Generalized Confocal Resonator Theory

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(Manuscript received March 5, 1962)

The theory of the confocal resonator is extended to include the effect of unequal aperture size and unequal radii of curvature of the two reflectors. The latter is equivalent to a periodic sequence of lenses with unequal focal lengths. This treatment is in Cartesian coordinates as previously used. In an appendix the modes and resonant formulas are written in cylindrical coordinates.

The effect of unequal aperture size of the two reflectors is shown to produce mode patterns of unequal size on the two reflectors of a confocal resonator. The previous computations for diffraction losses are found to be applicable. Generalization of the theory to the case of reflectors of unequal curvature shows the existence of low-loss regions and high-loss regions as the reflector spacing is varied. One of the high diffraction loss regions occurs when the reflector spacing is between the two unequal radii of curvature. Such a region is interpretable in terms of instabilities in a periodic sequence of lenses of unequal focal length. An estimate of diffraction losses is obtained for the low-loss regions. The presence of a high diffraction loss region or unstable region should be of importance in the design of resonators or of a periodic sequence of lenses.

I. INTRODUCTION

The existence of modes in an open structure such as the confocal Fabry-Perot type resonator has been demonstrated by Boyd and Gordon[1] and by Fox and Li.[2] This resonator consists of two spherical reflectors separated by their common radius of curvature, as shown in Fig. 1. The reflectors were assumed to be of equal aperture and square[1] or circular[2] if viewed in the z-direction. Uniform reflectivity over the reflecting surface was postulated. Goubau and Schwering[3, 4] have also reported on this problem and have obtained similar results.

A mode may be defined as a field distribution that reproduces itself in spatial distribution and phase, though not in amplitude, as the wave bounces back and forth between the two reflectors. Because of losses

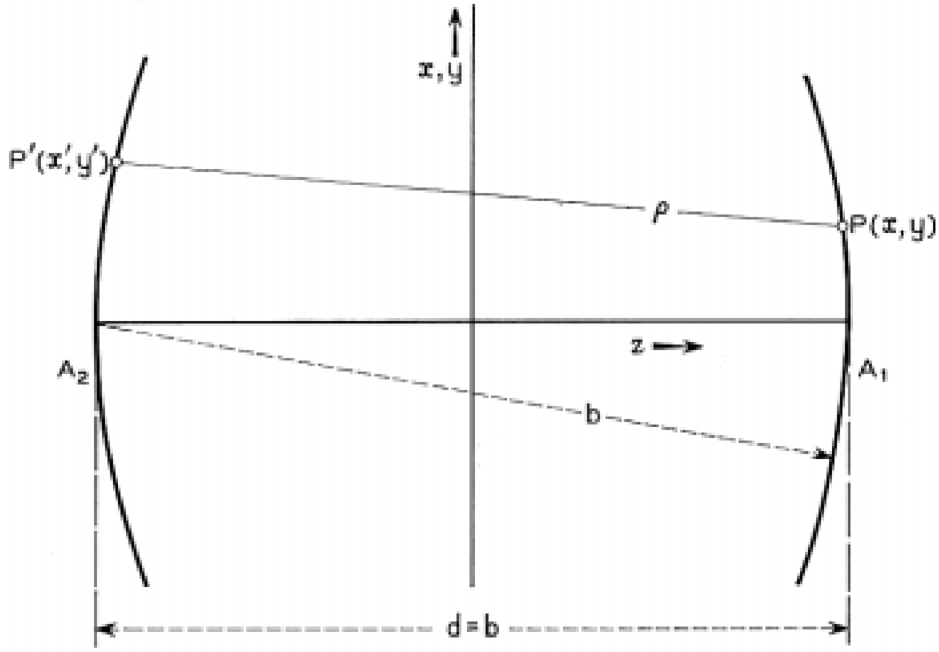

Fig. 1 — Confocal resonator with spherical reflectors.

due to diffraction and reflection, the reproduced pattern is reduced in intensity on each succeeding traversal of the resonator. The above-mentioned authors have shown that there is a set of modes which will reproduce themselves over the equal apertures A_1 and A_2 of the resonator.

Mathematically the modes of the confocal resonator form a complete orthogonal set of functions. For the confocal resonator these modes are highly degenerate in frequency; that is, many modes have the same resonance frequency. The degeneracy is split when the resonator is made nonconfocal by varying the plate spacing, though new degeneracies do appear at certain other spacings. Because of this frequency degeneracy, the modes of the confocal resonator are not unique unless the effects of loss are considered. Any linear combination of the degenerate frequency modes (the Hermite-Gaussian functions described in Ref. 1) is still a mode of the resonator.

When one includes the effect of diffraction losses due to finite apertures, the modes become unique, for then the eigenvalue degeneracy is split and each mode has its own characteristic rate of decay or Q. For the case of low diffraction losses, the eigenfunctions of the modes are still given with good approximation by the Hermite-Gaussian functions, which are exact only for the lossless case of infinite apertures. The frequency degeneracy is unaffected by the inclusion of diffraction losses.

Boyd and Gordon, in including diffraction losses, considered only the case where both reflectors, A_1 and A_2, are of *equal size*. This imposes a certain symmetry on the system. If, however, the two reflectors are of different sizes, one might expect for the confocal resonator, with its high frequency degeneracy, stationary field configurations that are *asymmetric* in the z-direction. This may be understood by considering the set of degenerate Hermite-Gaussian modes which are resonant at a frequency given by $2q + m + n$ equal a constant. Combinations of these modes may be superimposed at one reflector to form various new field patterns. The field patterns on the two reflectors can now be different since the original modes with even and odd q change their relative phase by 180° in going from one reflector to the other. It is reasonable that the lowest-loss mode for an unequal aperture resonator will be such a combination that the field patterns will be asymmetrical. This also turns out to be true for all higher-order modes.

For the case of the nonconfocal resonator with spacing such that there are no frequency degeneracies, this asymmetry in the stationary field configurations is not possible. The field distribution is forced to be symmetrical between the two reflectors. The diffraction losses are then determined mainly by the smaller of the reflectors.

Resonators with reflectors of *different* radii of curvature are investigated also. A region of high diffraction loss is found for a range of separation of reflectors with unequal curvature near the confocal separation. This has some practical significance for resonators and transmission systems in that one must be sure to operate in only the low-loss region. Due to the possibility of slightly unequal radii of curvature in the fabrication of resonators, it is desirable to space the reflectors to obtain a nonconfocal condition. The existence of "stable regions" of low loss and "unstable regions" of high loss as the reflector spacing is varied is interpretable in terms of the stable and unstable regions of a periodic sequence of lenses of unequal focal length.

II. MODES IN A LOSSLESS CONFOCAL RESONATOR

It was pointed out in the introduction that the modes of the lossless confocal resonator are highly degenerate in frequency and thus not unique. Boyd and Gordon, in describing the modes of the lossless confocal resonator in terms of Hermite-Gaussian functions, considered only the symmetrical situation of identical field patterns and spot sizes over each aperture. Because of the high degeneracy of the lossless resonator, asymmetric field patterns between the two reflectors are just as possible. In this section the relation between asymmetric spot sizes is obtained. Only the introduction in the following section of unequal apertures and the resulting diffraction losses will allow one to state which combination of asymmetric spot sizes is a unique mode of the system.

