The Tunnel Diode as a Linear Network Element

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Theorems are proved which completely characterize in an explicit manner the class of immittance matrices realizable with lossless reciprocal elements and a tunnel diode represented by the three-parameter "LC, −R" model. Techniques are presented for the synthesis of any immittance matrix within this class.

Considered first, from a scattering matrix viewpoint, are the so-called degenerate cases in which the immittance matrices of the lossless network do not exist. Throughout the remainder of the discussion it is assumed that the lossless network possesses an immittance matrix. Necessary and sufficient conditions, involving in a complicated manner the existence of a certain strict Hurwitz polynomial, are derived for realization with a wide class of terminations. A study of the existence of this polynomial for the particular terminations of interest leads to explicit realizability conditions.

I. INTRODUCTION

The small signal "C, −R" model of the tunnel diode (Fig. 1) provides a fairly good representation over a wide range of frequencies, and is much simpler to use in a general study of network properties than the "LC, −R" model (Fig. 2) which includes, in addition, the series inductor. The simpler model has been used extensively by network theorists.[1-10]

The primary purpose of this paper is to define in an explicit manner the class of $n \times n$ open-circuit impedance and short-circuit admittance matrices that are realizable with lossless reciprocal elements and a tunnel diode characterized by the "LC, − R" model. The results constitute an extension of the theory presented in Ref. 10* for the "C, − R" model. The main interest in this problem to date relates to the

* Although the present paper is self-contained, some familiarity with the earlier work would be of assistance to the reader.

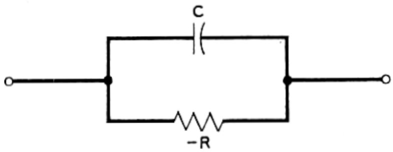

Fig. 1 — "C, −R" model of the tunnel diode.

special case $n = 2$. From a practical viewpoint our results anticipate the development of approximation techniques that lead to the specification of lossless-network tunnel-diode substructures which are to operate between prescribed sources and loads. Indeed an objective of this paper is to encourage research in this direction by presenting a complete solution to the realization problem.

The basic structure under consideration is shown in Fig. 3 in which the $(n + 1)$-port network is assumed to be a lossless reciprocal configuration containing inductors, capacitors, and ideal transformers. While we shall be particularly concerned with the case in which port $(n + 1)$ is terminated with the "LC, − R" model of the tunnel diode, many of the arguments to be presented are applicable to a much wider class of terminations. The overall network is assumed to possess either a short-circuit admittance matrix $\mathbf{Y}(s)$ or an open-circuit impedance matrix $\mathbf{Z}(s)$ relating currents and voltages at the ports $(1, 2, \cdots, n)$.

The realizability study is initiated in the following section where we discuss the cases in which the immittance matrices of the lossless network fail to exist. Throughout the remainder of the paper we consider the realizability of $\mathbf{Z}(s)$ and assume that the $(n + 1)$-port lossless network possesses an open-circuit impedance matrix $\hat{\mathbf{Z}}(s)$. This involves no loss of generality, of course, since results for the short-circuit admittance matrix $\mathbf{Y}(s)$ are identical with those for the open-circuit impedance matrix with the termination replaced with its reciprocal. In Section III necessary and sufficient conditions are presented, in terms of an unknown strict Hurwitz polynomial, for the realization of $\mathbf{Z}(s)$ with a wide class of terminations. The following sections utilize these results to obtain explicit realizability conditions for the particular termination of interest.

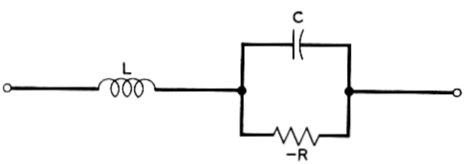

Fig. 2 — "LC, −R" model of the tunnel diode.

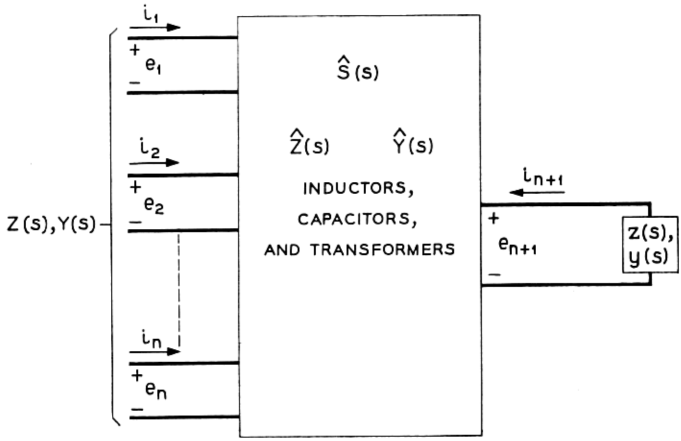

Fig. 3 — Most general structure defining $\mathbf{Z}(s)$ and $\mathbf{Y}(s)$.

In an interesting recent paper,[11] Schoeffler considers a problem similar to that discussed in Section III for the special case $n = 1$ under an assumption equivalent to supposing that the unknown polynomial is unity. In order to obtain explicit conditions, he further assumes that both $\mathbf{Z}(s)$ and the termination are regular at infinity and that $\mathbf{Z}(s)$ has no singularities on the entire $j\omega$-axis. Of course, for our purposes, these assumptions cannot be made. Indeed, for the particular problem considered here, the most interesting realizability conditions arise from a possible pole at infinity of the termination and from the influence of the unknown strict Hurwitz polynomial.

II. REALIZABILITY CONDITIONS WHEN THE IMMITTANCE MATRIX OF THE LOSSLESS NETWORK DOES NOT EXIST

The $(n + 1)$-port lossless network in Fig. 3 invariably possesses a symmetric regular para-unitary scattering matrix[12] which we shall denote by $\hat{\mathbf{S}}(s)$. However, the corresponding short-circuit admittance matrix $\hat{\mathbf{Y}}(s)$ exists if and only if* $\det[\mathbf{1}_{n+1} + \hat{\mathbf{S}}(s)]$ does not vanish identically in s. Similarly $\hat{\mathbf{Z}}(s)$ exists if and only if $\det[\mathbf{1}_{n+1} - \hat{\mathbf{S}}(s)]$ does not vanish identically in s. In this section the following theorem is proved which completely characterizes $\mathbf{Y}$ or $\mathbf{Z}$ in the event that $\hat{\mathbf{Y}}$ or $\hat{\mathbf{Z}}$ fails to exist.

Theorem 1: If $\mathbf{Y}$ [$\mathbf{Z}$] *in Fig. 3 exists with port* $(n + 1)$ *terminated with an admittance* y [*impedance* z] *but* $\hat{\mathbf{Y}}$ [$\hat{\mathbf{Z}}$] *does not exist,* $\mathbf{Y} = y\mathbf{C} + \mathbf{Y}'$ [$\mathbf{Z} =$

* The identity matrix of order $(n + 1)$ is denoted by $\mathbf{1}_{n+1}$.

