Spectral Density and Autocorrelation Functions Associated with Binary Frequency-Shift Keying

By W. R. BENNETT and S. O. RICE

(Manuscript received May 15, 1963)

General equations are derived for the spectral density and autocorrelation functions of a wave train consisting of sine-wave segments with constant amplitude. The frequency of a segment may be either f_1 or f_2. At regularly spaced intervals the frequency is switched or not switched according to a random choice. This type of wave occurs when a random series of marks and spaces is sent by frequency-shift keying. The results fall into two main classes — namely, that of discontinuous phase at the transitions, which is the typical situation in switching between two independent oscillators; and that of continuous phase at the transitions, which is more usually applicable when the frequency of a single oscillator is changed. Individual treatment is given of the various special cases which arise when integral relationships between the marking, spacing, signaling, and shift frequencies exist. No restriction is made on the relative magnitudes of the different frequencies involved.

I. INTRODUCTION

The spectral density function, or power spectrum, of a random sequence of signals defines the distribution of average signal power versus frequency. This information is useful in system design because it indicates the frequency band of most importance, the amount of average total power in any frequency interval, and the interference which may result in other systems. It does not tell us how much distortion the signals suffer when the channel does not pass all the frequencies represented, nor does it tell us about important spectral components which may be associated with unlikely but possible specific signal sequences. Keeping these limitations in mind, we still find the spectral density to have merit as a descriptive parameter of the system.

Another important function is the autocorrelation, which is the time domain analog of the spectral density. Because of the Fourier transform relationship between the two functions, either can be used as an auxiliary step in computing the other. The autocorrelation function is useful in its own right in signal analysis and can be made the basis of control operations.

The present paper presents results on the spectral density and autocorrelation functions associated with binary frequency modulation systems. There are two general types of operation, which in terms of apparatus may be classified as (a) switching between two oscillators, and (b) changing the frequency of a single oscillator. The mathematical distinction between these two cases will be considered here as shifting the frequency with discontinuous or continuous phase respectively.

The case of discontinuous phase, which is appropriate when we switch between independent marking and spacing oscillators, is the simpler one to analyze. We assume that the oscillators deliver equal amplitude and that each oscillator preserves its own coherence in time, i.e., that the two frequencies are constant. The waveforms in intervals containing the marking frequency, say, are segments of a sine wave having the marking frequency and extending throughout all time. One would intuitively expect, therefore, to find discrete spectral lines at the marking and spacing frequencies. Analysis verifies that if mark and space signals have independent equal probabilities, each of the two discrete components has half the amplitude of the complete FSK wave.

In addition there is a continuous spectrum consisting of switching function spectra centered at the marking and spacing frequencies. Since the signal wave is discontinuous at the switching instants, the associated voltage spectra fall off only as $1/f$ at high frequencies. This means that the spectral density function ultimately falls off as the inverse square of the frequency. Degenerate cases arise when there are commensurable relationships among the marking, spacing, and signaling frequencies. If these special relations are such as to produce continuous phase at the transitions, the resulting continuity in the signal wave leads to an ultimate inverse fourth-power variation of spectral density with frequency. If in addition the derivative of the phase is continuous, an inverse sixth power is obtained at remote frequencies.

In the case in which the frequency shifting is done with continuous phase, the signal wave is continuous at all times. The spectral density function must, therefore, fall off at least as fast as the inverse fourth power at frequencies remote from the center of the signal band. This is in accordance with the well-known fact that frequency-shift keying with continuous phase does not produce as much interference outside the signal band as the discontinuous case. The analysis of the continuous-phase case is considerably more difficult, even though the final results are of fairly simple form.

It was found necessary to distinguish between four possible cases. In the most general of these, in which there are no degenerate relationships among the three frequencies involved, the line spectra completely disappear and the spectral density function is continuous at all frequencies. When the difference between the marking and spacing frequencies is a multiple of the signaling frequency, defined as the sum of the number of marks and number of spaces per second, line spectral terms appear at the marking and spacing frequencies, and the continuous part of the spectral density function changes its form. The continuous part of the spectrum in this case is found to depend also on whether or not the sum of marking and spacing frequencies is a multiple of the signaling rate. A curious behavior occurs when the frequency shift is an odd multiple of half the signaling rate. Here no discrete components appear in the spectrum, but the continuous spectral distribution undergoes a sudden change relative to that at infinitesimally close values of frequency shift not possessing the critical property.

Table II given in Section VI lists the various cases together with the corresponding equation numbers for the associated spectral densities and autocorrelation functions. Section V gives illustrative curves of these functions for various relations among the marking, spacing, and signaling frequencies.

II. DISCONTINUOUS PHASE

Before considering the frequency-shift keying problem of this section, it is helpful to develop some general results applicable to random switching between two waves. Let $y(t)$ be a given function of time which is bounded for all $t \geqq 0$ and such that the limits later encountered exist. Let $x(t)$ be a random telegraph wave defined by

$$x(t) = x_n\,, \qquad nT \leqq t < (n+1)T \tag{1}$$

where $n = 0, 1, 2, \cdots$, and the x_n's are independent random variables which assume the values ± 1 with equal probability. We calculate the spectral density $w_v(f)$ and the autocorrelation $R_v(\tau)$ of

$$v(t) = x(t)y(t). \tag{2}$$

We note that $v(t)$ is generated from the source of $y(t)$ by inserting a reversing switch for which a random choice between positions is made at instants T apart.

The spectral density of $v(t)$ is given by the limit of $2\langle | S_N(f) |^2 \rangle / NT$ as $N \to \infty$. Here $\langle \quad \rangle$ denotes "ensemble average" and

$$\begin{aligned} S_N(f) &= \int_0^{NT} e^{-j\omega t}\, v(t)\, dt \qquad \omega = 2\pi f \\ &= \sum_{n=0}^{N-1} x_n e^{-j\omega n T} \int_0^T dt\, e^{-j\omega t}\, y(nT + t). \end{aligned} \tag{3}$$

Since $\langle x_n x_m \rangle$ is 1 if $n = m$ and 0 if $n \neq m$

$$w_v(f) = \lim_{N\to\infty} \frac{2}{NT} \sum_{n=0}^{N-1} \left| \int_0^T dt\, e^{-j\omega t}\, y(nT + t) \right|^2 . \tag{4}$$

The autocorrelation function $R_v(\tau)$ may be calculated either by averaging $\langle v(t) v(t + \tau) \rangle$ over all $t \geqq 0$ with τ held fixed, or by taking the Fourier transform of (4); thus

$$R_v(\tau) = \int_0^\infty w_v(f) \cos \omega\tau \, df. \tag{5}$$

Both methods show that $R_v(\tau) = 0$ for $|\tau| \geqq T$ and

$$R_v(\tau) = \frac{1}{T} \int_0^{T-\tau} dt \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} y(nT + t)\, y(nT + t + \tau) \tag{6}$$

for $0 \leqq \tau < T$.

Return now to the frequency-shift keying problem and consider the signal wave

$$\begin{aligned} u(t) &= \begin{matrix} u_1(t) \\ \text{or} \\ u_2(t) \end{matrix} \qquad \begin{matrix} nT \leqq t < (n+1)T \\ \\ n = 0, 1, 2, \cdots \end{matrix} \\ u_k(t) &= A \cos(\omega_k t + \theta_k) \qquad k = 1, 2 \end{aligned} \tag{7}$$

where the choice is made independently and with equal probability for each interval of length T. The signaling frequency is $\omega_s = 2\pi/T$ rad/sec. The wave $u(t)$ may be written as

$$u(t) = u_+(t) + x(t) u_-(t) \tag{8}$$

where $x(t)$ is defined by (1) and

$$\begin{aligned} u_+(t) &= \tfrac{1}{2} u_1(t) + \tfrac{1}{2} u_2(t) \\ u_-(t) &= \tfrac{1}{2} u_1(t) - \tfrac{1}{2} u_2(t). \end{aligned} \tag{9}$$

Thus, $u(t)$ is the sum of two steady-state cosine waves, given by $u_+(t)$, and a random component given by

$$v(t) = u(t) - u_+(t) = x(t)u_-(t). \tag{10}$$

The random component assumes the values $\pm u_-(t)$.

When $u_-(t)$ is identified with $y(t)$, the spectral density $w_v(f)$ of $v(t)$ can be obtained from (4). Since the algebra is rather tedious the procedure will be merely sketched. The integral in (4) is now

$$\int_0^T dt\, e^{-j\omega t}\, u_-\,(nT + t) = \frac{A}{4}\, e^{-j\omega T/2} \\ \{g(\omega - \omega_1) \exp\,[j\theta_1 + j\omega_1 T(n + \tfrac{1}{2})] + \cdots\} \tag{11}$$

where the braces contain four terms similar to the first and

$$g(a) = \frac{\sin\,(aT/2)}{(a/2)}. \tag{12}$$

The cosines in $u_-(nT + t)$ are expressed in exponential form before integrating.