Boyd and Gordon have computed surfaces of constant phase within and without the confocal resonator. These surfaces have approximately a spherical shape. Any of these surfaces may be replaced by spherical reflectors to form a new resonating structure of arbitrary spacing and curvature. Except for the obvious special case, such a resonator was termed nonconfocal. Boyd and Gordon have shown that each confocal system of radius of curvature and separation equal to b generates a set of surfaces of constant phase of radius b' and separation d linked by the relation*

$$d^2 - 2db' + b^2 = 0. \tag{1}$$

For a given b and b' there are two possible reflector separations, d_1 and d_2 :

$$d_1 = b' + \sqrt{b'^2 - b^2},$$

and (2)

$$d_2 = b' - \sqrt{b'^2 - b^2}.$$

The field distribution of the modes of these nonconfocal systems is *symmetric* with respect to the system center (as are the fields of the generating confocal system). The fundamental modes of all these nonconfocal systems have a spot size of radius w_o at the center of the resonator, given by

$$w_o = \sqrt{\frac{b\lambda}{2\pi}}, \tag{3}$$

where λ is the wavelength. The spot size of the fundamental mode of the confocal system at the reflectors is $w_s = w_o\sqrt{2}$. In general, the spot size at a distance $d/2$ from the center is given by

$$w_s' = w_o \sqrt{1 + \frac{d^2}{b^2}}. \tag{4}$$

The surface of constant phase at the center ($z = 0$) is a plane, and the whole family of surfaces is symmetric with respect to it. So far we

* The notation used here is consistent with that of Boyd and Gordon but unfortunately not with that of Fox and Li, who use b for the spacing of the plane parallel resonator. We use d for the spacing of the reflectors, b for the confocal radius of curvature and thus its spacing, and b' for the radius of curvature of a surface of constant phase and for the radius of curvature of the reflectors of a nonconfocal resonator with identically curved reflectors. For the nonconfocal resonator with unequal radii of curvature we use b_1 and b_2 .

have arranged the reflector pair symmetrically to this plane, and symmetric mode configurations have resulted. It is possible, however, to construct a field configuration which is *asymmetric with respect to the reflector system* by placing one of the reflectors (with radius of curvature b') at $z = d_1/2$ and the other at $z = -d_2/2$. At these locations both surfaces of constant phase have the same radius of curvature b'. This is indicated in Fig. 2(a). The resulting reflector separation can be computed from (2) as

$$d = \tfrac{1}{2}(d_1 + d_2) = b'. \qquad (5)$$

Since the spacing d equals the reflector radius of curvature b', it is apparent that a new confocal system has been formed. Apart from the surfaces of constant phase, consideration of spot sizes assures us that the

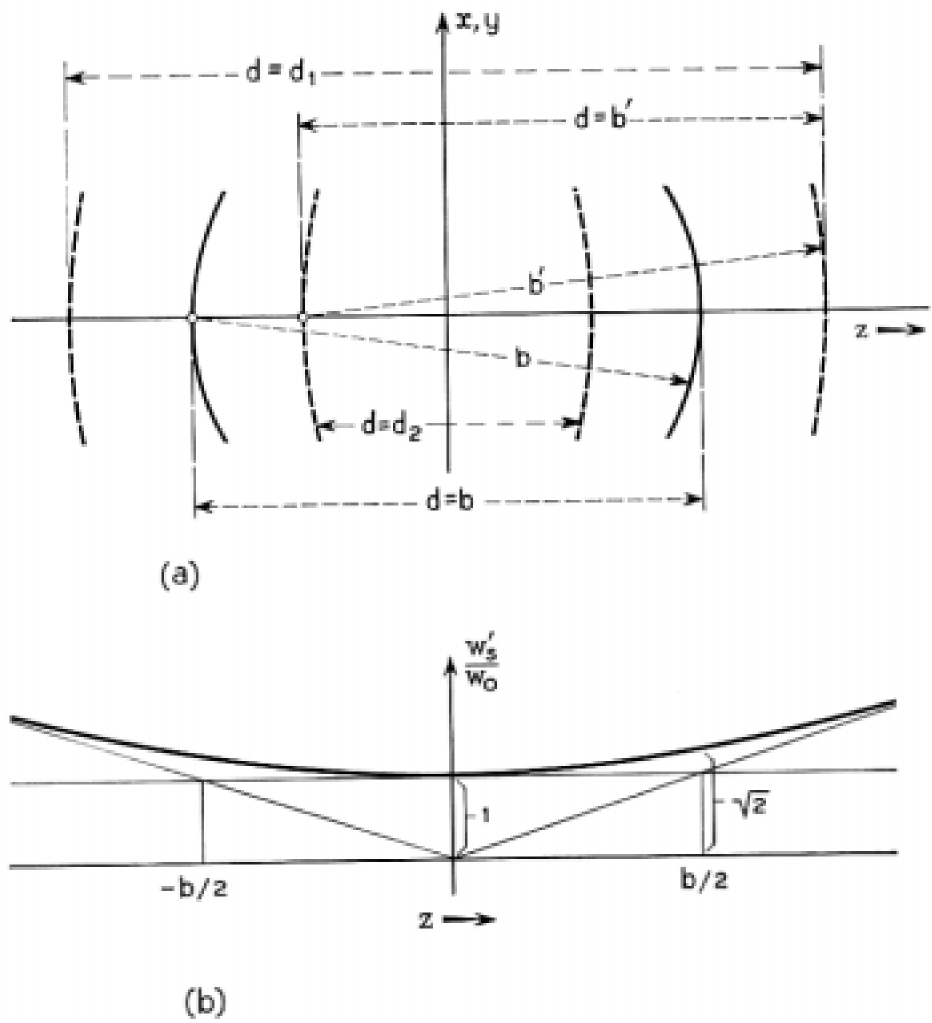

Fig. 2 — (a) Surfaces of constant phase including asymmetric confocal systems; (b) relative spot size $\frac{w_s'}{w_o} = \sqrt{1 + \frac{d^2}{b^2}}$.

modes obtained for this confocal system are, indeed, asymmetric. As indicated in Fig. 2(b) the spot size reaches its minimum value of w_o at $z = 0$ and is not by any means in the center of the reflector system. The spot sizes at the new reflectors can be computed from (4) as

$$w_1 = w_o \sqrt{1 + \frac{d_1^2}{b^2}}, \qquad w_2 = w_o \sqrt{1 + \frac{d_2^2}{b^2}}. \tag{6}$$

This combined with (2) yields the relation

$$w_1 w_2 = \frac{\lambda b'}{\pi}, \tag{7}$$

where $\sqrt{\lambda b'/\pi}$ is the spot size at the reflectors, which we would expect for the symmetrical set of modes of a confocal system of spacing b'.

The resonance condition for this system is obtained via Boyd and Gordon's equation (20) after some computation as

$$\frac{4b'}{\lambda} = 2q + (1 + m + n), \tag{8}$$

where m, n, and q are the mode numbers as defined in Boyd and Gordon's work. By comparing their equation (14) with our result, we find that the resonance conditions for the symmetric and the asymmetric modes of the confocal system are identical, as expected.

