Stimulated Emission of Bremsstrahlung

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(Manuscript received May 31, 1962)

A formula for the probability of stimulated emission of Bremsstrahlung is derived. It is shown that stimulated emission occurs if the incident electrons travel in a direction roughly parallel to the electric field vector of the wave stimulating the emission.

Emission from free electrons is used in electron tube devices. The purpose of this paper is to show that stimulated emission occurs already in the elementary process of the encounter of one electron and one nucleus or ion. There is no need for a slow-wave structure or elaborate electron bunching and no need to consider phase relationships. This elementary effect of stimulated emission should lead to a type of oscillator and broadband amplifier working without slow-wave structures or need for the close mechanical tolerances of high-frequency klystrons. A device of this kind may be noisier than a conventional maser.

It may be that the effect discussed in this paper is responsible for some of the hitherto unexplained semiconductor diode oscillations which have been reported in the literature.

I. INTRODUCTION

Stimulated emission of radiation from atoms or molecules is the process by which a maser operates. All masers use the transitions between bound states for their operation.

However, it is also well known that radiation can be emitted from a free electron in the presence of a static electric or magnetic field. The presence of this static field—for example the field of a nucleus—is necessary to simultaneously conserve energy and momentum during the emission or absorption process. Free electrons far from any other field can neither emit nor absorb photons of an infinitely extending radiation field because if they did, conservation of energy and momentum would be violated, as can easily be shown. Free electrons can only scatter photons, a phenomenon known as the Compton effect.

The emission of radiation from an electron passing by a nucleus is known as Bremsstrahlung. It will be shown in this paper that stimulated emission of Bremsstrahlung is possible if the incident electron travels more or less parallel to the electric field vector of the stimulating radiation field. The electron absorbs radiation if it travels more or less perpendicular to the electric field vector of the stimulating field.

Since stimulated emission of Bremsstrahlung exists, amplifiers and oscillators may be constructed using this effect. Stimulated emission of Bremsstrahlung does not require any bunched electron beams or observation of phase relationships. Moreover, it works better with slow than with fast electrons, and no traveling-wave structure is required since the fields for the stimulation process can be confined in a cavity.

II. STIMULATED EMISSION OF BREMSSTRAHLUNG

The theoretical principles required to derive the probabilities for stimulated emission or absorption from free electrons in the presence of a Coulomb field can be found in the textbook by Heitler.[1] We limit ourselves to nonrelativistic electron velocities and derive the probability for transitions between the following two states. The initial state consists of a free electron represented by a plane wave existing in the presence of a Coulomb and a radiation field with a certain number n of photons in a particular mode, while all other modes are empty. The final state consists of the same electron with different energy and momentum and with a number of $n + 1$ photons in the case of stimulated emission, or $n - 1$ photons in the case of absorption.

The transition probability per unit time is given by[2]

$$w = \frac{2\pi}{\hbar} \mid K_{F,0} \mid^2 \rho_F \tag{1)*$$

with $\hbar = 1.05 \times 10^{-27}$ erg·sec

$$K_{F,0} = \sum \left\{ \frac{V_{FI} H_{I0}}{E_o - E'} + \frac{H_{F\text{II}} V_{\text{II}0}}{E_o - E''} \right\}. \tag{2}$$

E_o is the initial energy of the whole system, while E' and E'' are the energies of the system in the intermediate virtual states. In the transitions to the intermediate states, energy need not be conserved. However, energy conservation is certainly required between the initial and final states of the whole system.

$$E_o = \frac{p_0^2}{2m} + n\hbar\omega \tag{3}$$

* A list of symbols is given below in Section VII.

$$E' = \frac{p'^2}{2m} + (n \pm 1)\hbar\omega \tag{4}$$

$$E'' = \frac{p''^2}{2m} + n\hbar\omega. \tag{5}$$

The (+) or (−) signs in (4) refer to emission and absorption respectively. $p = mv$ is the momentum of the electron and ω is the angular frequency of the electromagnetic field.