$z\mathbf{C} + \mathbf{Z}'$] where $\mathbf{C}$ is a nonnegative definite symmetric real matrix of constants of rank not exceeding unity and $\mathbf{Y}'$ [$\mathbf{Z}'$] is the short-circuit admittance matrix [open-circuit impedance matrix] of a lossless reciprocal network.

The proof is based on the following lemma which is adapted from a result of Youla et al.[12]

*Lemma 1: Let $\hat{\mathbf{S}}(s)$ be a regular symmetric para-unitary scattering matrix of order $(n + 1)$ such that the normal ranks of $[\mathbf{1}_{n+1} + \hat{\mathbf{S}}(s)]$ and $[\mathbf{1}_{n+1} - \hat{\mathbf{S}}(s)]$ are r and r' respectively. Then there exist two orthogonal constant matrices $\mathbf{T}$ and $\mathbf{T}'$ such that**

$$\mathbf{T}^t\ \hat{\mathbf{S}}(s)\ \mathbf{T} = \begin{bmatrix} -\mathbf{1}_{n+1-r} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{S}}_r \end{bmatrix} \tag{1}$$

$$\mathbf{T}'^t\ \hat{\mathbf{S}}(s)\ \mathbf{T}' = \begin{bmatrix} \mathbf{1}_{n+1-r'} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{S}}_{r'} \end{bmatrix}. \tag{2}$$

where $\hat{\mathbf{S}}_r$ and $\hat{\mathbf{S}}_{r'}$ are symmetric regular para-unitary scattering matrices of orders r and r' respectively. Moreover $det[\mathbf{1}_r + \hat{\mathbf{S}}_r]$ and $det[\mathbf{1}_{r'} - \hat{\mathbf{S}}_{r'}]$ do not vanish identically in s.

Suppose now that $\mathbf{Y}(s)$ in Fig. 3 exists but that $\hat{\mathbf{Y}}$ does not exist. Then the normal rank of $[\mathbf{1}_{n+1} + \hat{\mathbf{S}}(s)]$ is $r < (n + 1)$. Equation (1) can be interpreted as a realization of $\hat{\mathbf{S}}(s)$ in terms of a $(2n + 2)$-port ideal transformer network, $n + 1 - r$ short-circuits, and a reactance r-port possessing a short-circuit admittance matrix $\hat{\mathbf{Y}}_r = [\mathbf{1}_r + \hat{\mathbf{S}}_r]^{-1}[\mathbf{1}_r - \hat{\mathbf{S}}_r]$, as shown in Fig. 4.

Since $\mathbf{E}_b = \mathbf{T}^t\mathbf{E}_a$, where $\mathbf{E}_b{}^t = [e_{b1}, e_{b2}, \cdots, e_{b(n+1)}]$ and $\mathbf{E}_a{}^t = [e_{a1}, e_{a2}, \cdots, e_{a(n+1)}]$, the number of independent linear relations among the components of $\mathbf{E}_a$ is equal to the number of zero components of $\mathbf{E}_b$. However, since $\mathbf{Y}$ exists but $\hat{\mathbf{Y}}$ does not exist, this number is equal to unity $(r = n)$, and the resulting single linear constraint is

$$\sum_{j=1}^{n+1} t_{j1}e_{aj} = 0, \qquad t_{(n+1)1} \neq 0 \tag{3}$$

in which the t_{j1} are the elements in the first column of $\mathbf{T}$. As a consequence, it is a simple matter to show that we may construct an $(n + 1) \times n$ matrix of real constants $\mathbf{A}$ such that

$$\begin{aligned} \tilde{\mathbf{E}}_b &= \mathbf{A}\tilde{\mathbf{E}}_a \\ \mathbf{A}^t\tilde{\mathbf{I}}_b &= \tilde{\mathbf{I}}_a \end{aligned} \tag{4}$$

* The superscript t denotes matrix transposition.

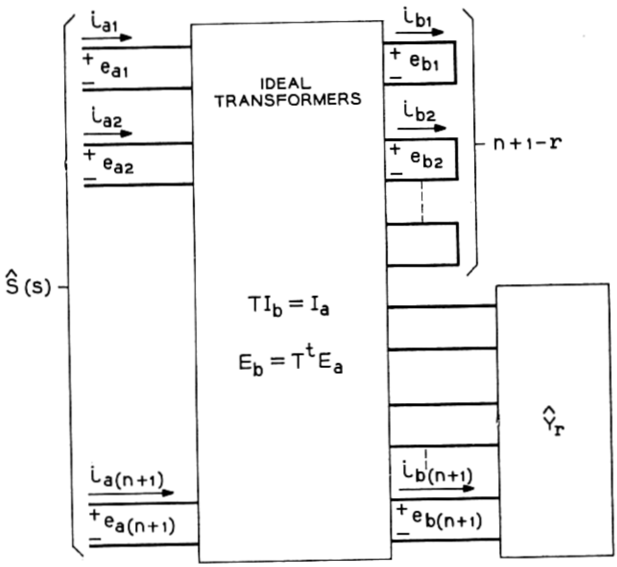

Fig. 4 — Realization of $\hat{\mathbf{S}}(s)$ when the normal rank of $[\mathbf{1}_{n+1} + \hat{\mathbf{S}}(s)]$ is r.

where

$$\begin{aligned}
\tilde{\mathbf{E}}_b{}^t &= [e_{a(n+1)}, e_{b2}, e_{b3}, \cdots, e_{b(n+1)}] \\
\tilde{\mathbf{I}}_b{}^t &= [i_{a(n+1)}, i_{b2}, i_{b3}, \cdots, i_{b(n+1)}] \\
\tilde{\mathbf{E}}_a{}^t &= [e_{a1}, e_{a2}, \cdots, e_{an}] \\
\tilde{\mathbf{I}}_a{}^t &= [i_{a1}, i_{a2}, \cdots, i_{an}].
\end{aligned}$$

But

$$\tilde{\mathbf{I}}_b = \begin{bmatrix} y & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Y}}_r \end{bmatrix} \tilde{\mathbf{E}}_b \tag{5}$$

which, together with (4), yields

$$\tilde{\mathbf{I}}_a = \mathbf{A}^t \begin{bmatrix} y & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Y}}_r \end{bmatrix} \mathbf{A}\tilde{\mathbf{E}}_a . \tag{6}$$

Thus,

$$\begin{aligned}
\mathbf{Y} &= \mathbf{A}^t \begin{bmatrix} y & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{Y}}_r \end{bmatrix} \mathbf{A} = \mathbf{B}^t\mathbf{B}y + \mathbf{D}^t\hat{\mathbf{Y}}_r\mathbf{D} \\
&= \mathbf{C}y + \mathbf{Y}'
\end{aligned} \tag{7}$$

where $\mathbf{B}^t$ is the n-vector of elements in the first column of $\mathbf{A}^t$ and $\mathbf{D}$ is the matrix of elements in the last n rows and columns of $\mathbf{A}$.

A very similar argument suffices to establish the result for the case in which $\hat{\mathbf{Z}}$ fails to exist.

