Intervals between Periods of No Service in Certain Satellite Communication Systems — Analogy with a Traffic System

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(Manuscript received May 11, 1962)

In satellite systems in which the relative positions of satellites are allowed to vary, there will be periods during which no service will be provided between given ground stations. Such periods are called "outages," and the intervals between successive outages are called "innages." Here the outage and innage time distributions are studied with the help of an analogy between a satellite system and a traffic system. The arrival of a customer in the traffic system corresponds to a satellite coming into view, and the service time of the customer corresponds to the time the satellite remains in view. In particular, the methods of analysis developed for traffic systems are applied to determine an approximation for the distribution of innage lengths.

I. INTRODUCTION

Several types of communication systems have been proposed which would use repeaters orbiting the earth as artificial satellites. The problem considered in this paper arises in systems employing a number of satellites at altitudes of several thousand miles. Typically, the orbit altitude might be of the order of 5000 miles with a period of revolution of about 5 hours.

The companion paper by Rinehart and Robbins[1] discusses the conditions under which a particular satellite will be visible to a given pair of ground stations. For the orbit altitudes considered here, the satellite will be visible intermittently. Conceivably the relative positions of the satellites could be maintained so that at least one satellite is mutually visible from the two ground stations at all times. However, at least for some of the early systems proposed, it is of interest to consider the case in which small differences in orbital period cause the relative positions of the satellites to vary with time. Attention is therefore directed to the statistical characteristics of satellite visibility.

We are especially interested in those periods during which no satellite is available for communication between a given pair of ground stations. For convenience these events are called *outages* although, as pointed out by Rinehart and Robbins, these occurrences need not imply an interruption of calls in progress. By analogy the intervals between outages are called *innages*.

For any particular system the most effective method of obtaining outage information appears to be that of simulation. The course of the system for a year or more is computed, with the help of a high-speed digital computer, and the outages and innages recorded. [A method for doing this and some sample results are presented in the paper by Rinehart and Robbins].

The present paper is concerned with determining the distribution of outage and innage lengths. The theory of outage length distribution has been studied by a number of people and some of their results, namely those which are needed here, are summarized in Section II. The traffic model is described in Section III. Sections IV and V contain results predicted by the model. In Section VI the predicted results are compared with those obtained by simulation. The work of Appendix B gives the basis for a model which is simpler but less accurate than the one described in Section III. The material in both appendices also appears to be of interest from the standpoint of traffic theory.

The general conclusion is that, for the cases examined, both the outage and innage distributions are approximately exponential. Their averages are related by the rather simple equation (1).

II. PRELIMINARY RESULTS

First note that the average innage length $\bar{t}_i$ and the average outage length $\bar{t}_o$ ("i" for innage and "o" for outage) are connected by the relation

$$\bar{t}_i = \frac{q}{1 - q}\,\bar{t}_o \tag{1}$$

where q is the quality of service; i.e., the fraction of time transmission is possible. This relation is always true and follows almost immediately from the definition of q. Typical values are $q = 0.99$ and $\bar{t}_o = 0.25$ hours; and it follows that the corresponding average innage length is $\bar{t}_i =$ 24.75 hours.

Next, let k be the number of satellites in the system. Consider a particular satellite. Every now and then it will pass over the region of mutual visibility. Let

T = average length of time between its reappearances. For polar orbits, T is the orbit time if the region of visibility includes one of the poles.

b^{-1} = average length of time the satellite is visible during one pass. The quantity b is a rate which occurs frequently in the analysis.

p = fraction of time the satellite is visible. From these definitions it follows that

$$b^{-1} = pT. \tag{2}$$

The satellite systems considered here are restricted to those for which T and b^{-1} are almost the same for all k satellites.

With this notation we have

$$q = 1 - (1 - p)^k, \tag{3}$$

$$\bar{t}_o = (1 - p)\, T/k, \tag{4}$$

$$P(t) = \left[1 - \frac{t}{(1 - p)T}\right]^{k-1}, \qquad 0 \leqq t \leqq (1 - p)T \tag{5}$$

where $P(t)$ is the probability that the length of an outage will exceed t. Also the expected number of satellites visible at a given time is kp. These formulas are based upon the assumption that the phase angles of the satellites are independent random variables and are distributed uniformly over the interval $(0,2\pi)$. Equation (3) is due to J. R. Pierce and R. Kompfner.[2] Equations (4) and (5) have been given by R. E. Mosher and R. I. Wilkinson, respectively, in unpublished memoranda. For values of t and k of interest, (5) may be approximated by

$$P(t) = \exp\,(-\, t/\bar{t}_o). \tag{6}$$

It will be observed that the constants of the satellite system enter the right-hand sides of (3) to (5) only through the three parameters k,p,T. Thus, as far as these formulas are concerned, the satellite system is specified by k,p,T.

III. TRAFFIC SYSTEM MODEL

The satellite system will be represented by a traffic system model which consists of k independent servers, each having an average service time b^{-1} and a service time distribution function $B(t)$. Customers are supplied to the servers by k independent Poisson sources, each producing customers at an average rate α. When a source produces a customer, that particular source is removed from the system while the customer is being served. Thus, when n servers are busy serving n customers, the average arrival rate is $(k - n)\,\alpha$. This type of input is a special case of a more general type (the limited source or "Engset input") which has been studied in traffic theory.