Multiplying (11) by its conjugate complex gives 16 terms. Substitution in (4) gives rise to expressions of the form

$$\gamma_N(\lambda) = \frac{1}{N}\sum_{n=0}^{N-1} e^{j\lambda n} = \frac{1 - e^{j\lambda N}}{N(1 - e^{j\lambda})} \\ \gamma(\lambda) = \lim_{N\to\infty} \gamma_N(\lambda) = \begin{cases} 1, \text{ if } \lambda = 2\pi m \\ 0, \lambda \text{ real and } \neq 2\pi m \end{cases} \tag{13}$$

where m is an integer. A typical value of λ is $2\omega_1 T$. At this stage $w_v(f)$ is given by

$$\begin{aligned} \frac{8Tw_v(f)}{A^2} &= g^2(\omega - \omega_1) + g^2(\omega + \omega_1) + g^2(\omega - \omega_2) + g^2(\omega + \omega_2) \\ &+ 2\cos\,(2\theta_1 + \omega_1 T) g(\omega - \omega_1) g(\omega + \omega_1)\gamma(2\omega_1 T) \\ &+ 2\cos\,(2\theta_2 + \omega_2 T) g(\omega - \omega_2) g(\omega + \omega_2)\gamma(2\omega_2 T) \\ &- 2\cos\left[\theta_2 + \theta_1 + \left(\frac{\omega_2 + \omega_1}{2}\right) T\right][g(\omega - \omega_1)g(\omega + \omega_2) \\ &+ g(\omega + \omega_1)g(\omega - \omega_2)]\gamma(\omega_2 T + \omega_1 T) - 2\cos\left[\theta_2 - \theta_1 \right. \\ &\left. + \left(\frac{\omega_2 - \omega_1}{2}\right) T\right][g(\omega + \omega_1)g(\omega + \omega_2) \\ &+ g(\omega - \omega_1)g(\omega - \omega_2)]\gamma(\omega_2 T - \omega_1 T). \end{aligned} \tag{14}$$

Some further reduction leads to the final result

$$
\begin{aligned}
2TA^{-2}w_v(f) = {} & \frac{G(\omega - \omega_1)}{(\omega - \omega_1)^2} + \frac{G(\omega + \omega_1)}{(\omega + \omega_1)^2} + \frac{G(\omega - \omega_2)}{(\omega - \omega_2)^2} + \frac{G(\omega + \omega_2)}{(\omega + \omega_2)^2} \\
& + \frac{2 \cos 2\theta_1 G(\omega - \omega_1)}{\omega^2 - \omega_1^2}\gamma(2\omega_1 T) + \frac{2 \cos 2\theta_2 G(\omega - \omega_2)}{\omega^2 - \omega_2^2}\gamma(2\omega_2 T) \\
& - 2 \cos(\theta_2 + \theta_1)\left[\frac{G(\omega - \omega_1)}{(\omega - \omega_1)(\omega + \omega_2)}\right. \qquad (15) \\
& \left. + \frac{G(\omega - \omega_2)}{(\omega + \omega_1)(\omega - \omega_2)}\right]\gamma(\omega_2 T + \omega_1 T) - 2\cos(\theta_2 - \theta_1) \\
& \cdot\left[\frac{G(\omega - \omega_1)}{(\omega - \omega_1)(\omega - \omega_2)} + \frac{G(\omega + \omega_2)}{(\omega + \omega_1)(\omega + \omega_2)}\right]\gamma(\omega_2 T - \omega_1 T)
\end{aligned}
$$

where

$$G(a) = \sin^2(aT/2) \qquad (16)$$

and $\gamma(2\omega_1 T)$, $\gamma(2\omega_2 T)$, $\gamma(\omega_2 T + \omega_1 T)$, $\gamma(\omega_2 T - \omega_1 T)$ are zero except when $2\omega_1/\omega_s$, $2\omega_2/\omega_s$, $(\omega_2 + \omega_1)/\omega_s$, $(\omega_2 - \omega_1)/\omega_s$, respectively, are integers (in which case the corresponding γ's are unity).

In the special case in which $2\omega_1/\omega_s = m$ and $2\omega_2/\omega_s = l$, with l and m integers and $l + m$ even, all of the γ's in (15) are equal to unity and the G's are equal to $\sin^2(\omega T/2)$ if m is even and to $\cos^2(\omega T/2)$ if m is odd. In particular, when $\theta_1 = \theta_2 = 0$ the following simple result is obtained

$$w_v(f) = \frac{2A^2(\omega_2^2 - \omega_1^2)^2\omega^2}{T(\omega^2 - \omega_1^2)^2(\omega^2 - \omega_2^2)^2} \times \begin{Bmatrix} \sin^2 \dfrac{\omega T}{2}, m \text{ even} \\ \cos^2 \dfrac{\omega T}{2}, m \text{ odd} \end{Bmatrix}. \qquad (17)$$

This spectral density function varies as $1/\omega^6$ for large ω, showing that the waveform of the signal is continuous and has a continuous derivative.

The spectral density function of $u(t)$ as defined by (7) differs from that of $v(t)$ by the presence of sinusoidal components of amplitude $A/2$ and frequencies ω_1 and ω_2. That is

$$w_u(f) = w_v(f) + \frac{A^2}{8}\delta(f - f_1) + \frac{A^2}{8}\delta(f - f_2). \qquad (18)$$

It can be shown from (6) and (18) that the autocorrelation functions $R_u(\tau)$ and $R_v(\tau)$ of $u(t)$ and $v(t)$ respectively are given by the following set of equations

$$R_u(\tau) = R_v(\tau) + \frac{A^2}{8}\cos\omega_1\tau + \frac{A^2}{8}\cos\omega_2\tau$$

$$R_v(\tau) = 0, \qquad |\tau| > T$$

$$8TA^{-2}R_v(\tau) = (T-\tau)(\cos\omega_1\tau + \cos\omega_2\tau)$$

$$-\frac{\sin\omega_1\tau}{\omega_1}\cos 2\theta_1\,\gamma(2\omega_1T) - \frac{\sin\omega_2\tau}{\omega_2}\cos 2\theta_2\,\gamma(2\omega_2T) \tag{19}$$

$$+\frac{2(\sin\omega_2\tau + \sin\omega_1\tau)}{\omega_2+\omega_1}\cos(\theta_2+\theta_1)\,\gamma(\omega_2T+\omega_1T)$$

$$+\frac{2(\sin\omega_2\tau - \sin\omega_1\tau)}{\omega_2-\omega_1}\cos(\theta_2-\theta_1)\,\gamma(\omega_2T-\omega_1T) \qquad 0 \leqq \tau \leqq T$$

$$R_v(-\tau) = R_v(\tau).$$

III. CONTINUOUS PHASE

Consider the signal wave

$$u(t) = \begin{matrix} A\cos(\omega_1 t + \theta_n) & nT \leqq t < (n+1)T \\ \text{or} & \\ A\cos(\omega_2 t + \phi_n) & n = 0, 1, 2, \cdots \end{matrix} \tag{20}$$

where the choice is made independently and with equal probability for each interval of length T. The initial values at $t = 0$ of the phase are $\theta_0 = \phi_0 = \phi$, and the succeeding values θ_n, ϕ_n are to be chosen so as to make the phase of $u(t)$ continuous at the transition points.

Let

$$\alpha = \tfrac{1}{2}(\omega_2 + \omega_1) \qquad \beta = \tfrac{1}{2}(\omega_2 - \omega_1). \tag{21}$$

Then

$$\omega_1 = \alpha - \beta \qquad \omega_2 = \alpha + \beta. \tag{22}$$

Set

$$u(t) = A\cos B_n(t), \quad nT \leqq t < (n+1)T, \qquad n = 0, 1, 2, \cdots \tag{23}$$

$$B_0(t) = (\alpha + x_1\beta)t + \phi \tag{24}$$

$$B_n(t) = \alpha t + x_{n+1}\beta(t - nT) + \phi + \beta T\sum_{r=1}^{n} x_r, \qquad n > 0. \tag{25}$$

We assume $x_1, x_2, \cdots$ to be independent random variables, each of which is equally likely to have the value $+1$ or -1. We verify that within the interval beginning at $t = nT$, the frequency is $\alpha + x_{n+1}\beta$, which is equal to ω_2 if $x_{n+1} = +1$ and equal to ω_1 if $x_{n+1} = -1$. Therefore, the function $B_n(t)$ satisfies the condition of an equiprobable choice between the two frequencies in each interval. We also note that the phase at the beginning of the typical interval is

$$B_n(nT) = \alpha nT + \phi + \beta T \sum_{r=1}^{n} x_r \tag{26}$$

while the phase at the end of the previous interval is

$$\begin{aligned} B_{n-1}(nT) &= \alpha nT + \phi + x_n\beta[nT - (n-1)T] + \beta T \sum_{r=1}^{n-1} x_r \\ &= B_n(nT). \end{aligned} \tag{27}$$

Thus the function $B_n(t)$ also satisfies the required condition of continuous phase.