III. REALIZABILITY OF $\mathbf{Z}(s)$ WHEN $\hat{\mathbf{Z}}$ EXISTS

Throughout the remainder of the paper we consider specifically the realizability of $\mathbf{Z}(s)$ under the assumption that $\hat{\mathbf{Z}}(s)$ exists. As mentioned earlier, this is equivalent to assuming that $\hat{\mathbf{Y}}$ exists and that the termination is replaced with its reciprocal in order to study the properties of $\mathbf{Y}(s)$.

We shall suppose that the impedance terminating port $(n + 1)$ is the positive-real function $z(s) = ab^{-1}$, where a and b are Hurwitz polynomials. Of course the impedance of the LC, − R model is not a positive-real function. However, it is convenient to replace the negative resistor with a positive one of equal magnitude so that we may state that $\mathbf{Z}(s)$ is a positive-real matrix, with the understanding that $\mathbf{Z}(s) = -\overline{\mathbf{Z}}(-s)$ where $\overline{\mathbf{Z}}(s)$ is the impedance matrix of the n-port with the resistor negative.[13] Further, it is sufficient to assume that the LC, + R termination comprises an inductor of value α Henries in series with a parallel combination of a unit resistor and unit capacitor, for any other values can be accommodated by impedance and frequency scaling. Thus, we shall be particularly interested in the results for $z(s) = (\alpha s^2 + \alpha s + 1)(s + 1)^{-1}$, $(\alpha \geqq 0)$. However in this section we shall merely require that*

$$z_e = \frac{a_e b_e - a_o b_o}{b_e^2 - b_o^2}, \tag{8}$$

the even part of z, have no zeros on the finite $j\omega$-axis and that $b(s)$ is a strict Hurwitz polynomial.

It is well known that

$$\mathbf{Z} = \mathbf{Z}_{11} - \mathbf{Z}_{12}\mathbf{Z}_{12}{}^t \frac{1}{Z_{22} + z} \tag{9}$$

where the submatrices in (9) are defined by the following partition of $\hat{\mathbf{Z}}(s)$:

$$\hat{\mathbf{Z}}(s) = \begin{array}{c} \begin{array}{cc} n & 1 \end{array} \\ \left[\begin{array}{cc} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{12}{}^t & Z_{22} \end{array}\right] \end{array} \begin{array}{c} n \\ 1 \end{array}. \tag{10}$$

* Throughout the paper we shall denote by the subscripts o and e respectively the odd and even parts of polynomials or matrices. Thus, for example, $\mathbf{A}_e = \frac{1}{2}[\mathbf{A}(s) + \mathbf{A}(-s)]$, $\mathbf{A}_o = \frac{1}{2}[\mathbf{A}(s) - \mathbf{A}(-s)]$.

The arguments to be presented center about a study of $\mathbf{Z}_e$, the even part of the matrix $\mathbf{Z}$. This matrix is given by

$$\mathbf{Z}_e = -\tfrac{1}{2}[z(s) + z(-s)]\mathbf{Z}_{12}\mathbf{Z}_{12}{}^t \frac{1}{[Z_{22} + z][Z_{22}(-s) + z(-s)]} \quad (11)$$

It is convenient to introduce the notation $Z_{22} = d^{-1}n_{22}$, $\mathbf{Z}_{12} = d^{-1}\mathbf{N}_{12}$ where d is an even polynomial, n_{22} is an odd polynomial and $\mathbf{N}_{12}$ is an n-vector of odd polynomials, with the understanding that d, n_{22}, and every element of $\mathbf{N}_{12}$ may have a common simple zero at the origin.* In this way it is unnecessary to treat separately the cases in which d is even or d is odd.

Accordingly,

$$\mathbf{Z}_e = -(a_eb_e - a_ob_o)\mathbf{N}_{12}\mathbf{N}_{12}{}^t \cdot \frac{1}{[bn_{22} + ad][b(-s)n_{22}(-s) + a(-s)d(-s)]}. \quad (12)$$

Note that the assumptions regarding z, n_{22}, and d require that the polynomial $[bn_{22} + ad]$ be strictly Hurwitz except possibly for a simple zero at the origin. Also, as one would expect,[11] the zeros of $[bn_{22} + ad]$ cannot coincide with any of those of $(a_eb_e - a_ob_o)$. This follows from the fact that the existence of a nontrivial solution for a and b satisfying $bn_{22} + ad = 0$ and $(a_eb_e - a_ob_o) = \frac{1}{2}[ab(-s) + ba(-s)] = 0$ at some point $s = s_1$ requires that s_1 satisfy $b(-s_1)n_{22}(-s_1) + a(-s_1)d(-s_1) = 0$, which contradicts the fact that the zeros of $bn_{22} + ad$ are restricted to a half-plane.

It is convenient to state the following

Definition: The matrix $\mathbf{Z}_e$ *is said to be in standard form if and only if*

$$\mathbf{Z}_e = -(a_eb_e - a_ob_o)\mathbf{U}\mathbf{U}^t \frac{1}{v(s)v(-s)}$$

where $v(s)$ *is a positive coefficient polynomial which is strictly Hurwitz except possibly for a simple zero at the origin and* $\mathbf{U}^t = [u_1, u_2, \cdots, u_n]$ *is a row matrix of odd real polynomials with the property that there is no factor* $\eta(s)\eta(-s)$ *common to all the* u_i *such that* $\eta^2(s)$ *divides* $v(s)$ *where* $\eta(s)$ *is a strict Hurwitz polynomial. Further,* $v(s)$ *and* $(a_eb_e - a_ob_o)$ *are relatively prime.*

In Section 3.1 the following result is proved.†

Theorem 2: Denote by $z(s)$ *the two-terminal positive-real impedance* $z(s) = ab^{-1}$, *with* b *a strict Hurwitz polynomial and* z_e *having no zeros on the*

* With this exception, n_{22} and d are assumed to be relatively prime.
† We shall use the notation $\lim_{s\to\infty} [\cdot] = [\cdot]_\infty$ throughout.

finite jω-axis. Then the rational positive-real open-circuit impedance matrix $\mathbf{Z}(s)$ *is realizable as shown in Fig. 3, with the understanding that the lossless reciprocal network possesses an open-circuit impedance matrix* $\hat{\mathbf{Z}}(s)$, *if and only if* $\mathbf{Z}_e$ *can be expressed in standard form and there exists a strict Hurwitz polynomial* $\eta(s)$ *defining*

$$\mathbf{X} = \pm \mathbf{U}\eta(s)\eta(-s), \qquad w = v\eta^2(s)$$

such that

(*i*) $(w_e a_o - w_o a_e)(w_o b_o - w_e b_e)^{-1}$ *is a reactance function, the degenerate case in which* $(w_o b_o - w_e b_e) \equiv 0$ *not permitted.*

(*ii*) $\left[\mathbf{X}\,\dfrac{b_o a_o - b_e a_e}{w_e a_o - w_o a_e}\right]_\infty$ *exists, and*

(*iii*) $\left[\dfrac{1}{s}\mathbf{Z}\right]_\infty - \left[\mathbf{Z}_e\,\dfrac{a(w_e - w_o)}{s(a_o w_e - a_e w_o)}\right]_\infty$ *is nonnegative definite when the reactance function in (i) has a pole at infinity.*