The instant a satellite becomes visible from both the receiver and transmitter corresponds to the arrival of a customer. The length of time a satellite remains visible corresponds to the time required to serve a customer. After a customer has been served he leaves the system. This corresponds to the satellite leaving the region of mutual visibility. The state in which n satellites are visible simultaneously (state n) corresponds to the state in which n servers are busy serving n customers. An outage corresponds to an idle period (state 0), i.e., a period during which all servers are idle. An innage corresponds to a busy period, i.e., to a period when one or more servers are busy.

Note that the constant orbit time of the satellite may introduce a regularity in the satellite arrivals. The traffic model has the shortcoming that there is no corresponding regularity in the customer arrivals.

The analogy between the satellite system and the model is established by taking k and b^{-1} to have the same values in both and setting

$$\alpha = 1/(1 - p)T. \tag{7}$$

To justify this choice for α, note that if a particular satellite is not visible at a time t selected at random, the chance that it will become visible in t, $t + dt$ is $dt/(T - pT)$. Comparison with the corresponding probability $\alpha\, dt$ for the traffic system gives (7). When the three satellite system parameters k,p,T are known, the three model parameters k, b^{-1}, α follow at once from (7) and $b^{-1} = pT$.

IV. OUTAGE AND QUALITY OF SERVICE PREDICTED BY MODEL

The values of the quality of service q and average outage length $\bar{t}_o$ predicted by the model agree exactly with (3) and (4) while the predicted outage length distribution is the exponential approximation (6) to Wilkinson's polynomial expression. A sketch of the proof of these statements is given in the following paragraphs.

First assume the service distribution $B(t)$ to be exponential, i.e., equal to $1 - e^{-bt}$. Then since the sources are Poisson, the behavior of the system is governed by the $k + 1$ state equations (Ref. 3, p. 30)

$$
\begin{aligned}
P_0' &= -k\alpha P_0 + bP_1 \\
P_1' &= k\alpha P_0 - [(k-1)\alpha + b]P_1 + 2bP_2 \\
P_2' &= (k-1)\alpha P_1 - [(k-2)\alpha + 2b]P_2 + 3bP_3 \\
P_3' &= (k-2)\alpha P_2 - [(k-3)\alpha + 3b]P_3 + 4bP_4 \\
& \cdots \\
P_k' &= \alpha P_{k-1} - kbP_k
\end{aligned}
\tag{8}
$$

where $P_n \equiv P_n(t)$ is the probability the system is in state n at time t and primes denote time derivatives. The steady state probability p_n that exactly n customers are present at a time picked at random is

$$p_n = \binom{k}{n}(\alpha b^{-1})^n p_0, \qquad p_0 = (1 + \alpha b^{-1})^{-k}. \tag{9}$$

This follows upon setting the derivatives in (8) to zero, taking $P_n(t) = p_n$, and solving the equations step by step. The expected number $\bar{n}$ of customers present is $k\alpha b^{-1}/(1 + \alpha b^{-1})$ and the average arrival rate is

$$\alpha(k - \bar{n}) = k\alpha/(1 + \alpha b^{-1}). \tag{10}$$

The quality of service is now

$$q = 1 - p_0 \tag{11}$$

and expression (3) for q may be obtained by using $\alpha = 1/(1 - p)T$, $b^{-1} = pT$. Since, (*i*) an outage corresponds to state 0, (*ii*) state 0 can end only through an arrival, and (*iii*) the arrivals in state 0 are Poisson with rate $k\alpha$, it follows that $\exp(-k\alpha t)$ is the probability (predicted by the model) that the length of an outage will exceed t. This agrees with the exponential approximation (6), and the average value $\bar{t}_0 = 1/(k\alpha)$ agrees with (4).

It is known that expression (9) for the steady state probability p_n holds not only for exponential service but also for the general service distribution $B(t)$ (Ref. 3., p. 90). Hence the model predicts that expressions (3), (4) and (6) still hold when the length of time a particular satellite stays in view has an arbitrary distribution $B(t)$.

V. INNAGE DISTRIBUTION PREDICTED BY MODEL

The average innage length (i.e., average busy period) predicted by the model when the service distribution $B(t)$ is arbitrary follows from $\bar{t}_o = 1/k\alpha$ and $q = 1 - p_0$:

$$\bar{t}_i = \frac{1 - p_0}{p_0} \bar{t}_o = [(1 + \alpha b^{-1})^k - 1]/k\alpha.$$

This much is easy. It is much more difficult to get the complete distribution, as the following work shows.

When the service lengths are exponentially distributed, the busy period distribution may be obtained by solving a "first passage" problem. State 0 is made absorbing and the system is started with an arrival at time 0. Thus the system starts in state 1 at time 0 and jumps from state to state in accordance with the arrivals and departures of customers. The system eventually lands in state 0 and stays there. This corresponds to the end of the busy period or innage.

When state 0 is made absorbing, the first two of the $k + 1$ state equations (8) are replaced by

$$\begin{aligned} P_0' &= bP_1 \\ P_1' &= -[(k-1)\alpha + b]P_1 + 2bP_2 . \end{aligned} \tag{12}$$

The modified equations (8) are to be solved subject to $P_1 = 1$ and $P_0, P_2, \cdots, P_k = 0$ at time 0. The probability that the length of an innage will exceed the length t is

$$G(t) = 1 - P_0 . \tag{13}$$

Step-by-step computation of the derivatives of P_0 at $t = 0$ from the differential equations gives the power series

$$\begin{aligned} G(t) = 1 - \frac{bt}{1!} &+ \frac{[(k-1)\,\alpha + b]bt^2}{2!} \\ &- \frac{[(k-1)^2\,\alpha^2 + 4(k-1)\,\alpha b + b^2]\,bt^3}{3!} \\ &+ [(k-1)^3\,\alpha^3 + (k-1)(9k-11)\,\alpha^2\,b \\ &\qquad + 11(k-1)\,\alpha b^2 + b^3]\,\frac{bt^4}{4!} + \cdots \end{aligned} \tag{14}$$

which is useful for small values of t.