The summation in (2) extends over all possible intermediate states. The matrix elements of the interaction Hamiltonian are given by[3]*

$$\begin{aligned} H_{I0} &= -\frac{1}{\sqrt{L^3}}\frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega}}\begin{Bmatrix}\sqrt{n+1}\\ \sqrt{n}\end{Bmatrix}\frac{1}{L^3}\int e^{-i\mathbf{k}'\cdot\mathbf{r}}\,p_e\,e^{\mp i\boldsymbol{\beta}\cdot\mathbf{r}}\,e^{i\mathbf{k}^o\cdot\mathbf{r}}\,d^3r \\ &= \frac{\hbar(k_x^o \mp \beta_x)}{\sqrt{L^3}}\frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega}}\begin{Bmatrix}\sqrt{n+1}\\ \sqrt{n}\end{Bmatrix}\delta_{k_x',k_x^o\mp\beta_x}\cdot\delta_{k_y',k_y^o\mp\beta_y}\cdot\delta_{k_z',k_z^o\mp\beta_z}\end{aligned} \tag{6}$$

$$\begin{aligned} H_{FII} &= -\frac{1}{\sqrt{L^3}}\frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega}}\begin{Bmatrix}\sqrt{n+1}\\ \sqrt{n}\end{Bmatrix}\frac{1}{L^3}\int e^{-i\mathbf{k}^F\cdot\mathbf{r}}\,p_e\,e^{\mp i\boldsymbol{\beta}\cdot\mathbf{r}}\,e^{i\mathbf{k}''\cdot\mathbf{r}}\,d^3r \\ &= -\frac{\hbar(k_x''\mp\beta_x)}{\sqrt{L^3}}\frac{e}{m}\sqrt{\frac{2\pi\hbar}{\omega}}\begin{Bmatrix}\sqrt{n+1}\\ \sqrt{n}\end{Bmatrix} \\ &\qquad \delta_{k_x'',k_x^F\pm\beta_x}\,\delta_{k_y'',k_y^F\pm\beta_y}\,\delta_{k_z'',k_z^F\pm\beta_z}.\end{aligned} \tag{7}$$

The δ's are the Kronecker δ-symbols. The k's and β's can assume only values of the form $2\pi n'/L$ with integer n' as a result of box normalization. e and m are the charge and mass of the electron, n is the number of photons in a large box of volume L^3 (box normalization). The upper values $\sqrt{n+1}$ and (−) signs belong to the emission case, while the lower values $\sqrt{n}$ and (+) signs belong to the case of absorption, $\boldsymbol{\beta}$ is the propagation vector of the plane electromagnetic wave with $\beta = |\boldsymbol{\beta}| = \omega/c$, $\mathbf{k}$ is the propagation vector of the plane electron wave with $\hbar|\mathbf{k}| = mv$, and p_e is the component of the momentum operator of the electron in the direction of the electric field vector of the radiation field. We choose as z-direction the direction of propagation of the stimulating light wave

$$\boldsymbol{\beta} = (0,0,\beta). \tag{8}$$

The direction of polarization (direction of electric field vector) is taken to be the x-direction so that

* Electrostatic c.g.s. units will be used throughout this paper.

$$p_e = -i\hbar \frac{\partial}{\partial x} . \tag{9}$$

In comparing the matrix elements (6) and (7) with Heitler's formulas, it has to be remembered that Heitler writes all momenta multiplied by the velocity of light c. He also drops the normalization factor L^3 taking his box as being of unit volume. The integration extends over the box of volume L^3.

The matrix elements of the Coulomb potential are given by

$$V_{\text{II},0} = \frac{1}{L^3} Ze^2 \int \frac{e^{-i\mathbf{k}''\cdot\mathbf{r}} \cdot e^{i\mathbf{k}^o\cdot\mathbf{r}}}{|\mathbf{r} - \mathbf{r}_c|} d^3r = \frac{Ze^2}{L^3} \frac{4\pi}{|\mathbf{k}^o - \mathbf{k}''|^2} \tag{10}$$

$$V_{F,I} = \frac{1}{L^3} Ze^2 \int \frac{e^{-i\mathbf{k}^F\cdot\mathbf{r}} \, e^{i\mathbf{k}'\cdot\mathbf{r}}}{|\mathbf{r} - \mathbf{r}_c|} d^3r = \frac{Ze^2}{L^3} \frac{4\pi}{|\mathbf{k}' - \mathbf{k}^F|^2} . \tag{11}$$