By suitably choosing b for a given reflector curvature b', almost any ratio of reflector spot sizes w_1/w_2 can be obtained. Thus, for a given lossless confocal system, the confocal geometry allows an infinite number of sets of modes (characterized by the spot sizes at each aperture). It is the *finite* size and shape of the reflector that selects one particular set, as we shall see in the following section.

III. MODES OF A CONFOCAL RESONATOR WITH REFLECTOR SIZES UNEQUAL

Consider a confocal resonator. Assume that the reflectors A_1 and A_2 are, in general, of different sizes and/or shapes. With this asymmetry in mind, it is no longer reasonable to postulate that the field pattern on A_1 be reproduced on A_2 when looking for self-consistent field configurations. Instead, as a more generalized definition of a mode let us require that an energy distribution launched with a certain pattern on A_1 reproduce this pattern *on* A_1 after bouncing back from A_2. No condition on the pattern on A_2 is imposed.

To express this mathematically we use the approximations of Boyd and Gordon's paper and the scalar formulation of Huygens' principle. A wave leaving reflector A_1 with a field pattern $E(x, y)$ arrives at A_2 with a pattern $E'(x', y')$ given by

$$E'(x', y') = \frac{ik}{2\pi b} \int_{A_1} d\bar{x}\, d\bar{y}\, E(\bar{x}, \bar{y}) e^{-ik\rho}, \tag{9}$$

where b is the mirror separation, $k = 2\pi/\lambda$, and

$$\rho = b - \frac{1}{b} (\bar{x}x' + \bar{y}y'). \tag{10}$$

Most of the energy is reflected from A_2 and travels back to A_1. The radii of curvature of the reflectors are assumed very large compared to the wavelength λ, and we can therefore assume that laws for the reflection of plane waves apply locally. Then we find that the reflected wave leaves A_2 with the pattern $-E'(x', y')$. It will arrive at A_1 with a certain distribution pattern which we shall call $-\sigma_m^2 \sigma_n^2 E(x, y)$. At this point we have introduced the postulate that the field patterns be reproduced, except for the amplitude factor $\sigma_m^2 \sigma_n^2$, after one complete return trip. Again, the energy is bounced back and leaves reflector A_1 with a field distribution which can be expressed in terms of $E'(x', y')$ as

$$\sigma_m^2 \sigma_n^2 E(x, y) = \frac{ik}{2\pi b} \int_{A_2} dx'\, dy'\, E'(x', y') e^{-ik\rho'}, \tag{11}$$

with

$$\rho' = b - \frac{1}{b} (xx' + yy'). \tag{12}$$

Substituting (9) into (11) to eliminate E', then inserting the expressions (10) and (12) for ρ and ρ', and, finally, interchanging integrals, one obtains an integral equation

$$\sigma_m^2 \sigma_n^2 E(x, y) = -\frac{k^2 e^{-2ikb}}{4\pi^2 b^2} \int_{A_1} d\bar{x}\, d\bar{y}\, E(\bar{x}, \bar{y}) K(x, \bar{x}; y, \bar{y}), \tag{13}$$

with the kernel

$$K(x, \bar{x}; y, \bar{y}) = \int_{A_2} dx'\, dy' \exp\left(i \frac{k}{b} [x'(x + \bar{x}) + y'(y + \bar{y})] \right). \tag{14}$$

This is the fundamental integral equation that yields as its solution the modes of our system and their diffraction losses. The kernel $K(x, \bar{x};$

$y, \bar{y}$) depends on the shape and size of reflector A_2 and will be evaluated for some special cases in the following sections.

Integral equations for the modes of asymmetric nonconfocal systems can be derived on the basis of arguments similar to the ones used in this chapter, but solutions for them are in general not available.

IV. CONFOCAL RESONATORS WITH UNEQUAL SQUARE AND RECTANGULAR APERTURES

In this section a confocal resonator with two reflectors of finite but *unequal* size is considered. Let reflector A_1 extend from $-a_1$ to $+a_1$ in the x direction and from $-A_1$ to $+A_1$ in the y direction as shown in Fig. 3(a). Reflector A_2 is chosen to be of a rectangular shape $2a_2$ by $2A_2$ correspondingly.

For the above reflector dimensions, the kernel of integral equation (13) takes the form

$$K(x, \bar{x}; y, \bar{y}) = \int_{-a_2}^{a_2} dx' \int_{-A_2}^{A_2} dy' \exp\left(i \frac{k}{b} [x'(x + \bar{x}) + y'(y + \bar{y})]\right). \quad (15)$$

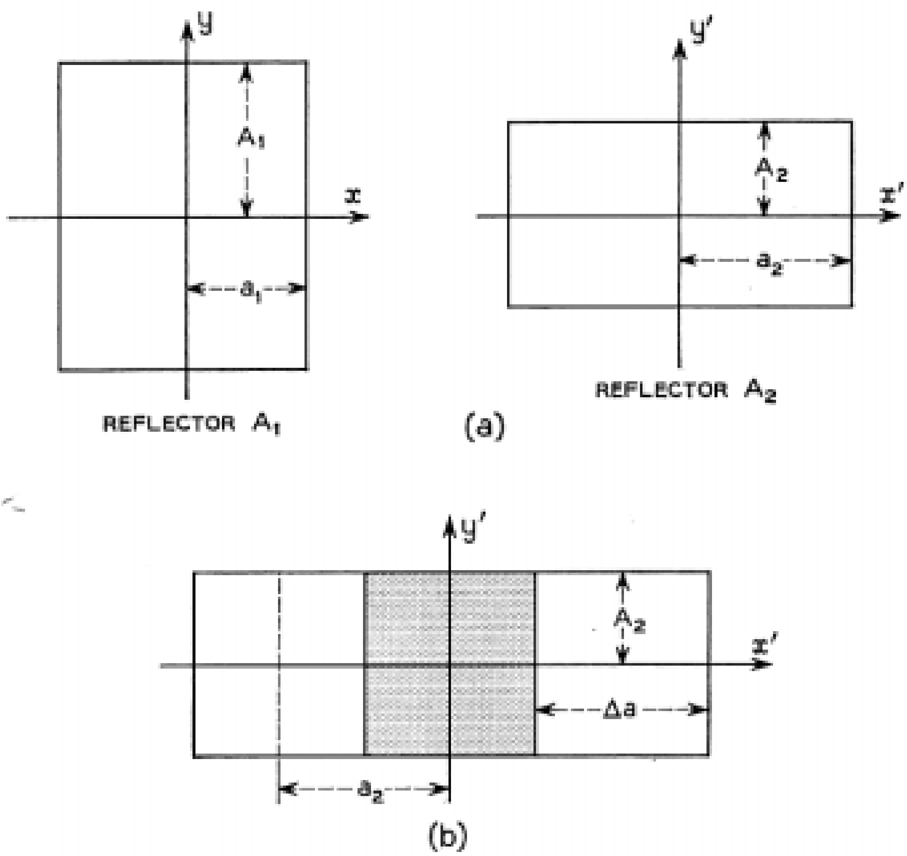

Fig. 3 — (a) Reflectors with rectangular aperture; (b) reflector A_2 blocked in center.