Since P_1 is determined by the last k differential equations of the modified set, and since the coefficients in these equations are constants, we may expect P_1 [and hence $G(t)$] to be expressible as the sum of k exponential terms. Indeed, when $\varphi_n(s)$ is used to represent the Laplace transform of $P_n(t)$, the k differential equations for $P_1(t), \cdots, P_k(t)$ go into k linear equations for $\varphi_1(s), \cdots, \varphi_k(s)$. By introducing the generating function

$$\Phi(x, s) = x\varphi_1(s) + \cdots + x^k\varphi_k(s)$$

in the usual way, the linear equations may be combined into

$$b\varphi_1(s) - x = (1 - x)(b + \alpha x)\frac{\partial\Phi(x, s)}{\partial x} - (s + \alpha k - \alpha kx)\Phi(x, s).$$

To obtain $\varphi_1(s)$, rewrite this equation as

$$[b\varphi_1(s) - x](1 - x)^{z-1}(b + \alpha x)^{-z-k-1} = \frac{\partial}{\partial x}(1 - x)^z(b + \alpha x)^{-z-k}\Phi(x, s) \tag{15}$$

where $z = s/(\alpha + b)$. Assume Re $(z) > 0$ and integrate (15) from $x = 0$ to $x = 1$. The right-hand side vanishes because $\Phi(0, s)$ is zero. Changing the variable of integration on the left-hand side from x to $y = (1 - x)/(b + \alpha x)$ gives

$$b\varphi_1(s)\int_0^{1/b} y^{z-1}(1 + \alpha y)^k\,dy = \int_0^{1/b}(1 - yb)y^{z-1}(1 + \alpha y)^{k-1}\,dy. \tag{16}$$

Expanding $(1 + \alpha y)^k$ and integrating termwise shows that the coefficient of $b\varphi_1(s)$ is $b^{-z}F_k(z)$ where

$$F_k(z) = \sum_{n=0}^{k}\binom{k}{n}\frac{(\alpha b^{-1})^n}{z + n}. \tag{17}$$

When the integrand on the right-hand side of (16) is replaced by its equivalent

$$\left[1 + \frac{z(\alpha + b)}{k\alpha}\right]y^{z-1}(1 + \alpha y)^k - \frac{\alpha + b}{k\alpha}\frac{d}{dy}y^z(1 + \alpha y)^k$$

(which is suggested by an integration by parts and adding and subtracting various terms) we obtain

$$b\varphi_1(s)F_k(z) = \left(1 + \frac{s}{k\alpha}\right)F_k(z) - \frac{b}{k\alpha}(1 + \alpha b^{-1})^{k+1}.$$

From $P_0' = bP_1$ it follows that $s\varphi_0(s) = b\varphi_1(s)$ and hence

$$\varphi_0(s) = \frac{1}{s} + \frac{1}{k\alpha} - \frac{b(1 + \alpha b^{-1})^{k+1}}{k\alpha sF_k(z)}.$$

As $s \to \infty$, $F_k(z)$ tends to $(1 + \alpha b^{-1})^k/z$ and $\varphi_0(s)$ is $O(1/s)$. Writing

$\varphi_0(s)$ as the sum of partial fractions, inverting to obtain $P_0(t)$, and using (13) gives the expression we seek:

$$G(t) = \sum_{m=0}^{k-1} \frac{b(1 + ab^{-1})^{k+1} \exp[(\alpha + b)z_m t]}{k\alpha z_m F_k'(z_m)} \tag{18}$$

where $F_k'(z) = dF_k(z)/dz$ and $z_0, z_1, \cdots, z_{k-1}$ are the zeros of $F_k(z)$. These zeros lie between the poles at $0, -1, -2, \cdots, -k$. When t is large $G(t)$ is given, effectively, by the term corresponding to $m = 0$ in (18). Usually z_0 is close to $z = 0$ and may be obtained by successive approximations from $F_k(z) = 0$, i.e., from

$$\frac{1}{z} = -\sum_{n=1}^{k} \binom{k}{n} \frac{(\alpha b^{-1})^n}{z + n}.$$

The foregoing method is the one originally used to obtain $G(t)$ as the sum of exponential terms. Subsequently, a more elegant method of obtaining (18) for $G(t)$ was suggested by L. Takács. His method removes the restriction that the service time distribution be exponential.

Takács' result is the following: Let $\beta(s)$, $\gamma(s)$ be the Laplace transforms of $B'(t)$, $-G'(t)$, the service time and busy period probability densities, respectively. The equation to determine $\gamma(s)$, given $\beta(s)$, is

$$\frac{1}{s + k\alpha - k\alpha\gamma(s)} = \int_0^\infty e^{-st}[P_{aa}(t)]^k \, dt \tag{19}$$

where $P_{aa}(t)$ has the Laplace transform $1/[s + \alpha - \alpha\beta(s)]$. It will appear later that the subscript a refers to the idle state (state 0). The expression (18) for $G(t)$ may be obtained from (19) by starting with $\beta(s) = b/(s + b)$. It turns out that $P_{aa}(t)$ is given by $[b + \alpha\, e^{-(\alpha+b)t}]/(b + \alpha)$ so that the Laplace transform of $[P_{aa}(t)]^k$ is not difficult to compute.

