown that only $n - 1$ determinants need be calculated for an nth-order system of equations. In addition (42) may be simplified because of the relation among the residues K_p to be demonstrated below.

To examine the poles and zeros of $d(\mu)$ it is convenient to consider the complex μ plane divided into horizontal strips of width 2π. The poles of $d(\mu)$ fall at $\lambda_p + j2\pi q$. Although the eigenvalues λ_p may lie in any of these strips, values of q always exist to give one pole in the fundamental strip $0 \leqq \text{Im } \mu < 2\pi$ representative of each λ_p. Hence $d(\mu)$ has exactly n poles in each strip. It will be seen shortly that $d(\mu)$

also has n zeros in each strip so that a pole-zero constellation for $d(\mu)$ might be illustrated by Fig. 1.

The desired relation among the residues K_p is obtained by noting that

$$\int_{abcd} d(\mu)\, d\mu = \sum_{p=1}^{n} K_p \tag{47}$$

where the integral is taken around the rectangular contour $abcd$ shown in Fig. 1 (or a congruent rectangle vertically displaced if a pole happens to fall at Im $\mu = 0$). The periodicity of $d(\mu)$ insures that the contributions to the integral from the horizontal sides ab and cd will cancel. The vertical sides bc and da are supposed to be displaced from the origin far enough to include all n poles in the rectangle so that (47) is valid. As their displacement approaches infinity the value of $d(\mu)$ approaches unity and the contributions from the vertical sides tend to cancel. Thus (47) implies

$$\sum_{p=1} K_p = 0. \tag{48}$$

This relation shows that (44) and (42) may be simplified to

$$K_\infty = 1 \tag{49}$$

and

$$d(\mu) = 1 + \frac{1}{2} \sum_{p=1}^{n} K_p \coth\left(\frac{\mu - \lambda_p}{2}\right). \tag{50}$$

It also allows one residue to be computed from the other $n - 1$, al-

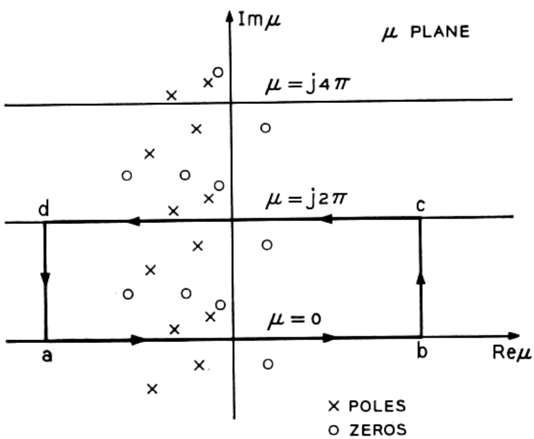

Fig. 1 — Pole-zero constellation.

though all n residues might be computed in practice and (48) used as a check for numerical accuracy.

Equation (50) expresses the characteristic determinant $d(\mu)$ in terms of the eigenvalues λ_p of the stationary part of the system and the residue determinants K_p. The characteristic exponents μ are thus by (38) and (50) the roots of the trigonometric equation

$$0 = 1 + \frac{1}{2} \sum_{p=1}^{n} K_p \coth \left(\frac{\mu - \lambda_p}{2} \right). \tag{51}$$

This trigonometric equation represents an explicit solution of the problem of finding characteristic exponents for an nth-order periodic system.

It is evident from (50) as well as from Fig. 1 and the periodicity of the function e^{μ} that the substitution

$$z = e^{\mu} \tag{52}$$

reduces (50) to a rational function in z. Zeros and infinities of z do not introduce superfluous poles or other singularities in this function because of (43). Thus, the poles and zeros of this rational function are mapped by (52) into the poles and zeros of $d(\mu)$ shown in Fig. 1. Any strip of vertical width 2π in the μ plane is mapped by (52) into the entire z plane so that the rational function of z has n poles in the z plane. The number of zeros of the rational function is also necessarily n. Hence, $d(\mu)$ has precisely n zeros in any horizontal strip of width 2π in the μ plane.