These integrals can be evaluated analytically and the kernel rewritten as

$$K(x, \bar{x}; y, \bar{y}) = \frac{4b^2}{k^2} \cdot \frac{\sin \frac{k}{b} a_2(x + \bar{x})}{(x + \bar{x})} \cdot \frac{\sin \frac{k}{b} A_2(y + \bar{y})}{(y + \bar{y})}. \tag{16}$$

Assume that the field pattern of a mode can be written in the form

$$E(x, y) = E_o f_m(x) g_n(y), \tag{17}$$

with E_o a constant amplitude factor, $f(x)$ a function of x only, and $g(y)$ a function of y only. Under these conditions integral equation (13) can be rearranged as

$$\begin{aligned} \sigma_m^2 \sigma_n^2 f_m(x) g_n(y) = -e^{-2ikb} \int_{-a_1}^{+a_1} d\bar{x}\, f_m(\bar{x}) \frac{\sin \frac{k}{b} a_2(x + \bar{x})}{\pi(x + \bar{x})} \\ \cdot \int_{-A1}^{A1} d\bar{y}\, g_n(\bar{y}) \frac{\sin \frac{k}{b} A_2(y + \bar{y})}{\pi(y + \bar{y})}. \end{aligned} \tag{18}$$

An integral relation satisfied by the angular prolate spheroidal wave functions $S_{0n}(c, s)$ is[5]

$$\frac{2c}{\pi} [R_{0n}^{(1)}(c, 1)]^2 S_{0n}(c, t) = \int_{-1}^{+1} ds \frac{\sin c(t - s)}{\pi(t - s)} S_{0n}(c, s), \tag{19}$$

where $R_{0n}^{(1)}(c, 1)$ is the radial prolate spheroidal wave function of nth order. Because

$$S_{0n}(c, s) = (-1)^n S_{0n}(c, -s), \tag{20}$$

it holds that

$$\frac{2c}{\pi} [R_{0n}^{(1)}(c, 1)]^2 S_{0n}(c, t) = (-1)^n \int_{-1}^{+1} ds \frac{\sin c(t + s)}{\pi(t + s)} S_{0n}(c, s). \tag{21}$$

We can compare this relation with integral equation (18) (with $c = (k/b)a_1a_2$ and $C = (k/b)A_1A_2$) and conclude that the field patterns of the TEM_{mnq} mode of the system are

$$\begin{aligned} &\text{on } A_1: \quad E(x, y) \propto S_{0m}\left(c, \frac{x}{a_1}\right) S_{0n}\left(C, \frac{y}{A_1}\right); \\ &\text{on } A_2: \quad E'(x', y') \propto S_{0m}\left(c, \frac{x'}{a_2}\right) S_{0n}\left(C, \frac{y'}{A_2}\right). \end{aligned} \tag{22}$$

As pointed out in Boyd and Gordon's work, the prolate spheroidal wave functions S_{0m} can be approximated by orthogonal Hermite functions if the apertures are large enough. This enables one to derive an approximate formula for the dimensions of the "spot" of the fundamental, which indicates the location where the field of this mode has decreased by a factor e^{-1} with respect to its maximum. For rectangular reflectors these quantities will, in general, be different for the x and the y directions. One obtains

$$\begin{aligned}
&\text{on reflector } A_1: \quad x_s = \sqrt{\frac{a_1}{a_2}}\sqrt{\frac{b\lambda}{\pi}}; \qquad y_s = \sqrt{\frac{A_1}{A_2}}\sqrt{\frac{b\lambda}{\pi}};\\
&\text{on reflector } A_2: \quad x_s' = \sqrt{\frac{a_2}{a_1}}\sqrt{\frac{b\lambda}{\pi}}; \qquad y_s' = \sqrt{\frac{A_2}{A_1}}\sqrt{\frac{b\lambda}{\pi}}.
\end{aligned} \tag{23}$$

Note that

$$x_s x_s' = y_s y_s' = w_s^2 = \frac{b\lambda}{\pi}, \tag{24}$$

which agrees with (7). From the above we see that, compared to a confocal resonator with equal apertures, the patterns of all modes on reflector A_1 are now magnified by a factor $\sqrt{a_1/a_2}$ in x direction and a factor $\sqrt{A_1/A_2}$ in y direction, if $a_1 > a_2$ and $A_1 > A_2$. The patterns on reflector A_2 are compressed correspondingly.

The center of the reflector system is no longer the position of maximum energy density. The position of maximum energy density will, in general, be different for the concentration in the x direction as compared to the concentration in the y direction. One computes displacements D_x and D_y of the positions of maximum concentration from the center in the direction of the smaller reflector:

$$D_x = \frac{b}{2}\cdot\frac{a_2^2 - a_1^2}{a_2^2 + a_1^2}; \qquad D_y = \frac{b}{2}\cdot\frac{A_2^2 - A_1^2}{A_2^2 + A_1^2}. \tag{25}$$

Comparison of (18) and (21) also yields the eigenvalues of the integral equation (20):

$$\sigma_m^2 \sigma_n^2 = -(-1)^{n+m} e^{-2ikb} \frac{4cC}{\pi^2} [R_{0m}^{(1)}(c, 1) \cdot R_{0n}^{(1)}(C, 1)]^2. \tag{26}$$

This shows that

(*i*) the resonance condition

$$\frac{4b}{\lambda} = 2q + (1 + m + n) \tag{27}$$

is not changed by making the apertures of the reflectors of a confocal system unequal, and

(ii) the diffraction losses of a confocal system with *unequal* reflector apertures of dimensions a_1, A_1, a_2, and A_2 are equal to the diffraction losses of a confocal system with *equal* aperture dimensions a_o, A_o if $a_o^2 = a_1a_2$ and $A_o^2 = A_1A_2$.

V. CONFOCAL RESONATOR WITH ONE REFLECTOR PARTIALLY BLOCKED

In this section we would like to quickly sketch the analytical treatment of a confocal reflector system in which one reflector — in our case A_2 — is blacked out in the center as shown in Fig. 3(b). The other reflector dimensions are assumed to be the same as in the previous section. The effective shape of reflector A_2 is now that of two rectangles of width Δa, extending from $x' = -a_2 - (\Delta a/2)$ to $-a_2 + (\Delta a/2)$, and from $x' = a_2 - (\Delta a/2)$ to $a_2 + (\Delta a/2)$. If we insert the corresponding limits into (14), we obtain the kernel

$$K(x,\bar{x};y,\bar{y}) = \frac{8b^2}{k^2(x+\bar{x})(y+\bar{y})}\sin\left[\frac{k\Delta a}{2b}(x+\bar{x})\right] \cdot\cos\left[\frac{ka_2}{b}(x+\bar{x})\right]\sin\left[\frac{k}{b}A_2(y+\bar{y})\right] \tag{28}$$

for the integral equation describing the system.