Further, any $\mathbf{Z}(s)$ *satisfying these conditions can be realized as shown in Fig. 3 with* $\hat{\mathbf{Z}}(s)$ *given by*

$$\hat{\mathbf{Z}}(s) = \begin{bmatrix} \mathbf{Z} + \mathbf{X}\mathbf{X}^t \dfrac{b(a_o b_o - a_e b_e)}{(b_o w_o - b_e w_e)w} & \mathbf{X}\dfrac{(a_o b_o - a_e b_e)}{(b_o w_o - b_e w_e)} \\ \mathbf{X}^t \dfrac{(a_o b_o - a_e b_e)}{(b_o w_o - b_e w_e)} & \dfrac{(a_o w_e - a_e w_o)}{(b_o w_o - b_e w_e)} \end{bmatrix}.$$

3.1 *Proof of Theorem 2*

It is evident that a prescribed $\mathbf{Z}_e$ can be expressed in standard form if $\mathbf{Z}$ is realizable.* Suppose that $\mathbf{Z}$ and $\mathbf{Z}_e$ in standard form are given and consider the problem of determining d, n_{22}, $\mathbf{N}_{12}$, and $\mathbf{Z}_{11}$. In particular, let us consider identifying d, n_{22}, and $\mathbf{N}_{12}$ by equating the standard form expression for $\mathbf{Z}_e$ with the right-hand side of (12). A common factor may have been cancelled in the expression for $\mathbf{Z}_e$ and hence an unknown factor must be reinserted before we can proceed. However, the unknown factor must be of the form $f^2(s) = g(s)g(-s)$ where $g(s)$ is a strict Hurwitz polynomial. Therefore $f(s) = \eta(s)\eta(-s)$ where $\eta(s)$ is a strict Hurwitz polynomial. It follows that the most general expression for $\mathbf{Z}_e$ of the form:

$$\mathbf{Z}_e = -(a_e b_e - a_o b_o)\mathbf{X}\mathbf{X}^t \frac{1}{w(s)w(-s)} \tag{13}$$

* The problem of factoring $\mathbf{Z}_e$ into this form is discussed elsewhere.[10]

in which

1. $w(s)$ is a positive coefficient polynomial which is strictly Hurwitz except possibly for a simple zero at the origin, and
2. $\mathbf{X}^t$ is a row matrix of real odd polynomials

is expressible in terms of the standard form expression with

$$\mathbf{X} = \pm \mathbf{U}\eta(s)\eta(-s) \tag{14}$$

$$w = v\eta^2(s) \tag{15}$$

where $\eta(s)$ is an arbitrary real strict Hurwitz polynomial.

Thus if $\mathbf{Z}$ is realizable, $\mathbf{X} = \mathbf{N}_{12}$ and $w = bn_{22} + ad$ for some $\mathbf{X}$ and w generated by (14) and (15) with n_{22} and d respectively odd and even polynomials that are relatively prime, except possibly for a common simple zero at the origin, such that $n_{22}d^{-1}$ is a reactance function. Equating the even and odd parts of w and $b\bar{n}_{22} + a\bar{d}$ yields:

$$\bar{d} = (b_o w_o - b_e w_e)\,(b_o a_o - b_e a_e)^{-1} \tag{16}$$

$$\bar{n}_{22} = (a_o w_e - a_e w_o)\,(b_o a_o - b_e a_e)^{-1}. \tag{17}$$

Suppose now that $\bar{n}_{22}\bar{d}^{-1}$ is a reactance function. Then the two functions:

$$\frac{\bar{n}_{22}}{\bar{d}} + z = \frac{w(a_o b_o - a_e b_e)}{b(b_o w_o - b_e w_e)} \tag{18}$$

and

$$\frac{\bar{d}}{\bar{n}_{22}} + z^{-1} = \frac{w(a_o b_o - a_e b_e)}{a(a_o w_e - a_e w_o)} \tag{19}$$

are required to be positive-real. Since the even polynomial $(a_o b_o - a_e b_e)$ is either a constant or has zeros in the right-half plane, it is evident from (18) and (19) that (16) and (17) are polynomials. Furthermore in view of the positive-real property of (18) and (19) and the fact (16) and (17) are respectively even and odd polynomials, it follows that the zeros of (16) and (17) are restricted to the $j\omega$-axis and that these zeros are simple, except possibly for a double zero of $\bar{d}$ at the origin which can occur when v and hence w has a simple zero at the origin. With this single permissible exception, it is also the case that (16) and (17) are relatively prime, for the condition that there exists a nontrivial solution for w_e and w_o in

$$\begin{aligned} b_o w_o - b_e w_e &= 0 \\ -a_e w_o + a_o w_e &= 0 \end{aligned}$$

is $(a_o b_o - a_e b_e) = 0$, which, by assumption, is not satisfied for pure imaginary values of s. Thus, if $\bar{n}_{22}\bar{d}^{-1}$ is a reactance function, $w = bn_{22} + ad$ with $n_{22} = \bar{n}_{22}$ and $d = \bar{d}$ where, as required, n_{22} and d are relatively prime polynomials except possibly for a common simple zero at the origin.

Next consider the determination of the submatrices $\mathbf{Z}_{12}$ and $\mathbf{Z}_{11}$. For $d = \bar{d}$ given by (16),

$$\mathbf{Z}_{12} = \mathbf{X}d^{-1} = \mathbf{X}\frac{(b_o a_o - b_e a_e)}{(b_o w_o - b_e w_e)}. \tag{20}$$

Using (9), $z(s) = ab^{-1}$, $d = \bar{d}$, $n_{22} = \bar{n}_{22}$, and (20), we find

$$\mathbf{Z}_{11} = \mathbf{Z} + \mathbf{X}\mathbf{X}^t \frac{b(a_o b_o - a_e b_e)}{(b_o w_o - b_e w_e)w}. \tag{21}$$

By substituting $\mathbf{Z} = \mathbf{Z}_o + \mathbf{Z}_e$ in (21) with $\mathbf{Z}_e$ given by (13), it is easy to show that (21) is a matrix of odd functions, as it should be. Furthermore, since $(b_o w_o - b_e w_e) = d(a_o b_o - a_e b_e)$, it is evident from (21) that $\mathbf{Z}_{11}$ is regular in the entire finite strict left-half plane and consequently has finite poles only on the $j\omega$-axis.