Because of the existence of well-developed techniques for polynomial manipulation, such as approximate solution methods, interpolation formulae, and stability criteria, it is practically convenient to utilize (50) and (51) in rational form. Accordingly let z be defined by (52) and x_p by

$$x_p = e^{\lambda_p}, \qquad p = 1,2, \cdots, n, \tag{53}$$

so that $d(\mu)$ is transformed to

$$D(z) = 1 + \frac{1}{2} \sum_{p=1}^{n} K_p \left(\frac{z + x_p}{z - x_p} \right) = d(\log z). \tag{54}$$

Further, let

$$g(z) = \prod_{p=1}^{n} (z - x_p) \tag{55}$$

be a characteristic polynomial defining the eigenvalues of the stationary part of $B(t)$. (This "characteristic polynomial" differs from the conventional one in that its roots are e^{λ_p} rather than λ_p.) Equation (51), the characteristic equation, then becomes

$$0 = f(z) + g(z) \tag{56}$$

where

$$f(z) = \frac{1}{2} \sum_{p=1}^{n} K_p(z + x_p) \prod_{\substack{q=1 \\ q \neq p}}^{n} (z - x_q). \tag{57}$$

These equations demonstrate that the characteristic polynomial for the periodic system is obtained by adding a certain interpolating polynomial to the characteristic polynomial of the stationary part of the system. The behavior of the interpolating polynomial is prescribed at the roots of the stationary part.

The interpolating polynomial $f(z)$ has the n assigned values

$$f(x_p) = K_p x_p \prod_{\substack{q=1 \\ q \neq p}}^{n} (x_p - x_q); \tag{58}$$

because of the relation (48) among the residues K_p the polynomial (57) is identical with the Lagrangian interpolating polynomial

$$f(z) = \sum_{p=1}^{n} K_p x_p \prod_{\substack{q=1 \\ q \neq p}}^{n} (z - x_q). \tag{59}$$

Evidently the interpolating polynomial $f(z)$ is the unique polynomial of degree $n - 1$ having the assigned values (58). Thus $f(z) + g(z)$, the characteristic polynomial of the periodic system, is the unique monic polynomial of degree n having the n assigned values given in (58). This point of view may give some insight into stability questions. For example, the classical criteria of Routh and Hurwitz, and other results on bounds of zeros of sums of polynomials may be useful here.

If all the residues K_p vanish, as in the stationary case, the limiting values of the characteristic exponents obtained from (51) and (56) are $u = \lambda_p$, $p = 1,2, \cdots, n$. In cases of small variations where all $|K_p|$ are small the characteristic exponents differ very little from the eigenvalues of the stationary part. Asymptotically they may be calculated from any of the approximate equations

$$0 \approx 1 + \frac{1}{2} K_p \coth\left(\frac{\mu - \lambda_p}{2}\right) \tag{60}$$

$$z \approx x_p(1 - K_p) \tag{61}$$

or

$$\mu \approx \lambda_p - K_p \,. \tag{62}$$

Although perturbation type solutions such as (62) probably are more easily calculated by less complicated techniques, characteristic exponents obtained from (61) or (62) may be useful as starting values for solving (51) or (56) by numerical methods.

VIII. EXAMPLE

The following example illustrates the technique for finding characteristic exponents. A second-order case is chosen for convenience because some digital computer programs needed for the efficient evaluation of the residue determinants are not yet available. However, higher-order examples are not different in principle nor will they require inordinately longer computations.

The Mathieu equation

$$\frac{d^2y}{dz^2} + (3 - 4\cos 2z)y = 0 \tag{63}$$

has the solution[12]

$$y = e^{j\beta z} \sum_{r=-\infty}^{\infty} c_{2r+1}\, e^{j(2r+1)z} \tag{64}$$

with

$$\beta = \pm 0.57943224 \cdots \,. \tag{65}$$

In vector form this equation is equivalent to

$$Y' = \pi \begin{bmatrix} 0 & 1 \\ -3 + 4\cos 2\pi t & 0 \end{bmatrix} Y \tag{66}$$

with the identifications

$$Y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \qquad y_1 = y, \qquad z = \pi t. \tag{67}$$