The result of the integration holds in the limit $L \to \infty$ and can be found in Ref. 4. Z is the number of charges of the nucleus, and $\mathbf{r} - \mathbf{r}_c$ is the distance between the point of integration and the nucleus. The factor ρ_F in (1) is the number of final states per unit energy range of the electron after scattering. It is

$$\rho_F = \frac{mL^3k^F}{\hbar^2(2\pi)^3} d\Omega \tag{12}$$

with $\hbar k^F = mv_F$ the momentum of the electron after scattering and

$$d\Omega = \sin\psi \, d\psi \, d\alpha \tag{13}$$

the element of solid angle of the electron scattered in the direction ψ and α. The relative orientations of ψ and α and the angles θ and φ of the incident electron are shown in Fig. 1.

The form of the matrix elements (6) and (7) contains a physical approximation. We have limited ourselves to an expansion of the electron wave function in terms of plane electron waves, as is apparent from the factors $e^{\pm i\mathbf{k}\cdot\mathbf{r}}$ appearing under the integrals. Using plane waves to describe the electron corresponds to the Born approximation, which holds if

$$2\pi \frac{Ze^2}{\hbar v} \ll 1.$$

Our probability function w, of (1), describes the differential probability per unit time that an electron incident with angles θ and φ emits (or absorbs) a photon into the existing plane wave carrying n photons, and that the electron is found with energy

$$\tfrac{1}{2}mv_F^2 = \tfrac{1}{2}mv_0^2 \mp \hbar\omega \tag{14}$$

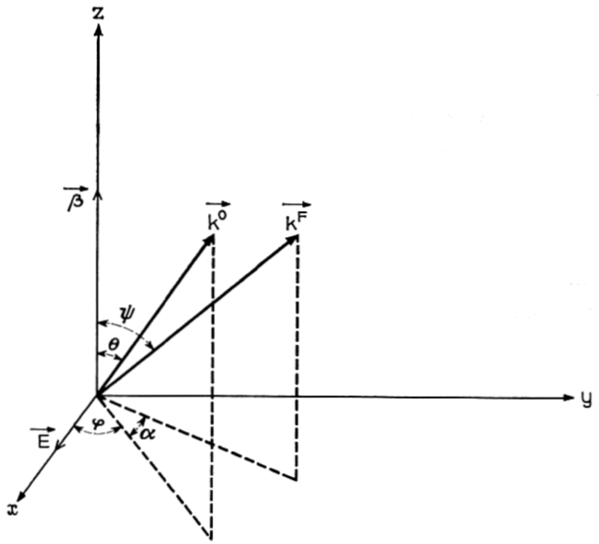

Fig. 1 — Relative orientation of the incident and scattered electron with respect to the direction of propagation and the electric vector of the radiation field.

traveling within the solid angle $d\Omega$ in the direction ψ,α. To obtain the total probability W per unit time regardless of the direction of the scattered electron, we have to integrate w over all directions of scattering.

Before proceeding further, we have to discuss the influence of the box normalization. The size of the box is arbitrary. The results become independent of the box if its sides L become infinitely long. It is apparent that as $L \rightarrow \infty$ we get $w \rightarrow 0$. To avoid this difficulty we consider that for a box of finite size we get as the number of emitted (or absorbed) photons per second

$$\Delta N = WN_e''$$

if N_e'' electrons are present in the box. Introducing the number N_e' of electrons per unit volume we get

$$\Delta N = WL^3N_e'.$$

Calling N_e the number of incident electrons which per second fly through the unit area at speed v_o , we obtain

$$\Delta N = W\frac{L^3}{v_o}N_e = \sigma N_e. \tag{15}$$

The quantity σ defined by (15) is the scattering or interaction cross section. This name is justified by the observation that σ has the dimension of an area.