Consider now the realizability of

$$\hat{\mathbf{Z}}(s) = \begin{bmatrix} \mathbf{Z} + \mathbf{X}\mathbf{X}^t \dfrac{b(a_o b_o - a_e b_e)}{(b_o w_o - b_e w_e)w} & \mathbf{X}\dfrac{(a_o b_o - a_e b_e)}{(b_o w_o - b_e w_e)} \\[2ex] \mathbf{X}^t \dfrac{(a_o b_o - a_e b_e)}{(b_o w_o - b_e w_e)} & \dfrac{(a_o w_e - a_e w_o)}{(b_o w_o - b_e w_e)} \end{bmatrix}. \tag{22}$$

We require the following lemma.[10]

Lemma 2: The symmetric matrix of real constants

$$\hat{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^t & A_{22} \end{bmatrix}, \qquad A_{22} > 0$$

partitioned as in (10) *is nonnegative definite if and only if* $\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{A}_{12}^t A_{22}^{-1}$ *is nonnegative definite.*

Let $\hat{\mathbf{K}}_i$ denote the residue matrix at a pole of $\hat{\mathbf{Z}}(s)$ which arises from a zero of $(b_o w_o - b_e w_e)$ at say $s = j\omega_i$, and let the residue matrix of $\mathbf{Z}$ at that pole be $\mathbf{K}_i$. Then,*

* When $w = su = s[u_e + u_o]$, where u is a strict Hurwitz polynomial, it is necessary to replace w_e, w_o and $\mathbf{X}$ respectively with u_o, u_e and the n-vector of even polynomials $s^{-1}\mathbf{X}$ before this argument is applied to verify the nonnegative definiteness of the matrix of residues associated with the pole at the origin.

$$\hat{\mathbf{K}}_i = \begin{bmatrix} \mathbf{K}_i + \mathbf{XX}^t \dfrac{b(a_ob_o - a_eb_e)}{\dot{q}w} & \mathbf{X}\dfrac{(a_ob_o - a_eb_e)}{\dot{q}} \\ \mathbf{X}^t \dfrac{(a_ob_o - a_eb_e)}{\dot{q}} & \dfrac{a_ow_e - a_ew_o}{\dot{q}} \end{bmatrix}_{s=j\omega_i} \tag{23}$$

where $\dot{q}$ is the derivative of $(b_ow_o - b_ew_e)$ with respect to s. Since $(\dot{q})^{-1}(a_ow_e - a_ew_o)$ evaluated at $s = j\omega_i$ is positive, $\hat{\mathbf{K}}_i$ is nonnegative definite if and only if

$$\begin{aligned} \mathbf{K}_i + \mathbf{XX}^t \frac{b(a_ob_o - a_eb_e)}{\dot{q}w} \bigg|_{s=j\omega_i} & \\ - \frac{(a_ob_o - a_eb_e)^2}{(\dot{q})^2} \mathbf{XX}^t \frac{\dot{q}}{(a_ow_e - a_ew_o)} \bigg|_{s=j\omega_i} & \\ = \mathbf{K}_i + \mathbf{XX}^t \frac{1}{\dot{q}} \left[\frac{(a_ob_o - a_eb_e)a(b_ew_e - b_ow_o)}{w(a_ow_e - a_ew_o)} \right] \bigg|_{s=j\omega_i} & \end{aligned} \tag{24}$$

is nonnegative definite, a condition which is clearly satisfied since $(b_ew_e - b_ow_o)$ vanishes at $s = j\omega_i$.

Finally, we require that $\hat{\mathbf{Z}}$ have at most a simple pole at infinity and that $\hat{\mathbf{K}}_\infty = [(1/s)\hat{\mathbf{Z}}]_\infty$ be nonnegative definite. It is clearly necessary that the limit

$$[Z_{22}^{-1}\mathbf{Z}_{12}]_\infty = \left[\mathbf{X} \frac{(a_ob_o - a_eb_e)}{(a_ow_e - a_ew_o)} \right]_\infty \tag{25}$$

exist. When $[Z_{22}^{-1}\mathbf{Z}_{12}]_\infty$ does exist with Z_{22} a reactance function, it follows from (9) that $\mathbf{Z}_{11}$ has at most a simple pole at infinity, since $\mathbf{Z}$ is assumed to possess this property, and consequently that $\hat{\mathbf{Z}}$ has at most a simple pole at infinity.

Suppose now that Z_{22} has a pole at infinity. According to Lemma 2 and (22), $\hat{\mathbf{K}}_\infty$ is nonnegative definite if and only if the following matrix of constants is nonnegative definite:

$$\begin{aligned} \mathbf{K}_\infty + \left[\frac{1}{s} \mathbf{XX}^t \frac{b(a_ob_o - a_eb_e)}{(b_ow_o - b_ew_e)w} \right]_\infty - \left[\frac{1}{s} \mathbf{X} \frac{(a_ob_o - a_eb_e)}{(b_ow_o - b_ew_e)} \right]_\infty & \\ \cdot \left[\frac{1}{s} \mathbf{X}^t \frac{(a_ob_o - a_eb_e)}{(b_ow_o - b_ew_e)} \right]_\infty \left[\frac{1}{s} \frac{(a_ow_e - a_ew_o)}{(b_ow_o - b_ew_e)} \right]_\infty^{-1}. & \end{aligned} \tag{26}$$

Some manipulation shows that (26) can be rewritten as simply

$$\mathbf{K}_\infty - \left[\mathbf{Z}_e \frac{a(w_e - w_o)}{s(a_ow_e - a_ew_o)} \right]_\infty. \tag{27}$$

If Z_{22} does not have a pole at infinity, $\hat{\mathbf{K}}_\infty$ is nonnegative definite if and only if $\mathbf{Z}_{12}$ does not have a pole at infinity and $[(1/s)\mathbf{Z}_{11}]_\infty$ is nonnegative definite. The first of these requirements is contained in the condition that (25) exist, while the second follows from the assumed positive-real property of $\mathbf{Z}$, for in this case $[(1/s)\mathbf{Z}]_\infty = [(1/s)\mathbf{Z}_{11}]_\infty$. This proves Theorem 2.

3.2 *Remarks Concerning Theorem 2*

Consider first the conditions under which the degenerate situation $(b_o w_o - b_e w_e) \equiv 0$ can arise. Assume that both b_o and b_e do not vanish identically in s. Note that a strict Hurwitz w cannot satisfy the equation, for b is assumed to be strictly Hurwitz and hence if w is also strictly Hurwitz $b_o w_o$ vanishes at the origin while $b_e w_e$ does not. The alternative possibility is that $w = su$ where u is a strict Hurwitz polynomial. In this event we have $(u_e b_o - u_o b_e) \equiv 0$ and therefore* $u = b$. This leads to the following expression for $\mathbf{Z}_e$:

$$\mathbf{Z}_e = -(a_e b_e - a_o b_o)\mathbf{X}\mathbf{X}^t \frac{1}{[s(b_e + b_o)][-s(b_e - b_o)]}$$

Since each element in the matrix $\mathbf{Z}_e$ must approach zero at least as fast as z_e, which is obvious from (11), the degree of each polynomial in the n-vector of odd polynomials $\mathbf{X}$ is at most unity. Thus $\mathbf{X}$ contains no non-$j\omega$-axis factors that can be cancelled and consequently $(b_o w_o - b_e w_e) \equiv 0$ implies that η^2 is a constant, say unity, and $w = s(b_e + b_o)$ or, equivalently, $(b_o v_o - b_e v_e) \equiv 0$. However given $(b_o v_o - b_e v_e) \equiv 0$, it is not clear that a nonconstant choice of η^2 could not render $\bar{d}^{-1}\bar{n}_{22}$ realizable as a reactance function. To resolve this question assume that $w = \eta^2 s(b_e + b_o)$ and consider $z + \bar{d}^{-1}\bar{n}_{22}$ which must be a positive-real function if $\bar{d}^{-1}\bar{n}_{22}$ is realizable. Some algebra yields

$$z + \bar{d}^{-1}\bar{n}_{22} = z_e \frac{\eta^2}{2\eta_e \eta_o}. \tag{28}$$

It is clear that (28) is not a positive real function for any choice of η when b_e, $b_o \not\equiv 0$ because of the right-half plane poles of z_e. Thus,

Lemma 3: When b_e, $b_o \not\equiv 0$, *condition* (*i*) *can be replaced with the statement*: $(w_e a_o - w_o a_e)(w_o b_o - w_e b_e)^{-1}$ *is a reactance function and* $(b_o v_o - b_e v_e) \not\equiv 0$.