Diagonalization by the transformation $PY = F$ where

$$P = \frac{1}{j2\sqrt{3}} \begin{bmatrix} j\sqrt{3} & 1 \\ -j\sqrt{3} & 1 \end{bmatrix} \tag{68}$$

yields (1), with

$$B(t) = \frac{\pi}{j\sqrt{3}}\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}e^{-j2\pi t} + j\pi\sqrt{3}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix} + \frac{\pi}{j\sqrt{3}}\begin{bmatrix}1 & -1\\1 & -1\end{bmatrix}e^{j2\pi t}. \tag{69}$$

The determinant $d(\mu)$ has the form shown by (35), and the residue at $\mu = j\pi\sqrt{3}$ is approximately $K = j0.562096$, a result obtained from a 42nd-order approximant. From (56) and (57) the characteristic exponents are solutions of

$$\cosh \mu = \cos \pi\sqrt{3} - jK \sin \pi\sqrt{3}, \tag{70}$$

which yields, for $K \approx j0.562096$

$$\mu \approx \pm j0.42059\pi. \tag{71}$$

Corresponding correct values of μ from (64) and (65) are $\pm j0.42057\pi$. A somewhat longer computation would be required to produce a result accurate enough for certain purposes. Such a computation was not employed here because a more fundamentally sound computing technique for band-limited periodic variations as in (69) would exploit the form of the residue determinant and its large number of zero elements. Specifically, it is possible to program a determinant evaluation technique for such cases so that the computation time is asymptotically proportional to the order of the truncated determinant rather than to its cube. This possibility is discussed further in Appendix C.

IX. CONCLUSIONS

A method has been developed for analysis and calculation of solutions of nth-order linear periodic differential systems. The system description employed is a set of n simultaneous first-order linear differential equations. The method allows the determination of characteristic exponents from polynomial equations the coefficients of which are linear combinations of $n-1$ convergent infinite-order determinants. Approximate computation of the determinants is feasible for problems of finite order. In addition to characteristic exponents the complete solutions may also be computed if desired.

APPENDIX A

Convergence

The validity of the analysis presented here depends upon the convergence of the infinite processes employed. It must be shown that the determinant $d(\mu)$ and the Fourier series for $Q(t)$ are convergent if the coefficient matrix $B(t)$ is suitably restricted. For this purpose (32) may be written as the infinite set of scalar equations

$$x_i + \sum_{j=-\infty}^{\infty} a_{ij} x_j = 0 \tag{72}$$

where a_{ij} and x_j are scalars, and the equations hold for all integral i. The coefficients a_{ij} actually are elements of the submatrices M_{kr}, and x_i elements of submatrices Q_r in (32). The determinant of coefficients of the scalar equations is

$$d(\mu) = | \delta_{ij} + a_{ij} | . \tag{73}$$

According to a theorem of St. Bobr[13] this determinant is absolutely convergent if

$$\sum_{i=-\infty}^{\infty} | a_{ii} | \tag{74}$$

and

$$\sum_{i=-\infty}^{\infty} \left[\sum_{\substack{j=-\infty \\ j \neq i}}^{\infty} | a_{ij} |^{p/(p-1)} \right]^{p-1} \tag{75}$$

converge for some value of p in the interval $1 < p \leqq 2$. (For $p = 2$, the case used here, the theorem was given by von Koch.) The expression in (74) obviously converges to zero since all a_{ii} in (72) are zero. Let the elements of the given matrix $B(t)$ be square integrable functions. Then Parseval's relation applies and the Fourier series coefficients for the matrix elements are surely square summable. Hence, the inside sum in (75) converges for $p = 2$. The outside sum also converges for $p = 2$, since its general term is asymptotically proportional to i^{-2} for large $| i |$ (as (33) and (34) indicate by their dependence on k). Of course, an exception occurs for values of μ given by Equation (39). The determinant $d(\mu)$ is singular at these points, but the convergence of the residue determinants K_p for simple poles is assured by St. Bobr's theorem. Thus, $d(\mu)$ converges absolutely and uniformly except for μ arbitrarily near $\lambda_p + j2\pi q$ and has poles at these values of μ.