The scattering cross section is, according to (15), defined by $\sigma = W(L^3/v_o)$. If we use the differential probability w of (1) instead of W, we obtain the differential scattering cross section

$$d\sigma = w\frac{L^3}{v_o}. \tag{16}$$

Using all the equations from (1) to (15), we obtain from (16) the differential cross section

$$d\sigma = \frac{8\pi e^6 Z^2 N k^F d\Omega}{m\omega v_o}\left\{\frac{k_x^{\,o}}{\left[\dfrac{p_o^{\,2}-p'^2}{2m} \mp \hbar\omega\right]\cdot\left[\mathbf{k}' - \mathbf{k}^F\right]^2} + \frac{k_x''}{\left[\dfrac{p_o^{\,2}-p''^2}{2m}\right]\cdot\left[\mathbf{k}^o - \mathbf{k}''\right]^2}\right\}^2. \tag{17}$$

The summation over all energy states reduces to one term because of the δ symbols in (6) and (7). We have taken

$$N = \frac{n}{L^3} \tag{18}$$

with N being the number of photons per unit volume. The normalization factor L has been eliminated from (17) and we can safely let $L \to \infty$. We see that the two factors $(n + 1)/L^3$ for the emission case and n/L^3 for the absorption case become identical as $L \to \infty$. The term 1 in $n + 1$ is related to the spontaneous emission of radiation. Since we calculate the transition probability (or scattering cross section) for emission of radiation into one well defined mode, without allowing for a spread in frequency or into several closely spaced modes traveling within a certain element of solid angle, the probability for spontaneous emission must be zero as $L \to \infty$.*

Only those terms give a contribution to the summation (2) for which

$$\mathbf{k}' = \mathbf{k}^o \mp \boldsymbol{\beta} \quad \text{or} \quad \mathbf{p}' = \mathbf{p}^o \mp \hbar\boldsymbol{\beta} \tag{19}$$

and

$$\mathbf{k}'' = \mathbf{k}^F \pm \boldsymbol{\beta} \quad \text{or} \quad \mathbf{p}'' = \mathbf{p}^F \pm \hbar\boldsymbol{\beta}. \tag{20}$$

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* If the radiation is confined to a cavity, L^3 is the volume of the cavity and the factor $1/L^3$ does not go to zero. The 1 in $(n + 1)/L^3$ gives rise to spontaneous emission of noise.

These relations, which follow from the δ-symbols in (6) and (7), mean that conservation of momentum is required even for the virtual states. We obtain with (19) and (20)

$$\mathbf{k}' - \mathbf{k}^F = \mathbf{k}^o - \mathbf{k}'' = \mathbf{k}^o - \mathbf{k}^F \mp \boldsymbol{\beta}. \tag{21}$$

With the help of Fig. 1 the following relation can be derived

$$\begin{aligned}[\mathbf{k}^o - \mathbf{k}^F \mp \boldsymbol{\beta}]^2 = k^{o2} + k^{F2} + \beta^2 \mp 2k^o\beta \cos\theta \pm 2k^F\beta\cos\psi \\ - 2k^ok^F(\cos\theta\cos\psi + \sin\theta\sin\psi\cos\alpha).\end{aligned} \tag{22}$$

Using (14) and (19) through (22) we obtain by integrating (20)

$$\begin{aligned}\sigma_{\substack{e\\a}} = \frac{2\pi m e^6 Z^2 N}{v_o\omega\beta^2 k^3\hbar^4}\int_o^\pi d\psi \int_{-\pi}^{\pi} d\alpha \sin\psi\sqrt{1\mp 2a\epsilon}\Bigg[\frac{\sin\theta\cos\varphi}{a-\cos\theta\pm\frac{1}{2}\epsilon} \\ + \frac{\sqrt{1\mp 2a\epsilon}\sin\psi\cos(\varphi+\alpha)}{\sqrt{1\mp 2a\epsilon}\cdot\cos\psi - a\pm\frac{1}{2}\epsilon}\Bigg]^2 \\ \cdot[1\mp a\epsilon + \tfrac{1}{2}\epsilon^2 \mp \epsilon\cos\theta \pm \epsilon\sqrt{1\mp 2a\epsilon}\cos\psi \\ - \sqrt{1\mp 2a\epsilon}(\cos\theta\cos\psi + \sin\theta\sin\psi\cos\alpha)]^{-2}.\end{aligned} \tag{23}$$

The upper subscript e of σ indicates emission and corresponds to the upper (+) or (−) signs in the equation. Conversely, the lower subscript a relates to absorption and corresponds to the lower signs.

We have used the abbreviations

$$\epsilon = \frac{\beta}{k^o} = \frac{\lambda_e}{\lambda} \tag{24}$$

with $\lambda_e = 2\pi/k^o$ being the wavelength of the electron wave and $\lambda = 2\pi/\beta$ the wavelength of the electromagnetic radiation.

$$a = \frac{c}{v_o} \quad \text{and} \quad a\epsilon = \frac{1}{2}\frac{\hbar\omega}{\frac{1}{2}mv_o^2}. \tag{25}$$

In going from emission to absorption, all we have to change is the sign of ϵ.