We have evaluated the spectral density function of $u(t)$ by two different methods, namely

(a) by taking the Fourier transform of the autocorrelation function, and

(b) by direct evaluation from the Fourier transform of the signal wave over a long time interval. The two methods are of comparable difficulty. The same results are finally obtained by both procedures, although the agreement is not immediately evident from the expressions which emerge naturally from the two sets of calculations.

To evaluate the autocorrelation function, we first calculate the average value of $u(t)u(t + \tau)$ over the ensemble at fixed t. Set $\tau \geqq 0$ and define k as the member of the set $0, 1, 2, \cdots$ satisfying the inequality

$$(n + k)T \leqq t + \tau < (n + k + 1)T \tag{28}$$

with n defined by $nT \leqq t < (n + 1)T$. Let $E_k(t, \tau)$ represent the mathematical expectation of $u(t)u(t + \tau)$ with t and τ fixed. If $k = 0$, the values of t and $t + \tau$ lie in the same signaling interval and the same function $B_n(t)$ applies to both. When $k > 0$ we use the function $B_n(t)$ for $u(t)$ and the function $B_{n+k}(t)$ for $u(t + \tau)$.

We calculate

$$\frac{2}{A^2} E_0(t, \tau)$$
$$= 2\langle \cos B_n(t) \cos B_n(t + \tau) \rangle$$
$$= \text{Re } \langle [\exp (jB_n(t + \tau) - jB_n(t)) + \exp (jB_n(t + \tau) + jB_n(t))] \rangle$$
$$= \text{Re } \left\langle \left[\exp (j\alpha\tau) \exp (jx_{n+1}\beta\tau) + \exp (j\alpha(2t + \tau) + j2\phi) \right.\right. \quad (29)$$
$$\left.\left. \cdot \exp (jx_{n+1}\beta(2t + \tau - 2nT)) \prod_{r=1}^{n} \exp (j2x_r\beta T) \right] \right\rangle .$$

Since the x's are independent, the average of a product of functions in which the variables appear separately is equal to the product of the averages of the individual functions. Since each x has only two possible values with a probability of one-half for each, we evaluate the expectations of individual terms by inserting the sum of the two possible functions with weighting factor one-half. Performing the necessary operations, we find

$$\frac{2}{A^2} E_0(t, \tau) = \cos \alpha\tau \cos \beta\tau \quad (30)$$
$$+ \cos (2\alpha\tau + \alpha\tau + 2\phi) \cos \beta(2t + \tau - 2nT) \cos^n 2\beta T.$$

The corresponding calculation for $k > 0$ leads to the result

$$\frac{2}{A^2} E_k(t, \tau) = \cos \beta(t + \tau - nT - kT)$$
$$\cos^{k-1} \beta T[\cos \alpha\tau \cos \beta(t - nT - T) + \cos (2\alpha t + \alpha\tau + 2\phi) \quad (31)$$
$$\cos \beta(t - nT + T) \cos^n 2\beta T].$$

The lag interval τ is bounded between adjacent multiples of T by defining m as the member from the set 0, 1, 2, $\cdots$ which satisfies $mT \leqq \tau < (m + 1)T$. We observe that the number k defined by (28) is related to m as follows

$$k = \begin{cases} m, & nT \leqq t < (m + n + 1)T - \tau \\ m + 1, & (m + n + 1)T - \tau \leqq t < (n + 1)T. \end{cases} \quad (32)$$

The autocorrelation function $R_u(\tau)$ is the average of $\langle u(t)u(t + \tau) \rangle$ taken from $t = 0$ to $t = \infty$, i.e.

$$
\begin{aligned}
R_u(\tau) &= \lim_{N\to\infty} \frac{1}{NT} \int_0^{NT} \langle u(t)u(t+\tau)\rangle \, dt \\
&= \lim_{N\to\infty} \frac{1}{NT} \sum_{n=0}^{N-1} \left[\int_{nT}^{(m+n+1)T-\tau} E_m(t,\tau)\, dt \right. \\
&\qquad \left. + \int_{(m+n+1)T-\tau}^{(n+1)T} E_{m+1}(t,\tau)\, dt \right] \qquad (33) \\
&= \lim_{N\to\infty} \frac{1}{NT} \sum_{n=0}^{N-1} \left[\int_0^{(m+1)T-\tau} E_m(t+nT,\tau)\, dt \right. \\
&\qquad \left. + \int_{(m+1)T-\tau}^{T} E_{m+1}(t+nT,\tau)\, dt \right].
\end{aligned}
$$

As indicated in (30) and (31), the case of $m = 0$, i.e., $0 < \tau < T$, requires a separate treatment from that of $m > 0$. In order to perform the integrations indicated in (33) it is convenient to expand (30) and (31) into the sum of terms in which t appears only once. The expanded equations are, with $0 \leqq t < T$

$$
\begin{aligned}
\frac{2}{A^2} E_0\,(t+nT,\,\tau) &= \cos\alpha\tau\cos\beta\tau \\
&\quad + \tfrac{1}{2}\cos^n 2\beta T \cos\left[(\alpha+\beta)(2t+\tau) + 2\phi + 2n\alpha T\right] \\
&\quad + \tfrac{1}{2}\cos^n 2\beta T \cos\left[(\alpha-\beta)(2t+\tau) + 2\phi + 2n\alpha T\right] \qquad (34)
\end{aligned}
$$

$$
\begin{aligned}
\frac{4}{A^2} E_k(t+nT,\,\tau) &= \cos\alpha\tau \cos^{k-1}\beta T\,[\cos\beta(\tau + T - kT) \\
&\quad + \cos\beta(2t+\tau-T-kT)] \\
&\quad + \cos^{k-1}\beta T\cos^n 2\beta T\,\{\cos\beta[\tau-(k+1)T]\cos[\alpha(2t+\tau) \qquad (35) \\
&\quad + 2\phi + 2n\alpha T] + \tfrac{1}{2}\cos[(\alpha+\beta)(2t+\tau) + 2\phi + 2n\alpha T - (k-1)\beta T] \\
&\quad + \tfrac{1}{2}\cos[(\alpha-\beta)(2t+\tau) + 2\phi + 2n\alpha T + (k-1)\beta T]\}.
\end{aligned}
$$

We note the possibility that some of the terms will not contribute anything to the result after the limiting process in (33) is performed. The expressions in (34) and (35) can be divided into two classes: those which do not contain n and those which contain n in the form of a factor of type $\cos^n 2\beta T \cos(\psi + 2n\alpha T)$. In the former group, we sum N equal terms and divide by NT; hence the summing and limiting operations are equivalent to a division by T. In the other group, the limit is zero if the sum remains finite as N becomes indefinitely large. The only case in which the sum does not remain finite is that in which the terms are equal for all n. Equality occurs if αT and βT are both multiples of π and also if αT and βT are both odd multiples of $\pi/2$. If neither of these conditions exists, the contributions of the terms which depend on n are zero. We shall call this the incommensurable case and shall treat it first. Afterward, we shall consider the effect of commensurable relations.

Except for the cases we have specifically excluded, we can now write for $0 < \tau < T$

$$\begin{aligned} R_u(\tau) &= \frac{A^2}{2T} \int_0^{T-\tau} \cos \alpha\tau \cos \beta\tau \, dt \\ &+ \frac{A^2}{4T} \int_{T-\tau}^{T} \cos \alpha\tau[\cos \beta\tau + \cos \beta(2t + \tau - 2T)] \, dt \\ &= \frac{A^2}{4\beta T} [\beta(2T - \tau) \cos \beta\tau + \sin \beta\tau] \cos \alpha\tau. \end{aligned} \tag{36}$$

For $\tau > T$

$$\begin{aligned} R_u(\tau) = \frac{A^2}{4T} \int_0^{(m+1)T-\tau} & \cos \alpha\tau \cos^{m-1} \beta T[\cos \beta(\tau + T - mT) \\ & + \cos \beta(2t + \tau - T - mT)] \, dt \\ + \frac{A^2}{4T} \int_{(m+1)\,T-\tau}^{T} & \cos \alpha\tau \cos^{m} \beta T \, [\cos \beta(\tau - mT) \\ & + \cos \beta(2t + \tau - 2T - mT)] \, dt \\ = \frac{A^2}{4\beta T} \cos \alpha\tau \cos^{m-1} & \beta T[\beta T \cos \beta(\tau + T - mT) \\ & + \beta(\tau - mT) \sin \beta T \sin \beta(\tau - mT) \\ & + \sin \beta T \cos \beta(\tau - mT)]. \end{aligned} \tag{37}$$