Since the determinant $d(\mu)$ has zeros at any of the n characteristic values of μ within the strip $0 \leqq Im\mu < 2\pi$, the deletion of the zeroth equation ($i = 0$) from (72) and the transposition of $a_{i0}x_0$ in each equation produces a nonzero determinant of coefficients in the equations

$$x_i + \sum_{\substack{j=-\infty \\ j \neq 0}}^{\infty} a_{ij} x_j = -a_{i0} x_0 = y_i , \qquad i \neq 0. \tag{76}$$

These equations with a_{ij} evaluated at a characteristic value of μ have as solutions the scalar quantities x_j needed to produce the matrices Q_p and thus the matrix $Q(t)$. That meaningful solutions to (76) exist for an arbitrary constant x_0 is shown by a theorem of L. W. Cohen.[14] This theorem (paraphrased) states that if (75) converges for the coefficients in (76), if the (convergent) determinant of (76) does not vanish, and if

$$\sum_{i=-\infty}^{\infty} | y_i |^p$$

converges, then the solutions exist, may be obtained by Cramer's rule (with infinite-order minor determinants), and have the property that

$$\sum_{i=-\infty}^{\infty} | x_i |^p$$

converges. Thus, if the elements of the given matrix $B(t)$ are square integrable functions, the coefficients a_{i0} are surely square summable, and the resulting trigonometric series for elements of $Q(t)$ have square summable coefficients. The Riesz-Fischer theorem[15] then states that the elements of $Q(t)$ are square integrable functions with Fourier coefficients given by the elements of Q_p and that the Fourier series for $Q(t)$ converges to $Q(t)$ in the mean. (Consequently there exists a sequence of partial sums of the Fourier series converging to $Q(t)$ "almost everywhere.")

In a more restricted case which might have more practical importance, it may be shown that if $B(t)$ is continuous so that the elements of B_p are $0(1/p^2)$, the solution matrix $Q(t)$ has the same property. Of course, the Fourier series for $Q(t)$ converges absolutely and uniformly in this case. This convergence condition and the more general one above demonstrate that the analysis technique is valid for a wide class of problems.

APPENDIX B

Multiple Poles of the Characteristic Determinant

If the matrix B_0, the stationary part of the coefficient matrix $B(t)$, has repeated eigenvalues, or if any of its eigenvalues differ by integral multiples of $j2\pi$, some denominators of rows of $d(\mu)$ are identical. In this case $d(\mu)$ has multiple poles, and the necessary analytical and computational procedures become more complicated. It is possible to treat the case of a single second-order pole of $d(\mu)$ by evaluation of $n - 1$

determinants as before, but greater multiplicities require considerably more extensive calculations.

When $d(\mu)$ has an m-fold pole at $\mu = \lambda_1$ the partial fractions expansion of $d(\mu)$ must contain the corresponding principal part of $d(\mu)$. The coefficients in the principal part involve derivatives of $(\mu - \lambda_1)^m d(\mu)$ evaluated at $\mu = \lambda_1$. These derivatives are more difficult to compute than the residue determinants of the simple case because they are linear combinations of most of the first minors of $d(\mu)$. The computation of such minors (not necessarily by direct methods) is also required if $Q(t)$ is to be determined (even when $d(\mu)$ has only simple poles). Appendix A shows this computation to be theoretically possible; it is equivalent to the inversion of a set of equations like (76). Nevertheless, the computation effort would be considerably greater than that required for computation of characteristic exponents when $d(\mu)$ has only simple poles.

When $d(\mu)$ has a single second-order pole, (48) may be utilized to make possible the calculation of characteristic exponents. It is convenient here to use the rational form of (54) for the infinite-order determinant $d(\mu)$. Let the repeated roots of B_0 be identified with λ_1, λ_2 and x_1, x_2 respectively. Define α_1 and α_2 by

$$\begin{aligned} \alpha_1 &= K_1(x_1 - x_2) \\ \alpha_2 &= K_2(x_2 - x_1) \end{aligned} \tag{77}$$

and allow x_1 to approach x_2. Substitution of (77) in (54) yields

$$\begin{aligned} D(z) = 1 &+ \frac{1}{2}\left(\frac{\alpha_1 - \alpha_2}{x_1 - x_2}\right)\frac{z^2 - x_1 x_2}{(z - x_1)(z - x_2)} \\ &+ \frac{\alpha_1 + \alpha_2}{2(z - x_1)(z - x_2)} + \frac{1}{2}\sum_{p=3}^{n} K_p \left(\frac{z + x_p}{z - x_p}\right). \end{aligned} \tag{78}$$