For very small reflector width $\Delta a \ll b\lambda/a$, the kernel is given with good approximation by

$$K(x,\bar{x};y,\bar{y}) = \frac{4b\Delta a}{k(y+\bar{y})}\cos\frac{ka_2}{b}(x+\bar{x})\sin\frac{kA_2}{b}(y+\bar{y}). \tag{29}$$

With this kernel, integral equation (13) can be separated into one equation containing functions of x and one containing functions of y only, as in the previous section. While the latter is the same as the integral equation treated in Section IV, the equation for $f(x)$ is of the form

$$\gamma f(x) = 2\Delta a\int_{-a_1}^{+a_1} d\bar{x}\, f(\bar{x})\cos\frac{ka_2}{b}(x+\bar{x}). \tag{30}$$

Applying standard procedures, this integral equation can be solved elementarily. The solutions are

$$f(x) = \cos\frac{ka_2}{b}x, \tag{31}$$

with the eigenvalue

$$\gamma = 2\Delta a \left(a_1 + \frac{b}{2ka_2} \sin 2 \frac{k}{b} a_1 a_2 \right), \tag{32}$$

and

$$f(x) = \sin \frac{ka_2}{b} x, \tag{33}$$

with the corresponding eigenvalue

$$\gamma = -2\Delta a \left(a_1 - \frac{b}{2ka_2} \sin \frac{2k}{b} a_1 a_2 \right). \tag{34}$$

The eigenvalues, of course, determine the diffraction losses and the resonance conditions for the reproducing patterns, as in the previous section. The resonance formula is

$$\frac{4d}{\lambda} = 2q + 1 + n \tag{35}$$

for the even cosine-function, and

$$\frac{4d}{\lambda} = 2q + 2 + n \tag{36}$$

for the odd sine-function.

One should note that the field distribution on reflector A_1 is simply the two-slit diffraction pattern one would expect from coherent excitation of the two narrow reflectors comprising A_2.

VI. RESONATORS WITH REFLECTORS OF UNEQUAL CURVATURE

To investigate resonator systems with concave reflectors of unequal radii of curvature, let us return to the consideration of a lossless system. From this model one can obtain information on spot sizes and resonance conditions. Diffraction losses will be estimated using the same approximation previously used by Boyd and Gordon for the nonconfocal resonator of equal curvature.

6.1 *Surfaces of Constant Phase*

Let reflector A_1 have a radius of curvature b_1, and reflector A_2 a radius of b_2. We shall base our argument on Boyd and Gordon's picture of surfaces of constant phase (Fig. 2), which we have already used in

Section II. In a set of surfaces, characterized by the confocal parameter b, reflector A_1 can be placed at the distances $\pm d_1/2$ from the center, and A_2 at $\pm d_2/2$ correspondingly. These distances can be computed from (2) as

$$d_1 = b_1 \pm \sqrt{b_1^2 - b^2}, \quad d_2 = b_2 \pm \sqrt{b_2^2 - b^2}. \tag{37}$$

With given concave reflectors, therefore, four different resonator systems can be found which fit this particular set of surfaces of constant phase. The four different reflector separations $d = \frac{1}{2}(d_1 + d_2)$ are given by

$$2d = b_1 + b_2 \pm \sqrt{b_1^2 - b^2} \pm \sqrt{b_2^2 - b^2}. \tag{38}$$

To obtain various other resonator systems the parameter b can be varied. But the range of this variation is restricted, since only real valued distances have physical meaning in this context. If we assume that $b_2 > b_1$, it follows from (38) that b can be varied in the range from 0 to b_1. One can thus obtain reflector separations d in the range from 0 to b_1 and from b_2 to $b_1 + b_2$ as shown in Fig. 4. No information on resonators with reflector separations in the range from b_1 to b_2 can be obtained. It is of interest that the confocal system for reflectors of unequal curvature, with $d = \frac{1}{2}(b_1 + b_2)$, is just in this "unstable region."

Let us restrict our discussion to systems of given b_1, b_2, and d in the range covered by the picture of surfaces of constant phase. We can

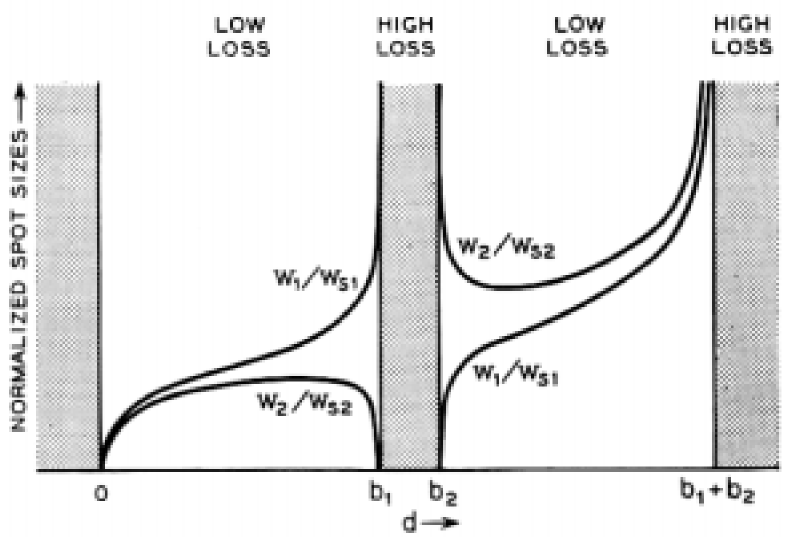

Fig. 4 — Spot sizes and high- and low-loss regions for a resonator with reflectors of unequal curvature and variable spacing.

inquire into the spot sizes w_1 on A_1 and w_2 on A_2. Combining (1), (4), (37), and (38) we obtain the relations

$$\left(\frac{w_1}{w_2}\right)^2 = \frac{b_1}{b_2} \cdot \frac{b_2 - d}{b_1 - d}, \tag{39}$$

$$(w_1 w_2)^2 = \left(\frac{\lambda}{\pi}\right)^2 \frac{b_1 b_2 d}{b_1 + b_2 - d}. \tag{40}$$

It also follows that the maximum concentration of energy occurs at the distance

$$\frac{d_1}{2} = d \frac{b_2 - d}{b_1 + b_2 - 2d} \tag{41}$$

from reflector A_1, and at

$$\frac{d_2}{2} = d \frac{b_1 - d}{b_1 + b_2 - 2d} \tag{42}$$

from A_2.

In Fig. 4 we have shown for a special case how the spot sizes w_1 and w_2 vary as a function of the reflector spacing d. In this figure the spot size on each reflector is normalized in terms of $w_{s1} = \sqrt{b_1 \lambda / \pi}$ and $w_{s2} = \sqrt{b_2 \lambda / \pi}$. These are the spot sizes at the reflectors of equal-radii confocal resonators with radii of b_1 and b_2 respectively.

Note that as d approaches b_1, the spot size w_2 on A_2 approaches zero, while the spot size w_1 on A_1 increases beyond limit. The corresponding effect occurs if d approaches b_2 from above. It should be remembered that the information obtained here can be applied usefully only so long as the spot sizes are somewhat smaller than the corresponding reflector dimensions.

The diffraction losses of the nonconfocal resonator of equal radii of curvature and aperture were previously estimated by Boyd and Gordon on the assumption that the diffraction loss is equal to that of its equivalent confocal resonator with reflector dimensions scaled up by the ratio of their spot sizes.