The spectral density function $w_u(f)$ is given by

$$w_u(f) = 4 \int_0^\infty R_u(\tau) \cos \omega\tau \, d\tau. \tag{38}$$

It is convenient to evaluate the integral in two parts

$$\begin{aligned} 4 \int_0^T R_u(\tau) \cos \omega\tau \, d\tau = \frac{A^2}{2\beta T} \int_0^T & [\beta(2T - \tau) \cos \beta\tau + \sin \beta\tau] \\ & \cdot [\cos (\omega + \alpha)\tau + \cos (\omega - \alpha)\tau] \, d\tau \end{aligned} \tag{39}$$

and

$$
\begin{aligned}
4 \int_T^\infty & R_u(\tau) \cos \omega\tau \, d\tau \\
&= \frac{A^2}{\beta T} \sum_{m=1}^{\infty} \int_{mT}^{(m+1)T} \cos \alpha\tau \cos^{m-1} \beta T [\beta T \cos \beta(\tau + T - mT) \\
&\qquad + \beta(\tau - mT) \sin \beta T \sin \beta(\tau - mT) \\
&\qquad + \sin \beta T \cos \beta(\tau - mT)] \cos \omega\tau \, d\tau \\
&= \frac{A^2}{\beta T} \int_0^T [\beta T \cos \beta(\tau + T) + \beta\tau \sin \beta T \sin \beta\tau \\
&\quad + \sin \beta T \cos \beta\tau] \sum_{m=1}^{\infty} \cos \alpha(\tau + mT) \cos \omega(\tau + mT) \cos^{m-1} \beta T \\
&= \frac{A^2}{2\beta T} \int_0^T [\beta T \cos \beta(\tau + T) + \beta\tau \sin \beta T \sin \beta\tau \\
&\qquad + \sin \beta T \cos \beta\tau][G(\omega + \alpha, \tau) + G(\omega - \alpha, \tau)] \, d\tau
\end{aligned} \tag{40}
$$

where

$$
G(y,\tau) = \frac{\cos y(\tau + T) - \cos \beta T \cos y\tau}{1 + \cos^2 \beta T - 2 \cos \beta T \cos yT}. \tag{41}
$$

The summation is performed by writing

$$
2 \cos \alpha(\tau + mT) \cos \omega(\tau + mT) \cos^{m-1} \beta T
$$

as the real part of

$$
[\exp (j(\omega + \alpha)(\tau + mT)) + \exp (j(\omega - \alpha)(\tau + mT))] \cos^{m-1} \beta T
$$

and thereby obtaining a geometric series. The series fails to converge when the absolute values of the individual terms are unity. For this reason, we must now exclude the case of βT equal to a multiple of π no matter what αT is. We have thus accumulated three special cases of commensurability to be given individual treatment later.

The remainder of the calculation is straightforward, but appears to lead into a morass of complication. The key to an end result of pleasing simplicity, which the autocorrelation method tends to conceal, is to arrange the work as follows

$$
\begin{aligned}
w_u(f) = \frac{A^2}{T} \Bigg[& \frac{H(\omega + \alpha)}{1 + \cos^2 \beta T - 2 \cos \beta T \cos (\omega + \alpha)T} \\
& + \frac{H(\omega - \alpha)}{1 + \cos^2 \beta T - 2 \cos \beta T \cos (\omega - \alpha)T} \Bigg].
\end{aligned} \tag{42}
$$

The integral (40) splits naturally into this form. In the integral (39) we associate the term $\cos(\omega + \alpha)\tau$ with the first part of (42) and the term $\cos(\omega - \alpha)\tau$ with the second part. Then

$$\begin{aligned} H(y) = \frac{1}{2\beta}\int_0^T \{[1 + \cos^2 \beta T - 2\cos\beta T \cos yT] \\ \cdot[\beta(2T - \tau)\cos\beta\tau + \sin\beta\tau]\cos y\tau \\ + [\beta T\cos\beta(\tau + T) + \beta\tau \sin\beta T \sin\beta\tau \\ + \sin\beta T\cos\beta\tau][\cos y(\tau + T) - \cos\beta T\cos y\tau]\}\, d\tau \\ = \frac{1}{2\beta}\int_0^T \{(1 + \cos^2\beta T)[\beta(2T - \tau)\cos\beta\tau + \sin\beta\tau] \\ - \cos\beta T[\beta T\cos\beta(\tau + T) + \beta\tau\sin\beta T\sin\beta\tau \\ + \sin\beta T\cos\beta\tau]\}\cos y\tau\, d\tau \\ + \frac{1}{2\beta}\int_0^T \{\beta T\cos\beta(\tau + T) + \beta\tau\sin\beta T\sin\beta\tau \\ + \sin\beta T\cos\beta\tau - \cos\beta T[\beta(2T - \tau)\cos\beta\tau \\ + \sin\beta\tau]\}\cos y(\tau + T)\, d\tau \\ - \frac{\cos\beta T}{2\beta}\int_0^T [\beta(2T - \tau)\cos\beta\tau \\ + \sin\beta\tau]\cos y(\tau - T)\, d\tau. \end{aligned} \tag{43}$$

In the second integral, substitute $\tau + T = \tau'$ and in the third integral substitute $T - \tau = \tau'$. Dropping the primes after the substitution and combining terms where possible, we then find that the result can be written in the form

$$H(y) = \frac{1}{2\beta}\int_0^T h_1(\tau)\cos y\tau\, d\tau + \frac{1}{2\beta}\int_T^{2T} h_2(\tau)\cos y\tau\, d\tau \tag{44}$$

where

$$h_1(\tau) = 2\beta(T - \tau)\cos\beta\tau - \beta\tau\cos\beta(\tau - 2T) + \sin\beta(\tau - 2T) + 2\sin\beta\tau \tag{45}$$

$$h_2(\tau) = \beta(\tau - 2T)\cos\beta(\tau - 2T) - \sin\beta(\tau - 2T). \tag{46}$$

The integration in (44) leads to the result

$$H(y) = 2\sin^2\frac{y+\beta}{2}T\sin^2\frac{y-\beta}{2}T\left(\frac{1}{y-\beta}-\frac{1}{y+\beta}\right)^2. \quad (47)$$

The complete equation for the spectral density can now be written in the form

$$\begin{aligned} w_u(f) &= \frac{2A^2\sin^2\left(\frac{\omega-\omega_1}{2}\right)T\sin^2\left(\frac{\omega-\omega_2}{2}\right)T}{T[1-2\cos(\omega-\alpha)T\cos\beta T+\cos^2\beta T]} \cdot\left[\frac{1}{\omega-\omega_1}-\frac{1}{\omega-\omega_2}\right]^2 \\ &+ \frac{2A^2\sin^2\left(\frac{\omega+\omega_1}{2}\right)T\sin^2\left(\frac{\omega+\omega_2}{2}\right)T}{T[1-2\cos(\omega+\alpha)T\cos\beta T+\cos^2\beta T]} \cdot\left[\frac{1}{\omega+\omega_1}-\frac{1}{\omega+\omega_2}\right]^2. \end{aligned} \quad (48)$$

The intermediate step converting the original integral (43) to the form (44) is very helpful in reducing the labor required to obtain the final form (47). Incidentally, (44) shows that the function $H(y)$ is the Fourier transform of a function of τ which is time-limited to the range 0 to $2T$. We also point out that discrete components do not appear in $u(t)$. The spectral density function is continuous at all frequencies and varies as the inverse fourth power of the frequency at frequencies remote from ω_1 and ω_2. The latter property must exist because the waveform of the signal is continuous at all times.

We now return to the three cases of commensurability which we found necessary to avoid in deriving the general result of (48). When βT is a multiple of π, an examination of (25) shows that $B_n(t)$ differs from $\alpha t + x_{n+1}\beta t + \phi$ by a multiple of 2π, and hence

$$u(t) = A\cos(\alpha t + x_{n+1}\beta t + \phi), \qquad nT \leqq t < (n+1)T. \quad (49)$$

Comparison with (7) shows that we now have one of the degenerate cases in which the generally discontinuous phase becomes continuous: i.e., that case in which $\theta_1 = \theta_2 = \phi$ and $\omega_2 - \omega_1 = r\omega_s$, r being an integer.