In order to decide if emission or absorption of radiation will actually occur, we have to take the difference

$$\sigma_T = \sigma_e - \sigma_a \tag{26}$$

between the cross sections for emission and absorption.

We do not have to evaluate the integral (23) exactly. In the range of physical interest we will always have $\epsilon \ll 1$ and also $a\epsilon \ll 1$ but $a \gg 1$

for nonrelativistic velocities. We limit ourselves to the case $a = c/v \gg 1$ and obtain approximately*

$$\sigma_{\substack{e\\a}} = \frac{2\pi m e^6 N Z^2 v}{\omega \beta^2 k^3 \hbar^4 c^2} \int_0^{2\pi} d\alpha$$
$$\int_0^{\pi} d\psi \frac{\sin\psi \sqrt{1 \mp 2a\epsilon}\,[\sin\theta\cos\varphi - \sqrt{1\mp 2a\epsilon}\sin\psi\cos(\alpha+\varphi)]^2}{[1 \mp a\epsilon - \sqrt{1\mp 2a\epsilon}\,(\cos\theta\cos\psi + \sin\theta\sin\psi\cos\alpha)]^2} \tag{27}$$

The double integral can be solved. The solutions have been expanded in terms of $a\epsilon$, and the difference between σ_e and σ_a has been taken.

We obtain to the lowest nonvanishing order of $a\epsilon = hf/mv^2$

$$\sigma_T = \frac{4e^6 N Z^2}{m^3 v^4 f^2}\left[(3\cos^2\varphi\sin^2\theta - 1)\ln\frac{2mv^2}{hf} - 2\cos^2\varphi\sin^2\theta\right]. \tag{28}$$

If $\varphi = 0$ or π and if $\theta = \pi/2$, the cross section σ_T assumes its largest positive value

$$\sigma_T = \frac{8e^6 N Z^2}{m^3 v^4 f^2}\left[\ln\frac{2mv^2}{hf} - 1\right]. \tag{29}$$

The direction of the incident electron defined by $\varphi = 0, \pi$, and $\theta = \pi/2$ is parallel to the electric vector of the radiation field (Fig. 1). Electron incidence parallel to the electric vector of the radiation field gives rise to stimulated emission of radiation.

Equation (28) shows that $\sigma_T < 0$ if $\varphi = \pi/2$. The electron is incident perpendicular to the radiation field and absorbs power from the field.

Let us take another look at (28) and determine the directions of incidence for which stimulated emission rather than absorption results.

Since an exact solution is hard to give, we will assume that the logarithmic factor in (28) is considerably larger than the remaining factor in the bracket consistent with the assumption $mv^2/hf \gg 1$.

We can then give approximate angles for which $\sigma_T = 0$.

If we let $\varphi = 0$, we obtain

$$\frac{\pi}{2} - \theta = 0.954 - \frac{0.236}{\ln\dfrac{2mv^2}{hf}}. \tag{30}$$

The angle 0.954 in radians corresponds to 54.6°.

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* Terms with ϵ (not $a\epsilon$) and terms of order $1/a$ are neglected. Terms with $a\epsilon$ have to be kept because the difference $\sigma_e - \sigma_a$ is proportional to $a\epsilon$.

If we let $\theta = \pi/2$ we obtain

$$\varphi = 0.954 - \frac{0.236}{\ln \frac{2mv^2}{hf}}. \tag{31}$$

All electrons incident on the ion within a cone with a cone angle of approximately 108°, whose axis is parallel to the electric vector of the stimulating radiation field, contribute to stimulated emission.