The discussion relating to the realization of $\mathbf{Z}$ when $z = (s + 1)$ shows that the assumption b_e, $b_o \not\equiv 0$ is necessary.

* We are ignoring a trivial constant multiplicative factor.

We wish to show now that condition (*iii*) is invariably satisfied when z is regular at infinity. This occurs because the limit

$$\left[\mathbf{Z}_e \frac{a(w_e - w_o)}{s(a_o w_e - a_e w_o)}\right]_\infty \tag{29}$$

vanishes when z is regular at infinity. To prove this it is sufficient to consider the limit obtained by evaluating (29) with $\mathbf{Z}_e$ replaced with z_e. Using $w = b\bar{n}_{22} + a\bar{d}$ to compute w_e and w_o, we find

$$\left[z_e \frac{a(w_e - w_o)}{s(a_o w_e - a_e w_o)}\right] = \frac{1}{s} z \left[\frac{\bar{d}}{\bar{n}_{22}}\left(\frac{a_e - a_o}{b_e - b_o}\right) - 1\right]$$

from which our assertion is obvious.* Thus,

Lemma 4: Condition (iii) is satisfied when $z(s)$ is regular at infinity.

In the following sections we shall use Theorem 2 and Lemmas 3 and 4 to obtain explicit realizability conditions for $z(s)$ or $z^{-1}(s) = (s + 1)^{-1}(\alpha s^2 + \alpha s + 1)$. *We assume throughout that $\mathbf{Z}(s)$ is known to be a rational symmetric $n \times n$ positive-real matrix, and that $\mathbf{Z}_e$ in standard form is given by*†

$$\mathbf{Z}_e = -\mathbf{U}\mathbf{U}^t \frac{1}{v(s)v(-s)}. \tag{30}$$

To further avoid repetition, the term "realizable with an impedance $z(s)$" is to be understood to refer to the realizability of the multiport matrix as a structure shown in Fig. 3 with the provision that $\hat{\mathbf{Z}}(s)$ exists. It is convenient to treat separately the cases in which $\alpha = 0$, and $\alpha > 0$.

IV. EXPLICIT REALIZABILITY CONDITIONS WHEN $z = (s + 1)^{-1}$ AND $z = (s + 1)$

When $z(s) = (s + 1)^{-1}$, conditions (*i*) and (*ii*) reduce to

(*i*) $w_o(w_e - sw_o)^{-1}$ must be a reactance function and $v_e \neq sv_o$ (see Lemma 3).

(*ii*) $[\mathbf{X}(1/w_o)]_\infty$ must exist.

According to Lemma 4 condition (*iii*) is satisfied.

* This result can be established in a more direct fashion by observing that the nonnegative definiteness of $\hat{\mathbf{K}}_\infty$ is implied by Lemma 2 and expression (9) when $z(s)$ is regular at infinity in view of the positive-real property of $\mathbf{Z}(s)$. Nevertheless, it is instructive to consider this matter from the viewpoint presented above.

† Here $(a_e b_e - a_o b_o) = 1$.

Condition (i) requires that $[(1/s)(w_e/w_o)]_\infty \geqq 1$. Observe that w must be of even degree and that $[(1/s)(w_e/w_o)]_\infty$ is simply the reciprocal of the negative of the sum of the zeros of the polynomial $w(s)$. It follows at once that (i) is satisfied for some $\eta(s)$ if and only if $[(1/s)(v_e/v_o)]_\infty \geqq 1$. Requirement (ii) is satisfied without additional restrictions on $\mathbf{Z}(s)$, for (i) implies that

$$[\mathbf{Z}_e]_\infty = \left[-\frac{1}{w(s)\,w(-s)}\,\mathbf{X}\mathbf{X}^t\right]_\infty = \mathbf{0} \tag{31}$$

since $\mathbf{X}$ is an n-vector of odd polynomials and $\mathbf{Z}_e$ is bounded at infinity. This proves

*Theorem 3:**

The matrix $\mathbf{Z}$ *is realizable with an impedance* $(s+1)^{-1}$ *if and only if* $v_e \not\equiv sv_o$ *and* $\left[\dfrac{v_e}{sv_o}\right]_\infty \geqq 1$.

4.1 $z(s) = (s + 1)$

In this more interesting case† the three conditions become

(i) $(w_o - sw_e)\ (w_e)^{-1}$ must be a reactance function and $w_e \not\equiv 0$.

(ii) $[\mathbf{X}\ (w_o - sw_e)^{-1}]_\infty$ must exist.

(iii) $\left[\dfrac{1}{s}\,\mathbf{Z}\right]_\infty - \left[\mathbf{Z}_e \dfrac{(s+1)\ (w_e - w_o)}{s(sw_e - w_o)}\right]_\infty$ must be nonnegative definite when $(w_o - sw_e)\ (w_e)^{-1}$ has a pole at infinity.

From (i), $k' = [w_o/sw_e]_\infty \geqq 1$. Since

$$\mathbf{Z}_e = -\,\mathbf{X}\mathbf{X}^t\,\frac{1}{w(s)\,w(-s)},$$

and (i) requires that $w(s)$ be of odd degree, (ii) is satisfied for $k' > 1$. However if $k' = 1$, (ii) is satisfied if and only if $[\mathbf{Z}_e]_\infty = \mathbf{0}$. In terms of k', condition (iii) is equivalent to the statement

$$\left[\frac{1}{s}\,\mathbf{Z}\right]_\infty - \frac{k'}{k'-1}\,[\mathbf{Z}_e]_\infty \tag{32}$$

must be nonnegative definite when $k' > 1$.

* This result was stated without proof in Ref. 10.

† This case together with the situation in which $\hat{\mathbf{Z}}$ does not exist is treated in detail from a somewhat different viewpoint in Ref. 10. It is included here primarily to illustrate the application of Theorem 2.

Consider now the influence of the strict Hurwitz polynomial $\eta(s)$. Note first that $\mathbf{Z}_e$ may be a matrix of constants; that is, $v(s)$ may be equal to γs where γ is a positive constant. Let β be the reciprocal of the negative of the sum of the zeros of $\eta(s)$. In this case $k' = \beta$ and β can be chosen arbitrarily large to minimize $k'(k' - 1)^{-1}$. Therefore when $\mathbf{Z}_e$ is a matrix of constants, (i), (ii), and (iii) reduce to the requirement that

$$\left[\frac{1}{s}\mathbf{Z}\right]_\infty - (1 + \epsilon)[\mathbf{Z}_e]_\infty$$

be nonnegative definite for some* $\epsilon > 0$.