When βT is a multiple of π and αT is not a multiple of π, we have $\omega_2 - \omega_1 = r\omega_s$, $\omega_2 + \omega_1 \neq l\omega_s$, and it follows that $2\omega_1/\omega_s$, $2\omega_2/\omega_s$ are not integers. Setting $\gamma(2\omega_1 T)$, $\gamma(2\omega_2 T)$, $\gamma(\omega_2 T + \omega_1 T)$ to zero in (15) and using

$$G(\omega - \omega_1) = \sin^2\left[\left(\frac{\omega - \omega_1}{2}\right)T\right] = \sin^2\left[\left(\frac{\omega - \alpha}{2}\right)T + \frac{r\pi}{2}\right]$$

$$= \begin{cases} \sin^2\left[\frac{\omega - \alpha}{2}\right]T, & r \text{ even} \\ \cos^2\left[\frac{\omega - \alpha}{2}\right]T, & r \text{ odd} \end{cases} \tag{50}$$

$$G(\omega - \omega_2) = G(\omega - \omega_1) \tag{51}$$

together with the corresponding expressions for $G(\omega + \omega_1)$, $G(\omega + \omega_2)$, gives $w_v(f)$ in

$$w_u(f) = w_v(f) + \frac{A^2}{8}\left[\delta(f - f_1) + \delta(f - f_2)\right] \tag{52}$$

where the notation is the same as in (18). When r is an even integer we calculate

$$w_v(f) = \frac{A^2}{2T}\left\{\sin^2\left(\frac{\omega - \alpha}{2}\right)T\left[\frac{1}{\omega - \omega_1} - \frac{1}{\omega - \omega_2}\right]^2 + \sin^2\left(\frac{\omega + \alpha}{2}\right)T\left[\frac{1}{\omega + \omega_1} - \frac{1}{\omega + \omega_2}\right]^2\right\} \tag{53}$$

and when r is odd

$$w_v(f) = \frac{A^2}{2T}\left\{\cos^2\left(\frac{\omega - \alpha}{2}\right)T\left[\frac{1}{\omega - \omega_1} - \frac{1}{\omega - \omega_2}\right]^2 + \cos^2\left(\frac{\omega + \alpha}{2}\right)T\left[\frac{1}{\omega + \omega_1} - \frac{1}{\omega + \omega_2}\right]^2\right\}. \tag{54}$$

In the next case both βT and αT are multiples of π; i.e., $\omega_2 - \omega_1 = r\omega_s$, $\omega_2 + \omega_1 = l\omega_s$, and $2\omega_1/\omega_s$, $2\omega_2/\omega_s$ are integers. Now all of the terms in (15) must be considered and

$$G(\omega - \omega_1) = G(\omega - \omega_2) = G(\omega + \omega_1) = G(\omega + \omega_2) = \begin{cases} \sin^2\frac{\omega T}{2} \text{ for } l - r \text{ even} \\ \cos^2\frac{\omega T}{2} \text{ for } l - r \text{ odd.} \end{cases} \tag{55}$$

Combining terms in (15) gives

$$w_v(f) = \frac{A^2}{2T}\begin{Bmatrix}\sin^2\frac{\omega T}{2}\text{ for } l - r \text{ even}\\ \cos^2\frac{\omega T}{2}\text{ for } l - r \text{ odd}\end{Bmatrix}\left[\left(\frac{1}{\omega-\omega_1}-\frac{1}{\omega-\omega_2}\right)^2 + \left(\frac{1}{\omega+\omega_1}-\frac{1}{\omega+\omega_2}\right)^2 + 2\left(\frac{1}{\omega-\omega_1}-\frac{1}{\omega-\omega_2}\right)\left(\frac{1}{\omega+\omega_1}-\frac{1}{\omega+\omega_2}\right)\cos 2\phi\right]. \tag{56}$$

In the last of the exceptional cases, both αT and βT are odd multiples of $\pi/2$. That is,

$$\begin{aligned} 2\alpha T &= (2l + 1)\pi, \qquad l = 0, 1, 2, \cdots \\ 2\beta T &= (2r + 1)\pi, \qquad r = 0, 1, 2, \cdots. \end{aligned} \tag{57}$$

Equivalently

$$\begin{aligned} \omega_2/\omega_s &= (l + r + 1)/2 \\ \omega_1/\omega_s &= (l - r)/2. \end{aligned} \tag{58}$$

The value of E_0 obtained by substituting (57) in (34) is

$$\frac{2}{A^2}E_0(t + nT, \tau) = \cos\alpha\tau\cos\beta\tau + \tfrac{1}{2}\cos[(\alpha+\beta)(2t+\tau)+2\phi] + \tfrac{1}{2}\cos[(\alpha-\beta)(2t+\tau)+2\phi]. \tag{59}$$

Substituting $k = 1$ in (35) and then inserting the special conditions of (57) we obtain

$$\frac{4}{A^2}E_1(t + nT, \tau) = \cos\alpha\tau[\cos\beta\tau - \cos\beta(2t+\tau)] - \cos\beta\tau\cos[\alpha(2t+\tau)+2\phi] + \tfrac{1}{2}\cos[(\alpha+\beta)(2t+\tau)+2\phi] + \tfrac{1}{2}\cos[(\alpha-\beta)(2t+\tau)+2\phi]. \tag{60}$$

Since βT is an odd multiple of $\pi/2$, the value of $\cos\beta T$ is zero. For $k > 1$ the right-hand member of (35) contains $\cos\beta T$ as a factor. Hence $E_k(t + nT, \tau)$ vanishes for $k > 1$. It follows that the autocorrelation function vanishes for $\tau > 2T$ and there can be no discrete sinusoidal components in the spectral density function.

A better understanding of this remarkable behavior can be obtained by examination of a particular signaling interval in which we are equally likely to find one of the two possible waves $A\cos(\omega_1 t + \psi_1)$ and

$A \cos (\omega_2 t + \psi_2)$. At the next switching instant, say $t = nT$, we either continue with the same wave or shift to the other frequency with continuous phase. There are thus four possible waves in the succeeding interval, viz.,

$$(i)\ A \cos (\omega_1 t + \psi_1)$$
$$(ii)\ A \cos [\omega_2 t + \psi_1 - (\omega_2 - \omega_1)nT]$$
$$(iii)\ A \cos [\omega_1 t + \psi_2 + (\omega_2 - \omega_1)nT]$$
$$(iv)\ A \cos (\omega_2 t + \psi_2).$$

Since $(\omega_2 - \omega_1)T$ is an odd multiple of π, the second and third terms can be written as $(-)^n A \cos (\omega_2 t + \psi_1)$ and $(-)^n A \cos (\omega_1 t + \psi_2)$ respectively. Waves (i) and (ii) are possible when the initial frequency is ω_1, and waves (iii) and (iv) are possible when the initial frequency is ω_2. Now examining the possibilities after the next subsequent switching instant, $t = (n + 1)T$, we find that for the original frequency equal to ω_1, the waves (i) and (ii) can change to the four possible waves:

$$(i)\ A \cos (\omega_1 t + \psi_1)$$
$$(ii)\ (-)^{n+1} A \cos (\omega_2 t + \psi_1)$$
$$(iii)\ (-)^n A \cos (\omega_2 t + \psi_1)$$
$$(iv)\ -A \cos (\omega_1 t + \psi_1).$$

It will be noted that the first and fourth waves are the same except for opposite signs and likewise for the second and third. In other words, for any observed value of frequency in a specified interval, the possible waveforms in the second succeeding interval can be divided into equally likely positive and negative matching pairs. This behavior, once established, must continue into all succeeding intervals. Hence the average lag product over the ensemble at fixed t and τ must vanish for τ greater than $2T$.

This may also be seen by noting that when $2\beta T = (2r + 1)\pi$, the quantity $\beta T(x_1 + \cdots + x_n - nx_{n+1})$ appearing in the definition (25) of $B_n(t)$ is an even or odd multiple of π according to whether $(x_1 + \cdots + x_n - nx_{n+1})/2 = r_n$ is even or odd. Hence

$$u(t) = \begin{cases} A(-)^{r_n} \cos (\omega_2 t + \phi), & x_{n+1} = 1 \\ A(-)^{r_n} \cos (\omega_1 t + \phi), & x_{n+1} = -1. \end{cases} \tag{61}$$

For any observed set of $x_1, \cdots x_{n+1}$ leading to a definite $u(t)$ in the nth interval, the waveforms in the second succeeding interval can be written as

$$\begin{aligned} &A(-)^{r_n+(r_{n+2}-r_n)} \cos(\omega_2 t + \phi), \qquad x_{n+3} = 1 \\ &A(-)^{r_n+(r_{n+2}-r_n)} \cos(\omega_1 t + \phi), \qquad x_{n+3} = -1. \end{aligned} \tag{62}$$

Since $r_{n+2} - r_n$ contains x_{n+2} only through the term $x_{n+2}/2$, the forms in (62) are of random sign independent of the original form (61). Hence the waveform in the second succeeding interval is entirely independent (in sign and frequency) of the waveform in the original interval.