One way to establish (19) is to regard the model as composed of k independent simple systems, each consisting of a source connected to a server. Consider a simple system. The lengths of the idle and busy periods have the respective probability densities $\alpha\, e^{-\alpha t}$, $B'(t)$ with Laplace transforms $\alpha/(\alpha + s)$, $\beta(s)$.

From (26) of Appendix A, the probability $P_{aa}(t)$ that an idle period will be in progress at time t, given that one is in progress at time 0, may be determined by inverting its Laplace transform $1/[s + \alpha - \alpha\beta(s)]$. The probability that all k servers are idle at time t, given that they are idle at time 0, is $[P_{aa}(t)]^k$. Equation (19) now follows upon using (26) again. This time the type (a) intervals correspond to the periods (outages) during which all k servers are idle. The arrival rate is αk and $p_a(t)$ is $\alpha k\, e^{-\alpha k t}$. The type (b) intervals correspond to the innages with probability density $-G'(t)$ and Laplace transform $\gamma(s)$.

VI. COMPARISON WITH SIMULATION

It is interesting to compare the results predicted by the model with those obtained by simulation. As an example, we shall take results obtained by Rinehart and Robbins for a Maine-Western Europe link. This link was assumed to have 18 satellites in random polar orbits at a height of 6,000 nautical miles.

The positions of the orbits, the locations of the receiver and transmitter, and some computations involving a number of representative passages over the region of mutual visibility lead to

1. the value $T = 6.35$ hours for the average interval between reappearances of a particular satellite,
2. the distribution function $B(t)$ for l, the length of time it is visible,
3. an average value of l equal to $b^{-1} = 1.46$ hours,
4. the value $p = b^{-1}/T = 0.230$ for the fraction of time the satellite is visible.

It turns out that the probability density $B'(t)$ can be approximated by the rectangle

$$B'(t) = \begin{cases} 0 \;, & t < 0.86 \text{ hours} \\ 0.833, & 0.86 < t < 2.06 \\ 0 \;, & 2.06 < t. \end{cases} \tag{20}$$

The system was required to furnish a quality of service close to 0.99. This requirement together with the expression $q = 1 - (1 - p)^k$ and the value $p = 0.230$ gives $k = 18$ and $q = 0.99094$. The model parameters are therefore taken to be

$$k = 18, \qquad b^{-1} = 1.46 \text{ hours}, \qquad \alpha = \frac{1}{(1-p)T} = \frac{1}{(0.770)(6.35)} = 0.205 \tag{21}$$

with the understanding that $p = 0.230$.

The values of q, $\bar{t}_o$, $\bar{t}_i$ obtained by simulation are compared with those predicted by the formulas of Section II (and also by the model) in Table I.

The second column gives values obtained by Rinehart and Robbins by a simulation which followed the system for 18 months. The third column gives values computed from the model parameters (21). It is

TABLE I

Quantity	Simulation	Model, p = 0.230	Model, p = 0.234
q	0.9918	0.99094	0.9918
$\bar{t}_o$	0.291 hours	$1/k\alpha = 0.271$	0.270
$\bar{t}_i$	35.1 hours	$q(1-q)^{-1}\bar{t}_o = 29.6$	32.6
b^{-1}	—	1.46 hours	1.48

seen that the values of the average outage length $\bar{t}_o$ agree fairly well. The discrepancy in $\bar{t}_i$ reflects the shortcomings of the model. The last column shows what happens when we hold k and T at their former values of 18 and 6.35, and fudge the value of p so as to make q have the simulation value 0.9918. This changes p from 0.230 to 0.234 and $\bar{t}_i$ from 29.6 to 32.6. It is seen that the value of $\bar{t}_i$ is very sensitive to such changes.

Fig. 1 shows three curves for the innage length distribution. The ordinate is $G(t)$, the probability that an innage length will exceed t. Curve A is the curve predicted by the model assuming exponential service, and is obtained by substituting the parameter values (21) in expressions (14) and (18) for $G(t)$. Curve B is the exponential approximation

$$G(t) = e^{-t/\bar{t}_i} \tag{22}$$

with $\bar{t}_i$ equal to the simulation value 35.1 hours. Curve C is the result obtained by simulation. During the 18 months simulated there were 122 innages, the longest lying between 250 and 260 hours, the next longest between 190 and 200 hours, and so on. The average innage lengths corresponding to curves A and C are $\bar{t}_i = 29.6$ and $\bar{t}_i = 35.1$, respectively, in agreement with the table above.

The curves shown in Fig. 1 agree moderately well. The agreement would be improved if curve A could be shifted so as to give an average value of 35.1 instead of 29.6. Some of the discrepancy between curves A and C around $t = 0$ can be ascribed to the assumption of exponential service. Better agreement in this region could be obtained by taking the model to have the (almost) true service distribution (20). The procedure for doing this is indicated by (19) but the task of carrying through the work seems to be difficult. Again, it should be possible to use the "Erlang service" approximation $4b^2t \exp(-2bt)$ for $B'(t)$ in (19) and also to solve the corresponding first passage problem [i.e., solve the equations corresponding to (8) and (12)]. However, this was not attempted.