When $\mathbf{Z}_e$ is not a matrix of constants, the most favorable choice of $\eta(s)$ is simply a constant, for then k' is maximized and $k'(k' - 1)^{-1}$ is minimized. Thus,

Theorem 4: The matrix $\mathbf{Z}$ *is realizable with an impedance* $z = (s + 1)$ *if and only if*

1. *When* $\mathbf{Z}_e$ *is a matrix of constants,* $[(1/s)\mathbf{Z}]_\infty - (1 + \epsilon)[\mathbf{Z}_e]$ *is nonnegative definite for some* $\epsilon > 0$.
2. *If* $\mathbf{Z}_e$ *is not a matrix of constants,* $k = [v_o/sv_e]_\infty \geqq 1$; *if* $k = 1$, $[\mathbf{Z}_e]_\infty = \mathbf{0}$; *if* $k > 1$, $[(1/s)\mathbf{Z}]_\infty - [k/(k - 1)][\mathbf{Z}_e]_\infty$ *is nonnegative definite.*

V. EXPLICIT REALIZABILITY CONDITIONS WHEN $z(s)$ OR $z^{-1}(s) = (\alpha s^2 + \alpha s + 1)(s + 1)^{-1}$, $\alpha > 0$

In these cases, as will become clear, the polynomial $\eta(s)$ plays a central role in determining the realizability conditions. We shall consider first the case: $z(s) = (\alpha s^2 + \alpha s + 1)(s + 1)^{-1}$. Here condition (i) requires that

$$(i) \quad \frac{\alpha s w_e - w_o(\alpha s^2 + 1)}{s w_o - w_e} = -\alpha s + \frac{w_o}{w_e - s w_o}$$

must be a reactance function and, using Lemma 3, $sv_o \not\equiv v_e$.

It is clearly necessary that $[w_e/sw_o]_\infty = 1$. Thus we may assume that v and w are of even degree. Let

$$v(s) = \sum_{k=0}^{2m} v_k s^k \tag{33}$$

* If the lossless network is not required to possess an open-circuit impedance matrix, ϵ can be taken to be zero.[10]

$$w(s) = \eta^2(s)v(s) = \sum_{k=0}^{2p} w_k s^k \tag{34}$$

where $p = m +$ (degree of η). Then, since the reactance function must have a nonnegative "residue" at infinity,

$$w_{2p-1}(w_{2p-2} - w_{2p-3})^{-1} \geqq \alpha$$

or since $w_{2p-1} = w_{2p}$,

$$1 - \alpha(w'_{2p-2} - w'_{2p-3}) \geqq 0 \tag{35}$$

where $w_k' = w_k(w_{2p})^{-1}$.

Condition (*ii*) reads:

$$(ii) \quad \left[\mathbf{X} \frac{\alpha s^2 - (\alpha s^2 + 1)}{\alpha s w_e - (\alpha s^2 + 1)w_o}\right]_\infty = \left[\mathbf{X} \frac{-1}{\alpha s(w_e - s w_o) - w_o}\right]_\infty$$

must exist.

Assume first that (35) holds with strict inequality in which case (*ii*) becomes

$$\left[\mathbf{X} \frac{-1}{\alpha(w_{2p-2} - w_{2p-3})\, s^{2p-1} - w_{2p}\, s^{2p-1}}\right]_\infty \tag{36}$$

must exist. But $[\mathbf{X}(1/s^{2p})]_\infty = \mathbf{0}$, since w is of even degree, and therefore the limit (36) does indeed exist. Suppose now that (35) holds with the equal sign. Then (*ii*) requires that

$$\left[\mathbf{X} \frac{-1}{\alpha(w_{2p-4} - w_{2p-5})\, s^{2p-3} - w_{2p-3}\, s^{2p-3}}\right]_\infty \tag{37}$$

exist.* Hence the degree of $\mathbf{X}$ must be $(2p - 3)$ at most. Since the degree of w is $2p$, (37) will exist if and only if

$$[s^2 \mathbf{Z}_e]_\infty = \mathbf{0} \tag{38}$$

Consider now condition (*iii*) which requires that

$$\left[\frac{1}{s}\mathbf{Z}\right]_\infty + [s^2\mathbf{Z}_e]_\infty \frac{\alpha}{1 - \alpha(w'_{2p-2} - w'_{2p-3})} \tag{39}$$

be nonnegative definite when (35) is satisfied with strict inequality.

From the form of conditions (35) and (39), it is clear that the most favorable realizability conditions for a given $\mathbf{Z}(s)$ are obtained when

* It can easily be shown that the reactance function property of the expression in (*i*) implies that the denominator in (37) does not vanish identically in s when (35) is satisfied with equality.

$\eta(s)$ is chosen to satisfy $[w_e/(sw_o)]_\infty = 1$ and to simultaneously minimize* $(w'_{2p-2} - w'_{2p-3})$.

This obviously requires that $[v_e/(sv_o)]_\infty \geqq 1$. We wish to establish

Lemma 5: The minimum value of $(w'_{2p-2} - w'_{2p-3})$ *is*

$$\varphi_1 = \frac{1}{3}\left[1 - \left(\frac{v_{2m-1}}{v_{2m}}\right)^3\right] - \frac{v_{2m-3}}{v_{2m}} + \left[\frac{v_{2m-1}}{v_{2m}}\right]\left[\frac{v_{2m-2}}{v_{2m}}\right] - \frac{1}{12}\left[1 - \frac{v_{2m-1}}{v_{2m}}\right]^3$$

and is attained when $\eta(s) = \lambda\left[s + \frac{1}{2}\left(1 - \frac{v_{2m-1}}{v_{2m}}\right)\right]^\delta$ *where* λ *is a positive constant and*

$$\delta = 1, \qquad \frac{v_{2m-1}}{v_{2m}} < 1$$

$$\delta = 0, \qquad \frac{v_{2m-1}}{v_{2m}} = 1.$$

5.1 *Proof of Lemma 5*

Denote by $s_1, s_2, \ldots, s_{2p}$ the zeros of $w(s)$. Using a result[14] due to Newton, we find that

$$\sum_{k=1}^{2p} s_k^3 = -(w'_{2p-1})^3 + 3(w'_{2p-1})(w'_{2p-2}) - 3(w'_{2p-3}) \tag{40}$$

Recalling that here $w'_{2p-1} = 1$,

$$\begin{aligned} w'_{2p-2} - w'_{2p-3} &= \tfrac{1}{3} + \tfrac{1}{3}\sum_{k=1}^{2p} s_k^3 \\ &= \tfrac{1}{3} + \tfrac{1}{3}\sum_{v} s_k^3 + \tfrac{2}{3}\sum_{\eta} s_k^3 \end{aligned} \tag{41}$$

where $\sum_v$ and $\sum_\eta$ denote respectively sums taken only over those indices corresponding to zeros of the polynomials v and η. Hence our problem reduces to determining the strict Hurwitz $\eta(s)$ such that $\sum_\eta s_k^3$ is minimized subject to

$$\sum_\eta s_k = \tfrac{1}{2}\left[\frac{v_{2m-1}}{v_{2m}} - 1\right] \leqq 0.$$

Of course when $\sum_\eta s_k = 0$, η is simply a constant. Assume then that

* Note that $[s^2\mathbf{Z}_e]$ is nonpositive definite.