For the nonconfocal resonator of unequal radii of curvature and square apertures of sides $2a_1$ and $2a_2$ respectively, the equivalent Fresnel numbers at reflectors A_1 and A_2, which determine the diffraction losses at each reflector, are obtained from Boyd and Gordon's equation (29) as

$$\begin{aligned} \left(\frac{a^2}{b\lambda}\right)_1 &= \frac{a_1^2}{d_1 \lambda} \left[2 \frac{d_1}{b_1} - \left(\frac{d_1}{b_1}\right)^2\right]^{\frac{1}{2}}, \\ \left(\frac{a^2}{b\lambda}\right)_2 &= \frac{a_2^2}{d_2 \lambda} \left[2 \frac{d_2}{b_2} - \left(\frac{d_2}{b_2}\right)^2\right]^{\frac{1}{2}}, \end{aligned} \tag{43}$$

where d_1 and d_2 are determined by (41) and (42). The diffraction loss at each reflector is then obtainable from Fig. 3 of Boyd and Gordon as α_{D1} and α_{D2}.

The resonator Q is given by

$$Q = \frac{2\pi d}{\alpha\lambda}, \tag{44}$$

where

$$\alpha = \tfrac{1}{2}(\alpha_{D1} + \alpha_{D2}) + \alpha_R \tag{45}$$

and α_R represents the reflection loss per bounce at a reflector plus the single-pass scattering and absorption loss between the reflectors.

On the basis of this estimate, one concludes that the diffraction losses increase sharply if the separation d approaches an "unstable" region. No similar estimate of diffraction losses is available for the "unstable" regions. However, a ray optical analysis which we present in the next section shows the divergent nature of "unstable" resonator systems. This indicates relatively high diffraction losses.

With results obtained here and the help of Boyd and Gordon's equation (20), the resonance condition for the resonator with reflectors of different curvature can be computed as

$$\frac{2d}{\lambda} = q + \frac{1}{\pi}(1 + m + n)\cos^{-1}\sqrt{\left(1 - \frac{d}{b_1}\right)\left(1 - \frac{d}{b_2}\right)}. \tag{46}$$

To compare this with Boyd and Gordon's resonance formula for resonators with equal curvature b', we rewrite their equation (31) in terms of d and b':

$$\frac{2d}{\lambda} = q + \frac{1}{\pi}(1 + m + n)\cos^{-1}\left(1 - \frac{d}{b'}\right). \tag{47}$$

We have found this to be a very convenient form in which to rewrite their resonance formula (31). But due to well known relations between the trigonometric functions, various other formulations are possible. One of these formulations was given by J. R. Pierce.[6]

A half nonconfocal resonator may be formed by a plane reflector and a spherical reflector of radius of curvature b_1 and spacing d between the reflectors with $d < b_1 < \infty$. The resonant condition may be obtained from (46) by letting $b_2 \rightarrow \infty$. The result is given by

$$\frac{2d}{\lambda} = q + \frac{1}{2\pi}(1 + m + n)\cos^{-1}\left(1 - \frac{2d}{b_1}\right). \tag{48}$$

6.2 *Equivalent Sequence of Lenses*

Let us call the regions $b_1 + b_2 > d > b_2$ and $0 < d < b_1$ "stable" or "low loss," and the regions $d > b_1 + b_2$ and $b_1 < d < b_2$ "unstable" or "high loss." We can understand these stable and unstable regions of a resonator system with reflectors of unequal curvature from another point of view if we replace the resonator by an equivalent sequence of lenses. These lenses are spaced at distances d and have focal lengths of $f_1 = b_1/2$ and $f_2 = b_2/2$ respectively. Lens systems of this type have been used in periodic focusing of long electron beams and instabilities have been observed.

Stability investigations of sequences of lenses of equal focal length are readily available.[7] These systems are stable if

$$0 < \frac{L}{f} < 4, \tag{49}$$

where L is the lens spacing and f the focal length.

A pair of lenses of focal lengths f_1 and f_2 spaced at the distance d can be replaced by an equivalent optical system[8] of a focal length f given by

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}. \tag{50}$$

The system's principal planes are found to be spaced at distances $h_1 = d(f/f_2)$ and $h_2 = d(f/f_1)$ from the corresponding lenses (see Fig. 5).

If we substitute such a thick lens for each pair of unequal lenses of our system, we obtain a sequence of equal optical systems of focal length f. If, furthermore, we define as their "effective" spacing

$$L = d + h_1 + h_2, \tag{51}$$

the arguments of Pierce's treatment[7] are applicable to our case.

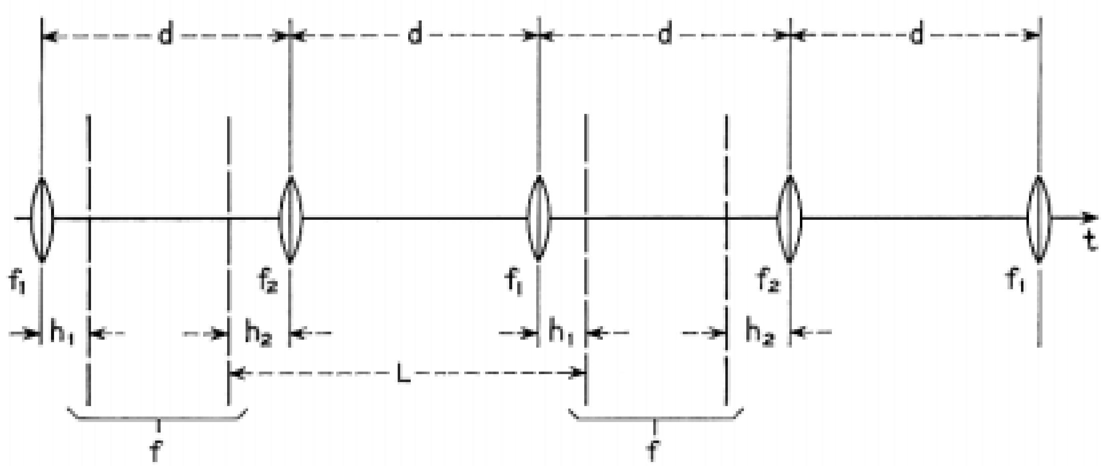

Fig. 5 — Sequence of lenses of alternating focal length

Combining equations we obtain

$$\frac{L}{f} = d\left\{\frac{2}{f_1} + \frac{2}{f_2} - \frac{d}{f_1 f_2}\right\}. \tag{52}$$

From (49) and (52) one can show that the boundaries of the stable regions are given by

$$\left(\frac{d}{b_1} - 1\right)\left(\frac{d}{b_2} - 1\right) \leqq 1, \tag{53}$$

and

$$\left(\frac{d}{b_1} - 1\right)\left(\frac{d}{b_2} - 1\right) \geqq 0. \tag{54}$$

These relations define the stable and unstable regions in agreement with the preceding discussion, but they are also valid for negative values of b_1 and b_2. If one allows for convex reflectors, one can also obtain this somewhat generalized result from the picture of surfaces of constant phase.

6.3 *Stability Diagram*

A. G. Fox and T. Li have suggested a two-dimensional diagram of the stable and unstable regions which is very instructive. Several choices of coordinates are possible. In Fig. 6 we have plotted d/b_1 and d/b_2 as coordinates. In this diagram the boundary lines described by (54) appear as straight lines, and the curve represented by (53) as a hyperbola, as shown. For confocal systems, i.e., systems with coinciding reflector foci, we have $2d = b_1 + b_2$, which may be written:

$$\left(\frac{d}{b_1} - \frac{1}{2}\right)\left(\frac{d}{b_2} - \frac{1}{2}\right) = \frac{1}{4}. \tag{55}$$

In our diagram, therefore, these systems are represented by points on another hyperbola and fall within the high-loss region. A transition from a "stable" to an "unstable" region means an extremely sharp increase of diffraction losses for reasonably large Fresnel numbers.