Some idea of the change produced in $G(t)$ when exponential service is replaced by other kinds of service may be obtained from Fig. 2. The curves of Fig. 2 show $G(t)$ for the simplified model based upon the results of Appendix B. Exponential service (curve D) and constant service time (curve E) are assumed, and Poisson arrivals are taken for both cases. The average arrival rate is $a = k/T = 18/6.35 = 2.84$ per hour and the average service time is $b^{-1} = 1.46$ hours. Substitution of these values in the expressions for $G(t)$ given in examples (a) and (b) of Appendix B give curves D and E, respectively.

The expanded scale at the top of Fig. 2 shows the behavior of $G(t)$ around $t = 0$. Both distributions predict the same average innage length, namely

$$\bar{t}_i = [e^{ab^{-1}} - 1]a^{-1} = 22.0 \text{ hours} \tag{23}$$

which is (43) in Appendix B. The discrepancy between 22.0 and the value $\bar{t}_i = 29.6$ given by the model of Section III shows the shortcomings

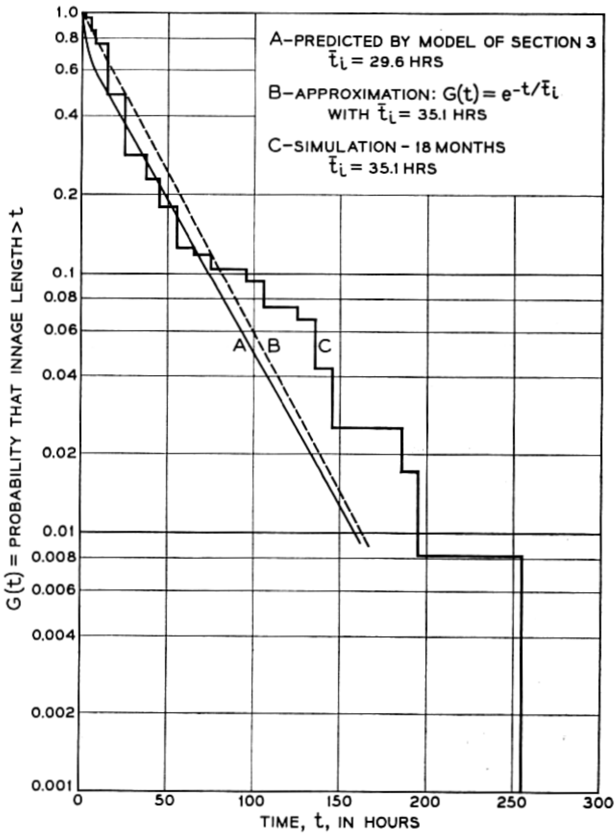

Fig. 1 — Innage length distribution for 18-satellite system with random polar orbits (Maine–Western Europe).

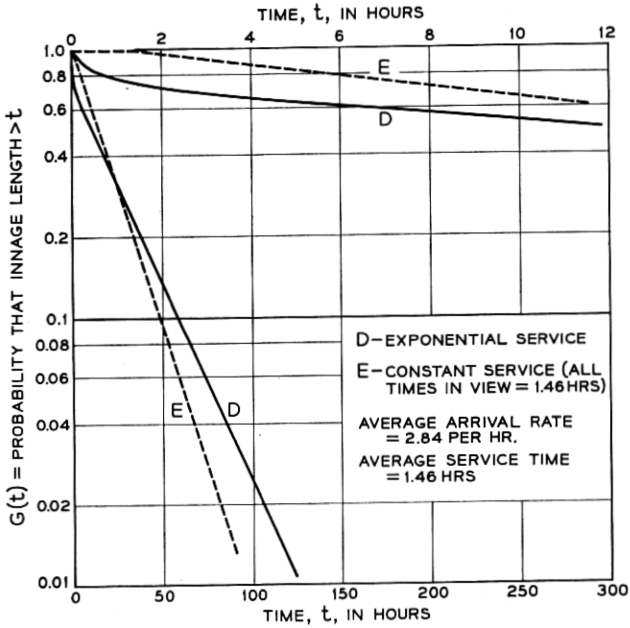

Fig. 2 — Innage length distributions.

of the simplified model. Nevertheless, it appears that the difference in shape between curves D and E illustrates the change in $G(t)$ produced by the different kinds of service. Support for this belief comes from the fact that curves A and D, both of which correspond to exponential service, have the same shape.

In view of the inaccuracies of the models and of the relatively good agreement shown by the curves B and C of Fig. 1, it seems that the simple exponential approximation for $G(t)$ is quite good.

ACKNOWLEDGMENT

I wish to express my thanks for the help I have received from my colleagues, especially J. D. Rinehart and M. F. Robbins, whose work on simulation gave rise to the present paper, and L. Takács, whose remarks led to a number of simplifications and generalizations. Many improvements in the presentation resulted from suggestions made by John Riordan. Additional information and suggestions were obtained from conversations with F. W. Sinden and W. H. Wise.

APPENDIX A

Probabilities Associated with Alternating Sequences

Expressions are recorded here for the Laplace transforms of two conditional probabilities. These transforms are of use in establishing Takács' result (19).