The number ΔN of emitted photons per second is obtained if we multiply σ_T by the number N_e of electrons which per second penetrate the unit area containing the nucleus with the charge Ze. If there is more than one nucleus interacting with the electron stream, we obtain the total number of emitted photons per second by multiplying the total number of nuclei N_n with the electron flux density N_e which per second interacts with them, provided the density of nuclei is low. For electrons incident parallel to the electric field vector $\varphi = 0$ or π, and $\theta = \pi/2$ we obtain from (29)

$$\frac{\Delta N}{N} = \frac{8e^6 Z^2 N_e N_n}{m^3 v^4 f^2}\left[\ln \frac{2mv^2}{hf} - 1\right]. \tag{32}$$

III. DISCUSSION

Equation (32) shows that the ratio of emitted photons per second to the stimulating photon density decreases rapidly as the electron velocity v increases. Within the limits set by $(mv^2/hf) \gg 1$ and by $2\pi(Ze^2/\hbar v) \ll 1$ we obtain more emission with slower electrons. The number of emitted photons increases also with decreasing frequency.

It may be useful to remark on the coherence of the process. We have to keep in mind that (32) gives the number of photons added precisely to the radiation mode which stimulates the emission. Let us assume that all other possible radiation modes are empty, $N = 0$. If we ask for the number of photons emitted into one of the empty modes, our results would be proportional to $1/L^3$, but it is proportional to $(hf/mv^2)\cdot(n/L^3)$ in case of emission into the radiation mode filled with n photons. (Remember the remark about $(n + 1)/L^3$ following (18). The factor hf/mv^2 arises by taking the difference $\sigma_e - \sigma_a$. The zero-order term drops out and the result is proportional to $a\epsilon = hf/mv^2$.) The radiation from the electron occurs preferentially into the occupied mode with a probability ratio equal to $[(hf)/(mv^2)]n$ (n is a big number even for moderate power). The power added to the occupied radiation mode is coherent with the radiation present in that mode because each photon adds precisely the energy hf. We have here the same situation which exists for stimulated emission from bound states, with the difference that the spontaneous emission of bound electrons has a well defined frequency while the free electron radiates into a very wide frequency band. Because of the ratio of spontaneous to stimulated emission, we would assume that the process discussed here is inherently noisier than stimulated emission from bound states. For the latter, the ratio of stimulated to spontaneous emission into the occupied mode is n while it is only $[(hf)/(mv^2)]n$ in our case. Apparently we get better results for slow electrons.

We want to state the oscillating condition for achieving self-sustained oscillation if the radiation is confined in a cavity. The cavity has a loaded Q defined by

$$Q_L = \omega \frac{NVhf}{n'hf} = \frac{\omega NV}{n'} \tag{33}$$

with cavity volume V, photon density N, and n' dissipated photons per second. The cavity oscillates if $n' = \Delta N$. With the help of (32), this leads to an expression for the minimum product necessary to achieve oscillation if we assume again that the electrons are incident parallel to the **E**-vector.

$$N_e N_n = \frac{\pi m^3 v^4 f^3 V}{4e^6 Z^2 Q_L \left[\left(\ln \frac{2mv^2}{hf}\right) - 1\right]}. \tag{34}$$

The oscillating frequency is determined only by the cavity. If many cavity modes can exist, oscillation will start in the mode with the smallest value of f^3/Q_L.

For use as an amplifier the bandwidth could be much larger than the bandwidth of conventional masers because there is no built-in resonance in this process to limit the frequency.

To build an amplifier or oscillator using stimulated emission from free electrons, we want to shoot dense electron beams of low energy through as dense an ensemble of ions as possible. This can be done by using ion beams rather than ions in a plasma because in a plasma scattering of the thermal plasma electrons by the plasma ions would have an adverse effect.

It is also possible to use electron scattering due to ionized impurities in a crystal to obtain stimulated emission. To apply our theory to electrons moving in conduction bands of crystals, we have to replace the electron mass m by the effective mass m_{eff} of the electron in the crystal. Furthermore, we have to multiply the scattering cross section σ by $1/\epsilon^3$ with ϵ being the dielectric constant of the crystal.

Experimentally, one has to take care that the conduction electrons move predominantly in the direction of the electric vector of the radiation field. This is achieved, for example, by lifting electrons into the conduction band with the help of the photoelectric effect using a crystal which in the dark is an insulator. The electrons will move predominantly in the direction of the electric vector of the pump light provided that this is polarized.

Another way of generating electrons moving in preferred directions is by the use of dc electric fields which exceed the breakdown voltage of an insulating or poorly conducting crystal.