The confocal system with reflectors of equal curvature is represented by a rather singular point in our diagram. We see that certain deviations from the ideal dimensions $d = b_1 = b_2$ will greatly increase the system losses. This should be taken into account when designing maser resonators or optical transmission systems, and it may be advisable to choose points of operation at a safe distance from the unstable region. The degenerate frequency characteristics of the confocal system can be

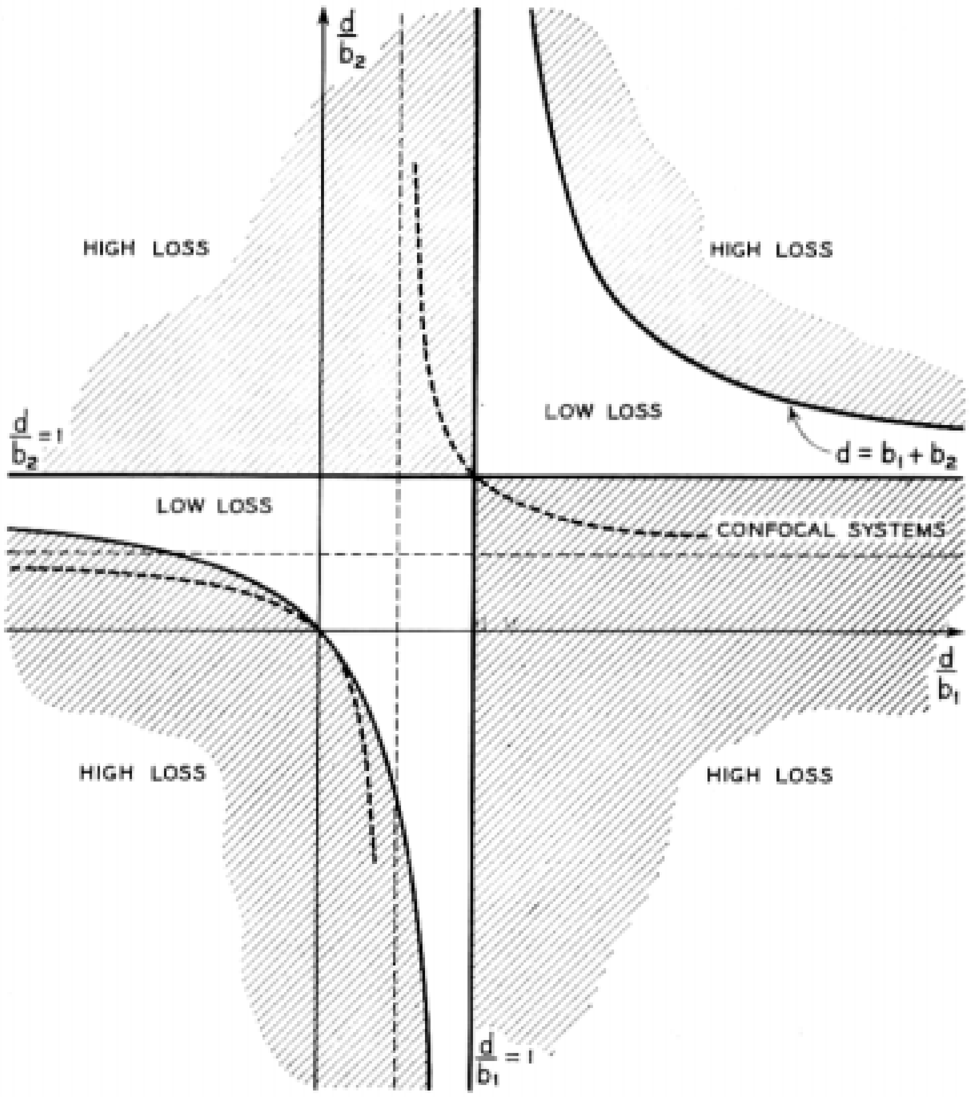

Fig. 6 — Two-dimensional diagram of stable and unstable regions.

obtained, if desired, by using a concave reflector ($b_1 = 2d$) and a flat one ($b_2 = \infty$) spaced at the distance $d = b_1/2$. This system is in a stable region.

6.4 *Equivalent Systems*

One may ask whether operation in one of the two low-loss regions is to be preferred to the other. We have found that the two regions are absolutely equivalent as far as diffraction losses, spot sizes, and resonance conditions are concerned. For a given reflector spacing d, we can find a corresponding pair of resonator systems, one operating in the lower stable region with reflector radii b_{1L} and $b_{2L} < d$, and one in the upper region with reflector radii b_{1U} and $v_{2U} > d$. Spot sizes, diffraction losses, and frequency response of both systems are the same if the conditions

$$\frac{1}{b_{1L}} + \frac{1}{b_{1U}} = \frac{2}{d}$$

and (56)

$$\frac{1}{b_{2L}} + \frac{1}{b_{2U}} = \frac{2}{d}$$

hold. In this case the mode patterns of one resonator correspond to the complex conjugates of the mode patterns of the other.

Some of these conclusions can be drawn on the basis of (39), (40), and (46) alone. To prove the correspondence of diffraction losses and mode patterns, however, the integral equations of the resonator systems have to be investigated using a procedure suggested by T. Li for the analysis of nonconfocal resonators with equal reflectors. Examination of the integral equations shows that unstable systems also are equivalent under the conditions given above.

VII. SUMMARY

The effects of unequal reflector apertures on the modes of a confocal resonator have been discussed. It was found that unequal apertures at the two reflectors have a large effect in determining the mode patterns. The resonant condition, however, is not changed by an asymmetry of this kind. Boyd and Gordon's picture of surfaces of constant phase does not contain *nonconfocal* systems (of equal curvature) with similar asymmetries in spot size. This is because lossless nonconfocal resonators are not, except for special cases, degenerate in frequency. This shows that the mode patterns of nonconfocal systems are not significantly changed if the reflectors are of unequal size but are larger than the spot size at the reflectors.

For resonators formed of two reflectors of unequal curvature, unstable regions of high loss are shown to exist in that an equivalent sequence of lenses becomes defocusing. The true confocal resonator is on the border of such unstable regions, though in fact it has minimum diffraction losses. Unfortunate deviations from the dimensions of the ideal confocal resonator can produce a system of high loss. The implications for the design of resonators for gaseous or solid optical masers or of long distance optical transmission systems are that the "equal" radii of curvature should be made slightly larger (or smaller) than the reflector spacing. One can also choose to simulate a confocal resonator with one curved and one flat reflector spaced at half the radius of curvature. This system is stable.

VIII. ACKNOWLEDGMENTS

Stimulating discussions with and constructive criticism from J. P. Gordon, W. W. Rigrod, A. G. Fox and T. Li are sincerely appreciated.