Consider a sequence comprised of two kinds of intervals which alternate with each other (for example, innages and outages). Let $p_a(t)$, $p_b(t)$ be the probability densities for the lengths of the two types of intervals, and let their respective Laplace transforms be $\alpha(s)$, $\beta(s)$. Suppose that the interval lengths are independent, let an interval of type (a) start at time 0, and let $P_{aa}(t)$ be the probability that an interval of type (a) is in progress at time t. Then the Laplace transform of $P_{aa}(t)$ is

$$\int_0^\infty e^{-st} P_{aa}(t)\, dt = \frac{1 - \alpha(s)}{s[1 - \alpha(s)\beta(s)]}. \tag{24}$$

Similarly, if an interval of type (b) starts at time 0, the Laplace transform of the probability $P_{ba}(t)$ that an interval of type (a) is in progress at time t is

$$\frac{[1 - \alpha(s)]\beta(s)}{s[1 - \alpha(s)\beta(s)]}. \tag{25}$$

These results are reminiscent of the relations between generating functions in the theory of recurrent events (Ref. 4, ch. 12).

The expression (24) may be obtained by noting that

$$\begin{aligned} P_{aa}(t) &= \Pr(t_{a1} > t) + \Pr(t_{a1} + t_{b1} + t_{a2} > t \\ &\qquad\qquad \text{and} \quad t_{a1} + t_{b1} < t) + \cdots \\ &= \Pr(t_{a1} > t) + [\Pr(t_{a1} + t_{b1} + t_{a2} > t) \\ &\qquad\qquad - \Pr(t_{a1} + t_{b1} > t)] + \cdots \end{aligned}$$

where t_{a1}, t_{b1}, t_{a2}, $\cdots$ are the lengths of the successive (a) and (b) type intervals. The Laplace transforms of $\Pr(t_{a1} > t)$, $\Pr(t_{a1} + t_{b1} > t)$, $\cdots$ are $[1 - \alpha(s)]/s$, $[1 - \alpha(s)\,\beta(s)]/s$, $\cdots$, and summation gives (24) when $|\,\alpha(s)\,\beta(s)\,| < 1$ (as it certainly is when $\mathrm{Re}(s) > 0$). Expression (25) may be obtained in a similar fashion or by convoluting $p_b(t)$ and $P_{aa}(t)$.

When $p_a(t) = a \exp(-at)$, the condition that a type (a) interval start at time 0 may be replaced by the condition that a type (a) interval be in progress at time 0. In this case $P_{aa}(t)$ is also the probability that a type (a) interval is in progress at time t, given that one was in progress at time 0. From (24) and $\alpha(s) = a/(a + s)$, the Laplace transform of this probability is

$$\frac{1}{s + a - a\beta(s)}. \tag{26}$$

APPENDIX B

Systems with an Infinite Number of Servers and Recurrent Inputs

This appendix will be concerned with systems containing an infinite number of servers and a "recurrent" input, i.e., an input in which the lengths of the intervals between successive arrivals are independent of each other and of the state of the servers. In many respects these systems are simpler than the limited source input introduced in Section III. Although they do not represent the satellite system as well, their greater simplicity enables us to estimate the shape of the innage distribution for cases which are difficult to handle by Takács' result (19).

In the following list of results, $A(t)$ is the distribution function for the distances between the arrivals of the recurrent input and $B(t)$ is the service time distribution for each one of the infinite number of servers. The expected distance between arrivals is a^{-1} and the expected service time is b^{-1}.

B.1 *The Conditional Probability* $P_{10}(t)$

Let all of the servers be idle at time -0 and let the first customer arrive at time 0 making one server busy at $+0$. Denote by $P_{1n}(t)$ the conditional probability that n servers are busy at time t. Consideration of the first arrival following time $+0$ leads to an integral equation which (in theory) may, be solved for $P_{10}(t)$, namely

$$P_{10}(t) = [1 - A(t)]B(t) + B(t) \int_0^t A'(t - v)P_{10}(v)\, dv, \tag{27}$$

where $A'(u) = dA(u)/du$. A corresponding equation for the generating function for $P_{1n}(t)$ is given by (44).

Example (*a*). For Poisson input $1 - A(t)$ is $\exp(-at)$. This corresponds to an unlimited source input. Substituting in (27), multiplying through by $[\exp(at)]/B(t)$, and differentiating with respect to t gives a differential equation for $P_{10}(t)$. Using the fact that $P_{10}(t) \to B(t)$ as $t \to 0$ to fix the constant of integration leads to

$$P_{10}(t) = B(t) \exp\left[-a \int_0^t [1 - B(\tau)]\, d\tau\right], \tag{28}$$

a result given by Refs. 5 and 6. The work may be simplified by starting with the assumption that $P_{10}(t)$ is of the form $B(t)P(t)$.

Example (*b*). For regularly spaced arrivals, $A'(u) = \delta(u - a^{-1})$ where $\delta(t)$ is the unit impulse. Equation (27) then gives

$$\begin{aligned} P_{10}(t) &= B(t), & 0 < t < a^{-1} \\ P_{10}(t) &= B(t)P_{10}(t - a^{-1}) & t > a^{-1} \\ P_{10}(t) &= B(t)B(t - a^{-1}) & a^{-1} < t < 2a^{-1} \\ P_{10}(t) &= B(t)B(t - a^{-1})B(t - 2a^{-1}) & 2a^{-1} < t < 3a^{-1} \end{aligned} \tag{29}$$

and so on.