The crystal will have to be cooled to increase the collision time for electron collisions with the vibrating lattice which have an adverse effect on the ordered electron motion and may not give rise to stimulated emission.*

Another way to achieve stimulated emission of Bremsstrahlung is the use of crossed electron and ion beams. These two beams can be made to cross in a capacitor which is part of an LC resonant circuit. The use of beams in vacuum will limit the application of our effect to low-frequency amplifiers and oscillators because of the limitation in available ion densities.

Equation (34) can be applied to the case of an LC circuit. In this case, V is the volume of the capacitor and Q_L the loaded Q of the LC circuit. An independent calculation has shown that (34) can be obtained if the voltage and current in the LC circuit rather than the electric and magnetic field in the capacitor are quantized. The photons in this case are the units of energy hf which are stored in the resonant circuit.

IV. NUMERICAL EXAMPLES

i. We assume an LC circuit with $Q_L = 30$ tuned to 70 mc, a capacitor volume $V = 12.5\text{ cm}^3$. The electrons are assumed to be accelerated by a potential of 10 volts, corresponding to a velocity of 2×10^8 cm/sec and to drift through the capacitor plates, which are made out of a wire mesh. To obtain oscillations, we obtain from (34) ($Z = 1$ is assumed)

$$N_e N_i = 4.6 \times 10^{29}\text{ sec}^{-1}\text{ cm}^{-2}.$$

If we assume that an electron beam with 10 ma/cm^2 current density, corresponding to $N_e = 6.3 \times 10^{16}\text{ sec}^{-1}\text{ cm}^{-2}$, is employed, $N_n = 7.3 \times 10^{12}$ ions are needed in the interaction region to make the circuit oscillate.

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* So far we have studied stimulated emission in the presence of Coulomb potentials. Whether other scattering potentials can be used will have to be determined by another study.

Assuming that the capacitor is made of a mesh of 5×5 cm^2 with a spacing of 0.5 cm, and assuming further that the ions are accelerated by a 10-volt potential, we find that the ion current of 54 ma has to pass through the capacitor (Cs ions assumed) in order to provide $N_n = 7.3 \times 10^{12}$ ions in the capacitor at any instant of time.

ii. We consider a solid with a dielectric constant $\epsilon = 10$, an effective electron mass $m_{\text{eff}} = 0.1\, m$ and assume that the material is contained in a cavity with volume $V = 20$ cm^3 which is resonant at 10 kmc with a $Q_L = 10{,}000$. The electron velocity is assumed to be $v = 10^7$ cm/sec. We obtain, taking $Z = 1$,

$$N_e N_n = 2.8 \times 10^{35} \text{ sec}^{-1} \text{ cm}^{-2}.$$

If we take the electron current density equal to the one used in the first example, $N_e = 6.3 \times 10^{16}$ sec^{-1} cm^{-2}, we obtain as the number of ions in the interaction region necessary to sustain oscillations, $N_n = 4.45 \times 10^{18}$ or a density of 2.2×10^{17} ions/cm^3.

The frequency could be increased further if it were possible to increase the ion and electron densities. Increasing the frequency to 100 kmc increases the product $N_e N_n$ by a factor of 1000. It may be possible to increase N_e as well as N_n by a factor of 30 and push the operating frequency to 100 kmc. Increasing the Q of the cavity would also help.

V. A COMPARISON WITH CLASSICAL THEORY

It is interesting to assume that electrons are incident from all possible directions with a uniform distribution over all angles of incidence.

We obtain from (28)

$$\bar{\sigma} = \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi \sigma_T = -\frac{32\pi e^6 N Z^2}{3m^3 v^4 f^2}. \tag{35}$$

Electrons incident with a random distribution over all directions lead to a net absorption of radiation. This absorption process gives rise to attenuation of waves traveling through plasmas or semiconductors. However, in a semiconductor our process gives only the contribution of impurity scattering but not of lattice scattering. Equation (35) does not contain h any more and must be equivalent to classical theories.

Equation (35) could be used to derive an expression for the attenuation constant α in an ionized gas. However, the approximations implied in this work do not allow the electron velocity to become arbitrarily small. We will, nevertheless, use (35) to derive the attenuation constant per centimeter of a plane wave in a plasma with electron density n_e and ion density n_i with the assumption that the electrons obey the Maxwell-Boltzmann velocity distribution.

$$f(v) = 4\pi v^2 \left[\frac{m}{2\pi kT}\right]^{3/2} \exp\left(-\frac{mv^2}{2kT}\right). \tag{36}$$

It is

$$\alpha = -\frac{\Delta N}{Nc} \tag{37}$$

where ΔN is the number of photons created per centimeter per second, c is the velocity of light, and N is the photon density.