APPENDIX

Cylindrical Coordinates

In this paper and in the paper of Boyd and Gordon[1] the mathematical analyses were based on a system of Cartesian coordinates. Resonators with reflectors of square or rectangular aperture have been investigated. For large apertures the authors were able to obtain approximations for the properties of the resonator modes on which many arguments of this paper are based. They showed that the mode patterns are describable in terms of Hermite-Gaussian functions.

For certain classes of problems, for instance if it is desired to obtain diffraction losses for circular apertures, it is preferable to use a cylindrical system of coordinates. Approximate solutions for the modes of resonators with reflectors of large circular apertures can be obtained from the work of Goubau and Schwering.[3] The results of these authors are presented in terms of hybrid waves, but by suitably combining two hybrid waves one obtains modes which for our purposes can be regarded as linearly polarized TEM waves. Goubau and Schwering show that the mode patterns are describable in terms of associated Laguerre-Gaussian functions. Fox and Li[9] have given asymptotic solutions for the mode patterns in terms of Sonine's polynomials, which can be shown to be equivalent to the above results.

One can obtain asymptotic solutions for the modes of the confocal resonator in cylindrical coordinates by making a scalar wave approximation and using Huygens' principle. This leads to an integral equation for the modes of the confocal resonator with reflector spacing and curvature b in cylindrical coordinates, which has been given in Appendix C of Fox and Li.[2] For an infinitely large aperture their result may be written in the form

$$\chi\{\sqrt{t}\,R_{pl}(t)\} = \exp\left[-ikb + i\frac{\pi}{2}(l+1)\right] \cdot \int_0^\infty \sqrt{t'}\,R_{pl}(t')\cdot J_l(tt')\cdot\sqrt{tt'}\,dt', \tag{57}$$

where $t = r\sqrt{(k/b)}$ and r is the radial distance in the plane of the aperture. As the solution of the above equation the function $R_{pl}(t)$ describes the radial dependence of the modes and the angular dependence is $e^{+il\varphi}$. The resonance condition for the individual modes is obtained from the eigenvalue χ of (57).

From Magnus and Oberhettinger[10] one observes that the associated Laguerre-Gaussian function is self-reciprocal under the Hankel transformation. Thus the solution of (57) is given by

$$R_{pl}(t) \propto t^l L_p^{\,l}(t^2) \cdot e^{-t^2/2} \tag{58}$$

where $L_p^{\,l}(t)$ is the associated Laguerre polynomial. The associated eigenvalue is found to be

$$\chi = \exp\left[-ikb + i\,\frac{\pi}{2}\,(2p + l + 1)\right] \tag{59}$$

which leads to the resonance condition for the confocal resonator with q as the longitudinal mode number:

$$\frac{4b}{\lambda} = 2q + 2p + l + 1. \tag{60}$$

The field distribution of the modes inside and outside the confocal resonator can be derived from the mode patterns on the reflectors by using Huygens' principle, as in Boyd and Gordon's paper. The field distribution can of course be obtained from Goubau and Schwering's work. Comparing Goubau and Schwering's equation (5a) with Boyd and Gordon's equation (20), one finds that the surfaces of constant phase of a Cartesian TEM_{mnq} mode are identical with the surfaces of constant phase of a cylindrical TEM_{plq} mode if

$$m + n = 2p + l. \tag{61}$$

The fields and therefore the spot size of the fundamental TEM_{00q} Cartesian mode and the fields of the fundamental cylindrical mode are identical throughout the resonator.

The resonance conditions and spot sizes of the modes of resonators with large circular apertures can therefore be deduced for nonconfocal resonators and resonators with reflectors of unequal radii of curvature, in exactly the same fashion as has been done for square apertures.

We do not propose to present this derivation again, but list as a reference some characteristic properties of the modes of resonators with large circular apertures, together with properties of the modes of resonators with large square apertures. It may be worth repeating that an aperture is considered "large" for a particular mode if the mode's energy, as calculated from the approximations below, is well concentrated within the aperture. Only under this condition are the mode's characteristics reasonably well described by the formulae listed below.

A.1 *Approximations for Resonators with Large Circular Apertures.*

A system of cylindrical coordinates (r, φ, z) is used, where the z-axis coincides with the resonator's optical axis. The corresponding modes are designated TEM_{plq}.

A.1.1 *Nonconfocal Resonators with Reflectors of Equal Radius of Curvature b' and a Reflector Spacing d.*

At the reflectors the spot size w_s' of the fundamental TEM_{00q} mode is given by

$$w_s' = \sqrt{\frac{\lambda b'}{\pi}} \left(\frac{d}{2b' - d}\right)^{\frac{1}{4}}. \qquad (62)$$

The relative field distribution (mode pattern) at the reflectors ($z = \pm d/2$) of a TEM_{plq} mode is given by

$$\frac{E(r, \varphi, \pm d/2)}{E_o} = \left(\frac{r}{w_s'}\sqrt{2}\right)^l \cdot L_p^{\,l}\left(2\frac{r^2}{w_s'^2}\right) \cdot e^{-r^2/w_s'^2} \cos l\varphi \qquad (63)$$

where $L_p^{\,l}$ are the associated Laguerre polynomials. The mode resonates at a wavelength given by

$$\frac{2d}{\lambda} = q + \frac{1}{\pi}(2p + l + 1)\cos^{-1}\left(1 - \frac{d}{b'}\right). \qquad (64)$$

A.1.2 *Resonators with Reflectors of Unequal Radii of Curvature b_1 and b_2 and a Reflector Spacing d.*

The spots of the fundamental mode are in general of different size on the two reflectors. We have a spot size w_1 on the reflector with radius of curvature b_1 and vice versa. In (39) and (40) these quantities have been expressed in terms of λ, d, b_1, and b_2. The TEM_{plq} mode patterns on the reflectors are obtained from equation (63) by substituting for w_s' the corresponding w_1 or w_2.

The resonance condition is

$$\frac{2d}{\lambda} = q + \frac{1}{\pi}(2p + l + 1)\cos^{-1}\sqrt{\left(1 - \frac{d}{b_1}\right)\left(1 - \frac{d}{b_2}\right)}. \qquad (65)$$

A.2 *Approximations for Resonators with Large Square Apertures.*

A system of Cartesian coordinates (x, y, z) is used with the z-axis coinciding with the resonator axis. The corresponding modes are designated TEM_{mnq}.

A.2.1 *Nonconfocal Resonators of Equal Radius of Curvature b' and a Reflector Spacing d.*

The spot size w_s' of the fundamental mode at the reflectors is again given by (62). The mode pattern of a TEM_{mnq} mode at the reflectors is given by

$$\frac{E(x, y, \pm d/2)}{E_o} = H_m\left(\frac{x\sqrt{2}}{w_s'}\right)\cdot H_n\left(\frac{y\sqrt{2}}{w_s'}\right)\cdot e^{-x^2+y^2)/w_s'^2} \quad (66)$$

where the H_m are the Hermitian polynomials. The resonance condition for this mode is given by (47).

A.2.2 *Resonators with Reflectors of Unequal Radii of Curvature b_1 and b_2 and a Reflector Spacing d.*

The spot sizes w_1 and w_2 of the fundamental mode at the two reflectors are the same as those discussed in Section A.1.2. The mode patterns of the TEM_{mnq} mode at the corresponding reflectors are obtained by substituting w_1 or w_2 for w_s' in (66). The resonance condition is given by (46).

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