Since $E_0(t + nT, \tau)$ and $E_1(t + nT, \tau)$ in (59) and (60) are found not to depend on n, we can evaluate the autocorrelation function by averaging over t in a single signaling interval as follows

$$\begin{aligned} R_u(\tau) &= \frac{1}{T}\int_0^{T-\tau} E_0(t + nT, \tau)\, dt, \\ &\quad + \frac{1}{T}\int_{T-\tau}^{T} E_1(t + nT, \tau)\, dt \qquad 0 < \tau < T \end{aligned} \tag{63}$$

$$R_u(\tau) = \frac{1}{T}\int_0^{2T-\tau} E_1(t + nT, \tau)\, dt, \qquad T < \tau < 2T. \tag{64}$$

For $0 < \tau < T$, we calculate

$$\begin{aligned} \frac{4T}{A^2} R_u(\tau) &= \left[(2T - \tau)\cos\beta\tau + \frac{\sin\beta\tau}{\beta}\right]\cos\alpha\tau \\ &+ \cos 2\phi\left[\frac{\cos\beta\tau\sin\alpha\tau}{\alpha} - \frac{\sin(\alpha+\beta)\tau}{2(\alpha+\beta)} - \frac{\sin(\alpha-\beta)\tau}{2(\alpha-\beta)}\right]. \end{aligned} \tag{65}$$

For $T < \tau < 2T$,

$$\begin{aligned} \frac{4T}{A^2} R_u(\tau) &= \left[(2T - \tau)\cos\beta\tau + \frac{\sin\beta\tau}{\beta}\right]\cos\alpha\tau \\ &- \cos 2\phi\left[\frac{\cos\beta\tau\sin\alpha\tau}{\alpha} - \frac{\sin(\alpha+\beta)\tau}{2(\alpha+\beta)} - \frac{\sin(\alpha-\beta)\tau}{2(\alpha-\beta)}\right]. \end{aligned} \tag{66}$$

The spectral density function is then found to be

$$\begin{aligned} w_u(f) &= 4\int_0^{2T} R_u(\tau)\cos\omega\tau\, d\tau \\ &= A^2\frac{\sin^2\omega T}{2T}\left[\left(\frac{1}{\omega-\omega_1} - \frac{1}{\omega-\omega_2}\right)^2 + \left(\frac{1}{\omega+\omega_1} - \frac{1}{\omega+\omega_2}\right)^2\right. \\ &\quad \left. + 2\left(\frac{1}{\omega-\omega_1} - \frac{1}{\omega-\omega_2}\right)\left(\frac{1}{\omega+\omega_1} - \frac{1}{\omega+\omega_2}\right)\cos 2\phi\right]. \end{aligned} \tag{67}$$

The difference between this result and the limit of the general expression (48) when the special case (57) is substituted consists of the term containing 2ϕ. This exceptional case thus has the property of remembering the initial phase angle even though the spectral density is continuous. The reason is that the phase of the wave with respect to the signaling interval must remain fixed throughout all time and is, therefore, not subject to averaging as in the more general case.

IV. FOURIER TRANSFORM METHOD

Expressions (48) and (67) for $w_u(f)$ have been obtained by first computing $R_u(\tau)$ and then taking its Fourier transform. As mentioned earlier, $w_u(f)$ can also be obtained by working with a Fourier-type integral of $u(t)$ taken over a long time interval. This method will now be sketched for the case in which $2\beta T$ is not a multiple of π. Most of the intermediate steps are omitted. If they were included, the length of the derivation would be comparable to the derivation based on $R_u(\tau)$.

The spectral density $w_u(f)$ of $u(t) = A \cos B_n(t)$ is the limit of $2\langle | S_N(f, NT) |^2 \rangle / NT$ as $N \to \infty$. In this expression

$$S_N(f, NT) = \int_0^{NT} e^{-j\omega t} u(t)\, dt = \sum_{n=0}^{N-1} s_n \tag{68}$$

$$s_n = A e^{-j\omega nT} \int_0^T e^{-j\omega t} \cos B_n(t + nT)\, dt \tag{69}$$

where s_n is a function of f. The ensemble average of

$$\begin{aligned} | S_N(f, NT) |^2 &= \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} s_n s_m^* \\ &= \sum_{n=0}^{N-1} s_n s_n^* + \sum_{k=1}^{N-1} \sum_{n=0}^{N-k-1} (s_n s_{n+k}^* + s_{n+k} s_n^*) \end{aligned} \tag{70}$$

is the sum of terms of the form

$$\begin{aligned} \langle s_{n+k} s_n^* \rangle = A^2 e^{-j\omega kT} \int_0^T e^{-j\omega t}\, dt \int_0^T e^{j\omega \tau} \\ \langle \cos B_{n+k}(t + nT + kT) \cos B_n(\tau + nT) \rangle\, d\tau. \end{aligned} \tag{71}$$

In these equations s_n^* denotes the conjugate complex of s_n.

The procedure used to obtain expressions (30) and (31) for $E_0(t, \tau)$ and $E_k(t, \tau)$ leads to

$$2\langle \cos B_n(t + nT) \cos B_n(\tau + nT)\rangle = \cos \alpha(t - \tau) \cos \beta(t - \tau) + \cos [\alpha(t + \tau + 2nT) + 2\phi] \cos \beta(t + \tau) \cos^n 2\beta T \tag{72}$$

$$2\langle \cos B_{n+k}(t + nT + kT) \cos B_n(\tau + nT)\rangle = \cos \alpha(t - \tau + kT) \cos \beta t \cos \beta(\tau - T) \cos^{k-1} \beta T + \cos [\alpha(t + \tau + 2nT + kT) + 2\phi] \cos \beta t \cos \beta(\tau + T) \cos^{k-1} \beta T \cos^n 2\beta T \tag{73}$$

where $k > 0$ in the last equation.

We shall consider only the case in which $2\beta T$ is not a multiple of π. Then the terms in (72) and (73) containing $\cos^n 2\beta T$ contribute nothing to the left-hand sides of

$$\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-1} \langle s_n s_n^*\rangle = 2^{-1}A^2 \int_0^T e^{-j\omega t}\, dt \int_0^T e^{j\omega\tau}\, d\tau \cos \alpha(t - \tau) \cos \beta(t - \tau) \tag{74}$$

$$\lim_{N\to\infty} N^{-1} \sum_{n=0}^{N-k-1} \langle s_{n+k} s_n^*\rangle = 2^{-1}A^2 e^{-j\omega kT} \cos^{k-1} \beta T \int_0^T e^{-j\omega t}\, dt \int_0^T e^{j\omega\tau} \cos \alpha(t - \tau + kT) \cos \beta t \cos \beta(\tau - T)\, d\tau. \tag{75}$$

Expression (48) for the spectral density $w_u(f)$ is now obtained by performing the integrations and then summing with respect to k as indicated in (70).

V. ILLUSTRATIVE CURVES

Fig. 1 shows a typical curve for the spectral density function when the phase is discontinuous at the instants of transition. The curve is calculated from

$$w_u(f)/A^2 = \frac{\delta(f - f_1)}{8} + \frac{\delta(f - f_2)}{8} + \frac{G(\omega - \omega_1)}{2T(\omega - \omega_1)^2} + \frac{G(\omega - \omega_2)}{2T(\omega - \omega_2)^2} \tag{76}$$

which is an approximation to (15) and (18). It holds when $\omega_2 - \omega_1$ is not a multiple of ω_s, and in addition ω_1 and ω_2 are so large that the portion of the spectrum folded back from $\omega = 0$, i.e., the portion de-

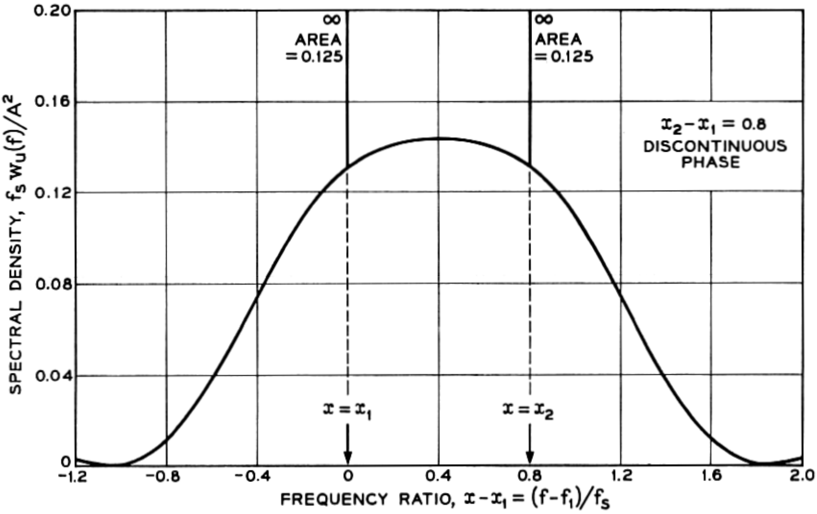

Fig. 1 — Spectral density of random binary FSK wave with discontinuous phase at transitions. Frequency shift = 0.8 times signaling frequency.

pending on inverse powers of $\omega + \omega_1$ and $\omega + \omega_2$, is negligible. The contribution of the neglected terms becomes appreciable only when the marking or spacing frequency is less than the signaling frequency. It is convenient to let $x = \omega/\omega_s = f/f_s$, $x_1 = f_1/f_s$, and $x_2 = f_2/f_s$.