Example (*c*). The case when $A(t)$ is arbitrary and $1 - B(t) = \exp(-t)$ has been studied by Takács[7] (see also Ref. 3, p. 33 et seq.). Multiplying (27) by $\exp(-st)$, integrating t from 0 to ∞, and introducing the Laplace transforms $\alpha(s)$, $\theta(s)$ of $A'(t)$, $P_{10}(t)$ leads to a recurrence relation between $\theta(s)$ and $\theta(s+1)$ which in turn gives

$$\begin{aligned} \theta(s) = \frac{1}{s} - \frac{1}{(s+1)(1-\alpha_s)} + \frac{\alpha_{s+1}}{(s+2)(1-\alpha_s)(1-\alpha_{s+1})} \\ - \frac{\alpha_{s+1}\alpha_{s+2}}{(s+3)(1-\alpha_s)(1-\alpha_{s+1})(1-\alpha_{s+2})} + \cdots \end{aligned} \tag{30}$$

where α_{s+n} is written for $\alpha(s+n)$. When $\theta(s)$ is known $P_{10}(t)$ may be obtained by inversion.

B.2 *The Busy Period Distribution $G(t)$ for Poisson Arrivals*

Let $-G'(t)$ be the probability density for the lengths of the busy periods (corresponding to innages) and consider the case of Poisson input and arbitrary service. The Laplace transform $\gamma(s)$ of $-G'(t)$ is given by

$$\gamma(s) = \frac{(s+a)\theta(s)}{1 + a\theta(s)} \tag{31}$$

where $\theta(s)$ is the Laplace transform of $P_{10}(t)$.

My original derivation of (31) was based on taking the Laplace transform of

$$-G'(t) = f(t) - a \int_0^t ds\, P_{10}(t - s) f(s) + a^2 \int_0^t ds\, P_{10}(t - s) \int_0^s dr\, P_{10}(s - r) f(r) - \cdots, \tag{32}$$

$$f(t) = P_{10}'(t) + aP_{10}(t), \tag{33}$$

where $P_{10}(t)$ is given by (28) and $f(t)\, dt$ is the probability that the system will jump from state 1 to state 0 in $(t, t + dt)$, given an arrival at time 0 which ends an idle period. The series (32) was obtained by an application of the method of inclusion and exclusion. Subsequently, Takács obtained a formula equivalent to (31) by an elegant method based upon results of the type stated in Appendix A. When this method is applied to obtain (31) the type (a) intervals are taken to be idle periods (outages) with $p_a(t) = a \exp(-at)$, $\alpha(s) = a/(a + s)$. The type (b) intervals become the busy periods so that $p_b(t)$, $\beta(s)$ become $-G'(t)$, $\gamma(s)$, respectively; and $P_{ba}(t)$ goes into $P_{10}(t)$. Expression (25) of Appendix A then says that the Laplace transform of $P_{10}(t)$ is

$$\theta(s) = \frac{\gamma(s)}{s + a - a\gamma(s)} \tag{34}$$

from which (31) follows.

Example (*a*). For Poisson arrivals and exponential service with $B = 1 - e^{-bt}$, (28) becomes (Ref. 3, p. 26)

$$P_{10}(t) = (1 - e^{-bt}) \exp[-\rho + \rho e^{-bt}], \qquad \rho = ab^{-1}.$$

The change of variable $y = e^{-bt}$ carries the integral for the Laplace transform of $P_{10}(t)$ into

$$b\theta(s) = F(z) - F(z + 1), \quad z = sb^{-1}$$

where

$$\begin{aligned} F(z) &= e^{-\rho} \int_0^1 y^{z-1} e^{\rho y}\, dy \\ &= \frac{1}{z} - \frac{\rho}{z(z+1)} + \frac{\rho^2}{z(z+1)(z+2)} - \cdots \\ &= z^{-1}[1 - \rho F(z + 1)] = \sum_{n=0}^{\infty} \frac{\rho^n e^{-\rho}}{n!(z + n)}. \end{aligned} \tag{35}$$

The Laplace transform of $G(t)$ may be shown to be

$$\frac{1-\gamma(s)}{s}=\frac{F(z+1)}{sF(z)}=\frac{1}{azF(z)}-\frac{1}{a} \tag{36}$$

and inversion gives

$$G(t)=\sum_{m=0}^{\infty}\frac{e^{z_m bt}}{\rho z_m F'(z_m)}, \qquad \rho=ab^{-1} \tag{37}$$

where $F'(z)=dF(z)/dz$. The zeros of $F(z)$ are real and (a) occur at $z_0, z_1, z_2, \cdots$, (b) are such that $-1<z_0<0$, $-2<z_1<-1, \cdots$, etc. and hence lie between the poles of $F(z)$ at $0, -1, -2, \cdots$, (c) may be computed by successive approximations with the help of the last series for $F(z)$ in (35).

A power series for $G(t)$ may be obtained by expanding $[1-\gamma(s)]/s$ in powers of $1/s$ and then replacing s^{-n-1} by $t^n/n!$. The same series may be obtained from the corresponding series (5.3) for the limited source case by letting $k\to\infty$, $\alpha\to 0$ in such a way as to keep $k\alpha$ equal to a. Replacing a by ρb then gives

$$\begin{aligned}G(t)=1-\frac{bt}{1!}+\frac{(\rho+1)(bt)^2}{2!}-\frac{(\rho^2+4\rho+1)(bt)^3}{3!}\\+\frac{(\rho^3+9\rho^2+11\rho+1)(bt^4)}{4!}-\cdots.\end{aligned} \tag{38}$$