We obtain with

$$\Delta N = \frac{n_i n_e}{4\pi} \int_0^\infty vf\,(v)\,\bar{\sigma}\,dv \tag{38}$$

from (37) with the help of (35), (36) and (38)

$$\alpha = \left\{\frac{4}{3}\sqrt{\frac{2\pi}{3}}\frac{1}{(kT)^{3/2}}\frac{n_i n_e e^6 Z^2}{m^{3/2} f^2 c}\right\}\frac{4\sqrt{3}}{2\pi}\int_0^\infty \frac{1}{x}\,e^{-x^2}\,dx. \tag{39}$$

The integral in (39) cannot be evaluated since it requires an integration over

$$x = v\sqrt{\frac{m}{2kT}}$$

starting from zero. The integral is logarithmically divergent. However, our theory does not allow us to apply (35) for arbitrarily small electron velocities. Nevertheless, it is interesting to note that the part of (39) inside the brackets equals exactly equation (5-48) of Ref. 5 if we follow Spitzer's indication and multiply his formula with

$$1 - e^{\frac{hf}{kT}} \approx \frac{hf}{kT}$$

to take stimulated emission into account. The divergent factor outside the brackets should be one. We cannot derive its value because of the limitations of our approximation.

VI. SUMMARY

We have presented an approximate theory of stimulated emission of Bremsstrahlung. The theory considers the process of stimulated emission which occurs if a stream of electrons is scattered by one individual ion or nucleus in the presence of a radiation field. The approximations require that

$$\frac{c}{v} \gg 1, \quad \frac{hf}{mv^2} \ll 1, \quad \text{and} \quad 2\pi \frac{Ze^2}{\hbar v} \ll 1.$$

We have found that stimulated emission occurs if the directions of the incident electrons lie within a cone whose axis is parallel to the electric vector of the stimulating radiation and whose cone angle is approximately 108°.

The effect is fairly weak, so that rather high ion and electron densities are required to achieve substantial amplification or oscillation.

It is predicted that the effect will work best in a solid when the conduction electrons are scattered by ionized impurities.

It may be that this effect is responsible for the recently reported oscillations which were obtained with the use of semiconductor diodes and which seem presently to be unexplained.[6,7,8]

It is possible that other scattering potentials may lead to stimulated emission. We have restricted ourselves to Coulomb scattering. It may be worthwhile to study other scattering potentials, for example electron scattering by the vibrating lattice in semiconductors.

VII. LIST OF SYMBOLS

α	used as azimuth of the scattered electron relative to the incident electron; also attenuation constant
$\beta = \omega/c$	propagation constant of electromagnetic wave
c	phase velocity of electromagnetic wave
$e = 4.803 \times 10^{-10}$ e.s.u.	charge of the electron
f	frequency of the electromagnetic wave
φ	azimuth of the incident electron
$h = 6.624 \times 10^{-27}$ erg·sec.	Planck's constant
$\hbar = h/2\pi = 1.054 \times 10^{-27}$ erg·sec.	
H_{10}	matrix element of radiation field
k	propagation constant of electron wave; also Boltzmann's constant $= 1.38 \times 10^{-16}$ erg·degree^{-1} (appears only in the combination kT)
L	length of fictitious box used for box normalization

$m = 9.107 \times 10^{-28}$ gram	mass of the electron
n_i	ion density
n_e	electron density
N_e	number of electrons penetrating the unit area per unit time
N_n	number of ions or nuclei
N	photon density
Ω	solid angle
$\omega = 2\pi f$	angular frequency of the electromagnetic wave
$\mathbf{p} = \hbar\mathbf{k}$	momentum of the electron
ψ	polar angle of the scattered electron
Q_L	loaded Q of the resonant cavity
ρ_F	density of physical states per unit energy
σ	interaction cross section
T	absolute temperature
θ	polar angle of the incident electron
v	electron velocity
V_{F0}	matrix element of the Coulomb potential
w	differential transition probability per unit time
W	transition probability per unit time
Z	number of elementary charges of the ion.

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