Fig. 1 is calculated for the case $\omega_2 - \omega_1 = 0.8\omega_s$, i.e., $x_2 - x_1 = 0.8$. The abscissa is $x - x_1$. The ordinate is $f_s w_u(f)/A^2$. The curve is symmetrical about $x - x_1 = 0.4$. The steady-state terms are represented by spikes of infinite height and infinitesimal width at $x = x_1$ and $x = x_2$. Each of these spikes has an area of $\frac{1}{8}$. The area under the continuous curve is $\frac{1}{4}$. The total area is $\frac{1}{2}$, which is the mean-square value of the signal wave per unit of squared amplitude.

Fig. 2 shows a case of continuous phase corresponding to a frequency shift equal to 0.8 times the signaling frequency. This curve was computed from (48) where, again, the terms containing inverse powers of $\omega + \omega_1$ and $\omega + \omega_2$ are assumed to be negligibly small. Since the infinite spikes representing steady-state components are absent, the total area under the curve must be $\frac{1}{2}$. We note that peaks of finite height and width appear just outside the interval bounded by the marking and spacing frequencies. These peaks become more pronounced and move toward the marking and spacing frequencies as we approach the commensurable case in which the frequency shift $f_2 - f_1$ is exactly equal to the signaling rate f_s. Fig. 3 shows the curve for frequency shift equal to 0.95 times

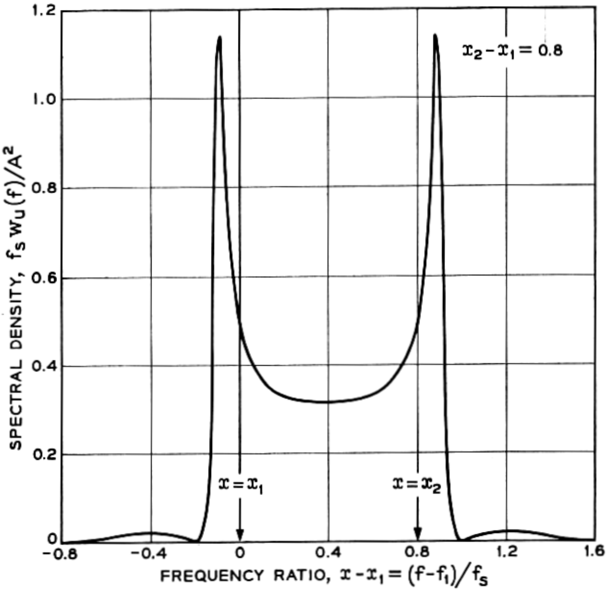

Fig. 2 — Spectral density of random binary FSK wave with continuous phase at transitions. Frequency shift = 0.8 times signaling frequency.

the signaling rate. Here the peaks are almost twenty times as high as in Fig. 2. The limiting case of $x_2 - x_1 = 1$ is exhibited in Fig. 4. The finite spikes of Fig. 2 and 3 have now become full-fledged impulses of infinite height, infinitesimal width, and area $\frac{1}{8}$. They represent the mean-square value of steady-state components at the marking and spacing frequencies. The continuous part of the curve was calculated from (54), noting that $(\omega - \alpha)T = 2\pi(x - x_1 - \frac{1}{2})$ in this case. Fig. 5 shows a representative curve on the other side of the limiting case, with the frequency shift taken equal to 1.2 times the signaling rate. The finite peaks now appear inside the interval between marking and spacing frequencies.

It is instructive to study the transition from Figs. 2 and 3 to Fig. 4. Since the curves are symmetrical about $x - x_1 = (x_2 - x_1)/2$, it is sufficient to consider the region of rapid change near $x = x_1$. Setting $x_2 - x_1 = x_d$, we approximate (48) for $w_u(f)$ in this region by

$$w_u(f) \approx \frac{A^2 \sin^2 (x - x_1)\pi \sin^2 (x - x_2)\pi}{2\pi^2 f_s (x - x_1)^2 [1 - 2 \cos (2x - x_2 - x_1)\pi \cos x_d\pi + \cos^2 x_d\pi]}. \tag{77}$$

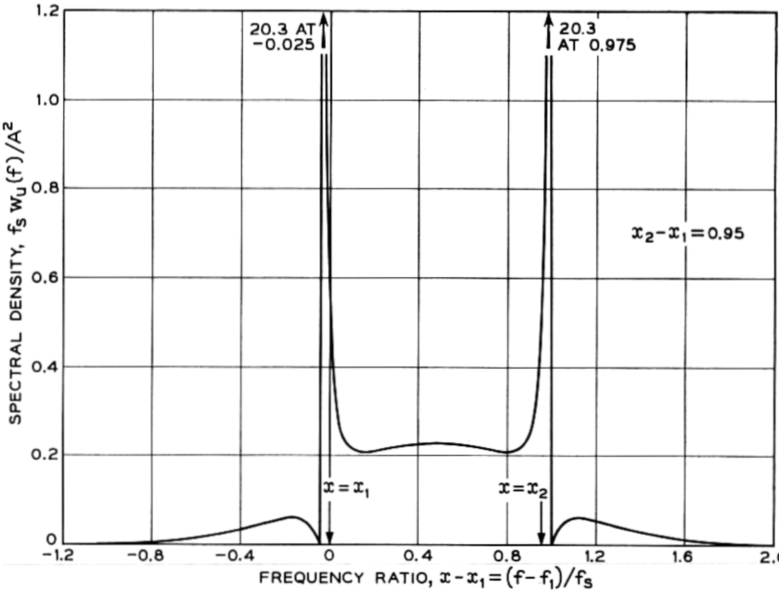

Fig. 3 — Spectral density of random binary FSK wave with continuous phase at transitions. Frequency shift = 0.95 times signaling frequency.

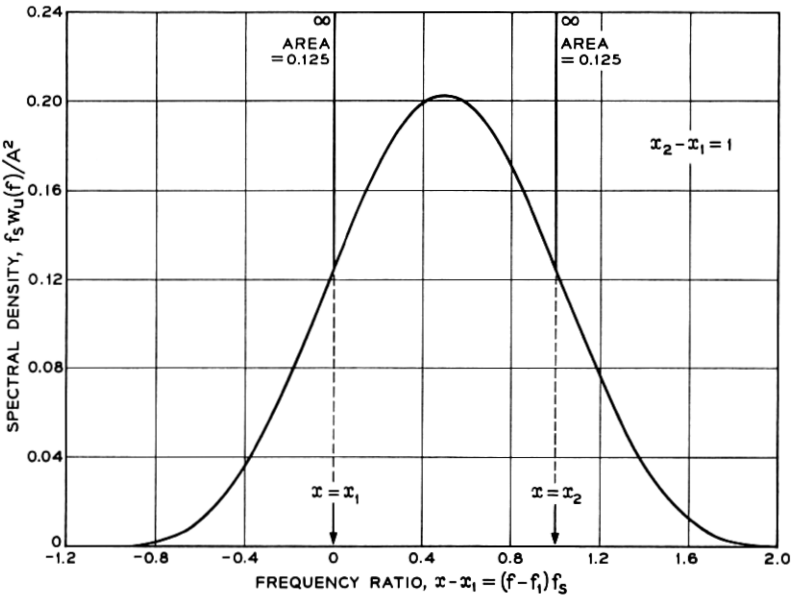

Fig. 4 — Spectral density of random binary FSK wave with continuous phase at transitions. Frequency shift = signaling frequency.

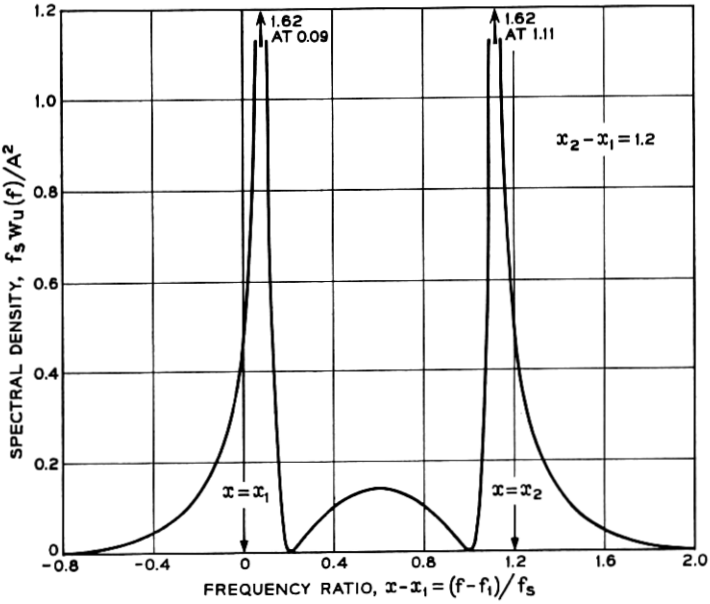

Fig. 5 — Spectral density of random binary FSK wave with continuous phase at transitions. Frequency shift = 1.2 times signaling frequency.