Example (*b*). For Poisson arrivals and constant service time, $B(t)$ jumps from 0 to 1 at time $t=b^{-1}$. Equation (28) shows that $P_{10}(t)$ jumps from 0 to $\exp(-ab^{-1})$ at $t=b^{-1}$. The Laplace transforms are readily computed and the one for $G(t)$ gives

$$G(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{[1-e^{-(a+s)b^{-1}}]e^{st}}{s+a\,e^{-(a+s)b^{-1}}}\,ds, \qquad c>0. \tag{39}$$

Taking c large enough, say $c=b$, to make $|x|<1$ where x is $as^{-1}\exp[-(a+s)b^{-1}]$ and expanding the denominator in powers of x leads to

$$G(t)=1+\sum_{n=1}^{N}(-\rho e^{-\rho})^n\left[\frac{(bt-n)^{n-1}}{(n-1)!\rho}+\frac{(bt-n)^n}{n!}\right]. \tag{40}$$

Here, as in example (a), $\rho=ab^{-1}$ and the upper limit of summation is determined from $N<bt<N+1$. In particular,

$$\begin{aligned} G(0) &= 1, \qquad G(b^{-1} - 0) = 1, \qquad G(b^{-1} + 0) = 1 - e^{-\rho} \\ G(2b^{-1}) &= 1 - (1 + \rho)\, e^{-\rho} \\ G(3b^{-1}) &= 1 - (1 + 2\rho)\, e^{-\rho} + \rho\left(1 + \frac{\rho}{2}\right) e^{-2\rho}. \end{aligned} \tag{41}$$

For large values of t

$$G(t) \sim \frac{(\sigma_0 + \rho)}{\rho(1 + \sigma_0)}\, e^{\sigma_0 b t} \tag{42}$$

where σ_0 is the rightmost root of

$$\sigma + \rho\, e^{-\rho-\sigma} = 0.$$

When ρ is large σ_0 is approximately $-\rho\, e^{-\rho}$.

The innages corresponding to examples (a) and (b) have the same average length, namely

$$\bar{t}_i = [e^{ab^{-1}} - 1]/a. \tag{43}$$

To see this, note that from traffic theory, or by letting $t \to \infty$ in (28), the fraction of idle time is $p_0 = 1 - q = \exp(-ab^{-1})$. The average outage time is $\bar{t}_0 = 1/a$, and (43) follows upon using the relation (1) between $\bar{t}_i$, $\bar{t}_o$, and q.

B.3 *Miscellaneous Results for Recurrent Input*

Except for the case in which arrivals occur at multiples of some fixed spacing, $P_{10}(t)$ approaches the steady state probability p_0 as $t \to \infty$, and the Laplace transform $\theta(s)$ of $P_{10}(t)$ has a simple pole at $s = 0$ with residue p_0.

If $P_{10}(t)$ tends to a periodic function the residue gives its average value. Application of this result to the case of regularly spaced arrivals and exponential service with $B(t) = 1 - e^{-t}$ leads to the rather curious expansion

$$\int_0^1 \prod_{n=0}^{\infty} (1 - x^{-\tau-n})\, d\tau$$
$$= 1 - \frac{1}{\ln x}\left[1 - \frac{1}{2(x-1)} + \frac{1}{3(x-1)(x^2-1)} - \cdots\right].$$

Both sides represent the average value of the periodic function to which

$P_{10}(t)$ tends as $t \to \infty$. The integral on the left (with $x = \exp a^{-1}, x > 1$) is the average value as computed from (29), while the series on the right is the residue of $\theta(s)$ at $s = 0$ obtained by setting $\alpha(s) = \exp(-sa^{-1})$ in (30) and letting $s \to 0$.

The generating function

$$P_1(x, t) = \sum_{n=0}^{\infty} x^n P_{1n}(t)$$

for the conditional probability $P_{1n}(t)$ that state n exists at time t, given that an arrival at time 0 ends an idle period, [see (27)] satisfies the integral equation

$$P_1(x, t) = [x + (1 - x)B(t)] \cdot \left[1 - A(t) + \int_0^t A'(t - v)P_1(x, v)\, dv\right]. \quad (44)$$

A formal step-by-step solution may be obtained by introducing the binomial moments $M_n(t)$ defined by

$$P_1(x, t) = \sum_{n=0}^{\infty} (x - 1)^n M_n(t),$$

$$M_n(t) = \sum_{k=n}^{\infty} \binom{k}{n} P_{1k}(t).$$

The value of $M_0(t)$ is one and the higher-order moments satisfy integral equations obtainable from (44). When the Laplace transforms of $A'(t)$, $B'(t)$, $M_n(t)$ are denoted by $\alpha(s)$, $\beta(s)$, $\mu_n(s)$ it is found that the integral equations lead to

$$\mu_0(s) = s^{-1}, \qquad \mu_1(s) = \frac{1 - \beta(s)}{s[1 - \alpha(s)]}$$

$$\mu_n(s) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{[1\ \beta(s - z)]}{s - z} \frac{\alpha(z)\mu_{n-1}(z)}{1 - \alpha(s)}\, dz, \qquad n > 1. \quad (45)$$

The singularities of $\alpha(z)\mu_{n-1}(z)$ are supposed to lie to the left and those of $[1 - \beta(s - z)]/(s - z)$ to the right of the path of integration, s being chosen so as to make this possible. In theory, the successive values of $\mu_n(s)$ may be obtained step by step and thus ultimately lead to an expression for $P_1(x, t)$. For exponential service the integrals in (45) may be evaluated and lead to results given by Takács[7] (see also Ref. 3, p. 33).

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