$\sum_{\eta} s_k < 0$. First observe that the real-part of $(-g + jh)^3$ (g, h real constants; g positive), exceeds $-g^3$ for all $h \neq 0$. Hence the optimum η has only real zeros. Next note that $\sum_{\eta} s_k^3 \geqq [\sum_{\eta} s_k]^3$ when each s_k is a negative constant. Thus the optimum η is a single linear factor

$$\lambda\left[s + \tfrac{1}{2}\left(1 - \frac{v_{2m-1}}{v_{2m}}\right)\right]$$

in which λ is a positive-real constant and the corresponding minimum value of $(w'_{2p-2} - w'_{2p-3})$ is φ_1, where

$$\varphi_1 = \frac{1}{3} + \frac{1}{3}\sum_{v} s_k^3 - \frac{1}{12}\left(1 - \frac{v_{2m-1}}{v_{2m}}\right)^3. \tag{42}$$

Expression (42) can be written in the more convenient form given in Lemma 5 by using (40).

The results of this section can be summarized as follows.

Theorem 5: The matrix $\mathbf{Z}(s)$ *is realizable with an impedance* $z(s) = (\alpha s^2 + as + 1)(s + 1)^{-1}$, *where* $\alpha > 0$, *if and only if*

1. $\left[\dfrac{v_e}{sv_o}\right]_\infty = \dfrac{v_{2m}}{v_{2m-1}} \geq 1, \quad v_e \not\equiv sv_o$

2. $1 - \alpha\varphi_1 \geqq 0$

3. If $1 - \alpha\varphi_1 = 0$, $[s^2\mathbf{Z}_e]_\infty = \mathbf{0}$

4. If $1 - \alpha\varphi_1 > 0$, $[(1/s)\mathbf{Z}]_\infty + [s^2\mathbf{Z}_e]_\infty[\alpha/(1 - \alpha\varphi_1)]$

must be nonnegative definite, where

$$\varphi_1 = \frac{1}{3}\left[1 - \left(\frac{v_{2m-1}}{v_{2m}}\right)^3\right] - \frac{v_{2m-3}}{v_{2m}} + \left[\frac{v_{2m-1}}{v_{2m}}\right]\left[\frac{v_{2m-2}}{v_{2m}}\right] - \frac{1}{12}\left[1 - \frac{v_{2m-1}}{v_{2m}}\right]^3$$

and the v_k *are defined by* $v(s) = \sum_{k=0}^{2m} v_k s^k$.

Further, if $\mathbf{Z}$ *satisfies these conditions,* $\hat{\mathbf{Z}}(s)$ *given by (22) is realizable with* $\eta = 1$ *when* $(v_{2m})(v_{2m-1})^{-1} = 1$ *and with*

$$\eta(s) = s + \tfrac{1}{2}\left(1 - \frac{v_{2m-1}}{v_{2m}}\right)$$

when $(v_{2m})(v_{2m-1})^{-1} > 1$.

5.2 *Realizability with* $z(s) = (s + 1)(\alpha s^2 + \alpha s + 1)^{-1}$

Here condition (i) requires that

$$\frac{sw_e - w_o}{\alpha s w_o - (\alpha s^2 + 1)\, w_e} = \left[-\alpha s + \frac{w_e}{w_o - sw_e} \right]^{-1}$$

must be a reactance function and $\alpha s v_o - (\alpha s^2 + 1)v_e \not\equiv 0$. Thus,

$$\left[\frac{w_o}{sw_e}\right]_\infty = 1, \quad \text{and} \quad 1 - \alpha\,(w'_{2p-1} - w'_{2p-2}) \geqq 0 \qquad (43)$$

where $w = \sum_{k=0}^{2p+1} w_k s^k$ and $w_k' = w_k(w_{2p+1})^{-1}$.

Condition (ii) requires that

$$\left[\mathbf{X}\,\frac{1}{w_o - sw_e}\right]_\infty = \left[\mathbf{X}\,\frac{1}{(w_{2p-1} - w_{2p-2})\, s^{2p-1}}\right]_\infty$$

exist, which is valid if and only if $[s^4\mathbf{Z}_e]_\infty$ exists. According to Lemma 4, condition (iii) is satisfied.

A moments reflection, in view of the two expressions in (43), will show that the determination of the polynomial $\eta(s)$ which leads to the weakest realizability conditions on $\mathbf{Z}(s)$ is essentially the same problem considered in the last section. The final result reads

Theorem 6:

The matrix $\mathbf{Z}$ *is realizable with an impedance* $z(s) = (s + 1)(\alpha s^2 + \alpha s + 1)^{-1}$, *where* $\alpha > 0$, *if and only if*

1. $\left[\frac{v_o}{sv_e}\right]_\infty = \frac{v_{2m+1}}{v_{2m}} \geqq 1, \quad \alpha s v_o - (\alpha s^2 + 1)v_e \not\equiv 0$
2. $[s^4\mathbf{Z}_e]_\infty$ *exists*
3. $1 - \alpha\varphi_2 \geqq 0$

where

$$\varphi_2 = \frac{1}{3}\left[1 - \left(\frac{v_{2m}}{v_{2m+1}}\right)^3\right] - \frac{v_{2m-2}}{v_{2m+1}} + \left[\frac{v_{2m}}{v_{2m+1}}\right]\left[\frac{v_{2m-1}}{v_{2m+1}}\right] - \frac{1}{12}\left[1 - \frac{v_{2m}}{v_{2m+1}}\right]^3$$

and $v(s) = \sum_{k=0}^{2m+1} v_k s^k$.

Further, if $\mathbf{Z}$ *satisfies these conditions, the corresponding* $\hat{\mathbf{Z}}$ *is realizable with* $\eta(s) = 1$ *when* $(v_{2m+1})(v_{2m})^{-1} = 1$ *and with*

$$\eta(s) = s + \tfrac{1}{2}\left[1 - \frac{v_{2m}}{v_{2m+1}}\right]$$

when $(v_{2m+1})(v_{2m})^{-1} > 1$.

VI. CONCLUDING OBSERVATION

It is of interest to note that the conditions presented in Theorem 5 $[z = (\alpha s^2 + \alpha s + 1)(s + 1)^{-1}]$ reduce to those of Theorem 3 $[z = (s + 1)^{-1}]$ as α approaches zero. However, a similar situation does not occur with respect to Theorems 4 and 6, for here the behavior of the matrix of even-parts at infinity is critically dependent upon whether $\alpha = 0$ or $\alpha > 0$.

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