Equation (48) may be written as

$$\frac{\alpha_1 - \alpha_2}{x_1 - x_2} + \sum_{p=3}^{n} K_p = 0 \tag{79}$$

so that

$$L + \sum_{p=3}^{n} \lim_{x_1 \to x_2} K_p = 0 \tag{80}$$

where

$$L = \lim_{x_1 \to x_2}\left(\frac{\alpha_1 - \alpha_2}{x_1 - x_2}\right) = \lim_{x_1 \to x_2}(K_1 + K_2) \tag{81}$$

is a finite limit. Evidently, as x_1 approaches x_2,

$$\lim_{x_1 \to x_2} D(z) = 1 + \frac{L}{2}\left(\frac{z + x_2}{z - x_2}\right) + \frac{\alpha_2}{(z - x_2)^2} + \frac{1}{2}\sum_{p=3}^{n} K_p \left(\frac{z + x_p}{z - x_p}\right). \quad (82)$$

The zeros of this limiting form of $D(z)$ correspond to the characteristic exponents in this case. The parameter α_2 may be determined by factoring $1/(\lambda_2 - \lambda_1)$ from the appropriate row of K_2 and computing the resulting determinant Δ, since

$$\alpha_2 = \Delta \cdot \lim_{x_1 \to x_2} \left(\frac{x_2 - x_1}{\lambda_1 - \lambda_2}\right) = \Delta. \quad (83)$$

The parameter L required in (82) may be computed from (80). Cases where two poles of $d(\mu)$ are almost coincident may be treated in a similar fashion, except that no limits are involved.

APPENDIX C

Approximate Computation of Residue Determinants

In practical cases where the number of terms in the Fourier series for $B(t)$ is limited, truncated approximants to the residue determinants may be evaluated by techniques that exploit the special form of these determinants. The form of a truncated residue determinant is illustrated by the scheme in Fig. 2, in which all elements outside of the shaded region are zero. Except for one submatrix near the center of the array the principal diagonal blocks represent nonsingular triangular sub-

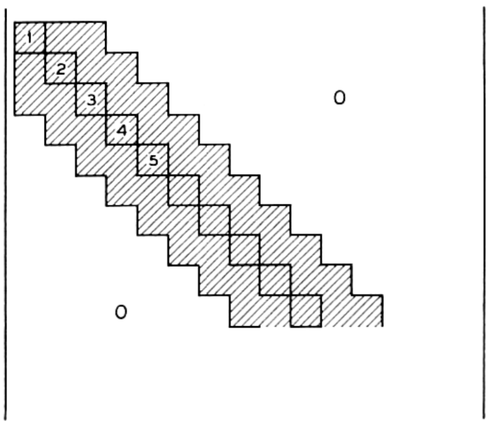

Fig. 2 — Form of truncated residue determinant.

matrices. To demonstrate the feasibility of computing truncated residue determinants of large order it will be shown that the computation time required for a reduction to triangular form is much smaller than for a general determinant of the same order. (The computation time required for a general determinant is asymptotically proportional to the cube of its order.)

To evaluate the determinant in Fig. 2 let zeros be produced below submatrix 1 by elementary operations with the rows passing through 1. Similar operations to produce zeros below 2 do not disturb the zeros already produced. Such operations may be continued in the usual manner to produce zeros below 3, 4, etc., until a triangular array of submatrices is realized. The number of arithmetic operations necessary in each step of zero production is essentially dependent only upon the order of the original system of equations and the number of terms in $B(t)$. Observation of Fig. 2 shows that the number of zero-producing steps for a truncated determinant of large order is asymptotically proportional to the order of the determinant. Thus, the computation time required for a reduction to triangular form is also asymptotically proportional to the order of the truncated determinant to be evaluated.

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