We are interested in the behavior of (77) as $x_2 - x_1$ approaches unity. Setting $x_2 - x_1 = 1 - \epsilon$ and $x - x_1 = y$, we find that when both ϵ and y are small compared with unity we can approximate (77) by

$$Y = f_s w_u(f)/A^2 \approx \frac{1}{8} \frac{(y + \epsilon)^2}{(\pi\epsilon^2/4)^2 + (y + \epsilon/2)^2}. \tag{78}$$

It is seen that Y depends on y approximately as shown in Table I. The value $y = \pm\infty$ corresponds to several positive or negative multiples of ϵ, and therefore actually becomes small in absolute value as ϵ approaches zero. It follows that Y hitches onto the value 0.125 shown at $x = x_1$ in Fig. 4. The curves shown in Figs. 2 and 3 correspond to $\epsilon = 0.2$ and $\epsilon = 0.05$. They show the behavior indicated by (78). In particular,

Table I — Approximate Ordinates of Spectral Density in the Neighborhood of Peak

y	$-\infty$	$-\epsilon$	$-\epsilon/2$	0	$+\infty$
Y	$\frac{1}{8}$	0	$1/(2\pi^2\epsilon^2)$	$\frac{1}{2}$	$\frac{1}{8}$

Y obtains its peak value of approximately $1/(2\pi^2\epsilon^2)$ near $y = -\epsilon/2$ and drops down to about half the peak value at $y = -\epsilon/2 \pm \pi\epsilon^2/4$. When $\epsilon = 0.05$, the peak value of Y is 20.3. The area under the peak, as measured by the integral of Y taken from $y = -\epsilon$ to $y = +\epsilon$, approaches $\frac{1}{8}$ as ϵ approaches zero. This agrees in the limit with the area of the impulses shown in Fig. 4.

The work of Sunde[1] has indicated that the special case in which the frequency shift is equal to the bit rate has a theoretical advantage in that intersymbol interference can be suppressed at the sampling instants in the output of an ideal frequency detector. The results presented here show one method by which such a frequency lock can be attained. Since the two principal peaks of the spectral density function reach maximum height when the condition $x_2 - x_1 = 1$ is attained, the output of a spectral analyzer can be used to determine the proper bias on the tuning control of the keyed oscillator. Another possible instrumentation can be devised by use of the autocorrelation function. When the signal wave with continuous phase transitions is multiplied by itself delayed by a large multiple of the bit interval, the average value of the product tends toward zero except when the frequency shift is locked to the bit rate. In practice, the advantage of a rigid lock-in to the theoretical optimum has not proved to be very significant. The actual reduction of intersymbol interference which could be achieved by choosing the best value of frequency shift would typically be masked by other departures from the ideal conditions.

Fig. 6 illustrates the case in which αT and βT are odd multiples of $\pi/2$. The significantly different properties exhibited are not very pronounced except when marking and spacing frequencies are sufficiently low to be comparable with the signaling rate. The case shown in Fig. 6 applies when the marking frequency is half the signaling frequency and the spacing frequency is equal to the signaling frequency. The frequency shift is half the signaling frequency. The curves are calculated from (67) for three different values of the initial phase angle. Since there are no steady-state components, the area under each curve must be 0.5. The peak of the spectral density function changes from 0.763 to 0.857 as the cosine of the initial phase angle is varied from -1 to $+1$. The ordinates become zero at $x = \frac{3}{2}, 2, \frac{5}{2}, \cdots$. As x approaches infinity the maximum values of the intervening loops decrease as x^{-6} when $\cos\phi = 1$ and as x^{-4} for other values of $\cos\phi$.

When the values $f_1 = f_s/2$, $f_2 = f_s$ corresponding to Fig. 6 are substituted in the general equation (48) for $w_u(f)$, the result is the case $\cos 2\phi = 0$ shown in Fig. 6. This is to be expected since when the con-

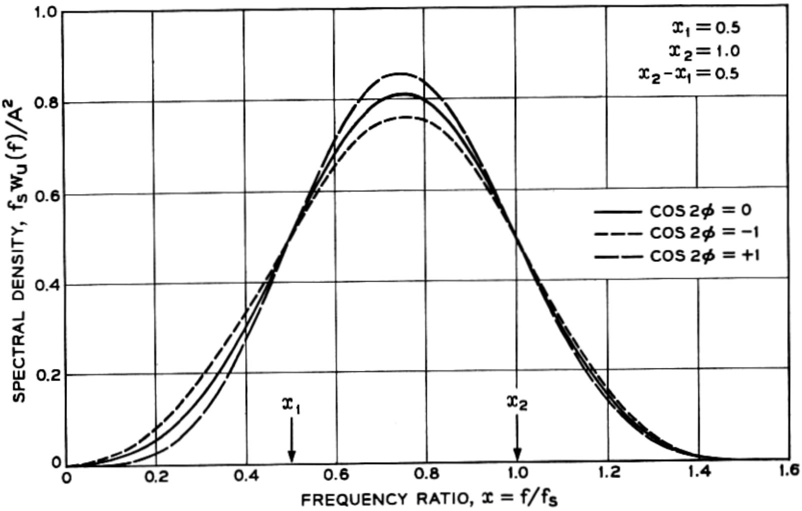

Fig. 6 — Spectral density of random binary FSK wave with continuous phase at transitions. Frequency shift = 0.5 times signaling frequency. Marking frequency = signaling frequency. Spacing frequency = 0.5 times signaling frequency.

ditions $f_1 = f_s/2$, $f_2 = f_s$ are almost (but not quite) satisfied, the phase of $u = A \cos B_n(t)$ changes by a small amount, or by π plus a small amount, from one transition point to another. Over a long period of time these small changes accumulate and have the same effect as replacing $\cos 2\phi$ by its average value 0.

Figs. 7–12 inclusive show the normalized autocorrelation functions corresponding to the cases of Figs. 1–6 respectively. To avoid making the curves depend on the values of marking and spacing frequencies, we have indicated the envelope of the high-frequency oscillations in Figs. 7–11. The autocorrelations are obtained by multiplying the solid line curves by $\frac{1}{2}A^2/\cos[(\omega_2 + \omega_1)\tau/2]$, which in terms of the lag time τ is a cosine wave at the midband frequency. The resulting oscillation has the value unity at $\tau = 0$ and is contained within the solid and dashed curves. Fig. 12, which is drawn for specified marking and spacing frequencies, shows an actual autocorrelation function.

The typical case of discontinuous phase, which is illustrated in Fig. 7, has a linearly damped envelope until τ reaches the value T. At time T the envelope changes continuously to that of the sum of two cosine waves at the marking and spacing frequencies. The latter envelope is a cosine wave at half the difference frequency and it persists with undiminished amplitude throughout all values of τ greater than T. Fig. 8

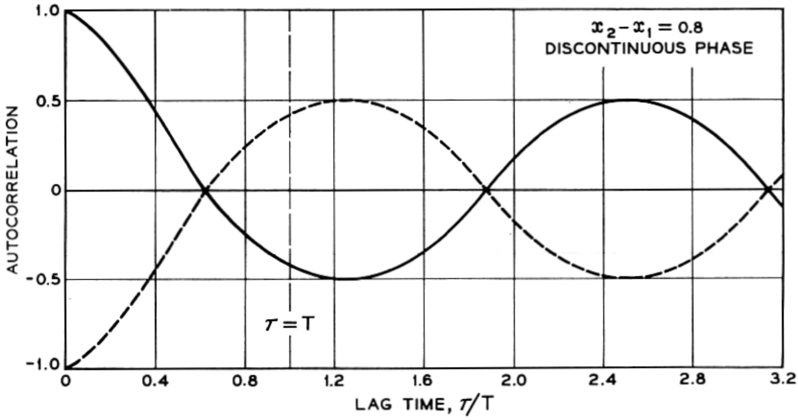

Fig. 7 — Envelope of autocorrelation function when phase is discontinuous at transitions and $f_2 - f_1 = 0.8f_s$. Actual autocorrelation is full line curve multiplied by $(A^2/2) \cos [(\omega_2 + \omega_1)\tau/2]$.

represents the same case as Fig. 7 except that the phase is continuous. The effect is that the envelope of the autocorrelation in Fig. 8 decays to zero at infinite lag time instead of oscillating with constant amplitude. The decay in each multiple of T after the second one is produced by a multiplication of the corresponding values in the preceding interval by $\cos (x_2 - x_1)\pi$, which has the value -0.809 in Fig. 8. As $x_2 - x_1$ approaches unity, the multiplying factor produces only a slight reduction

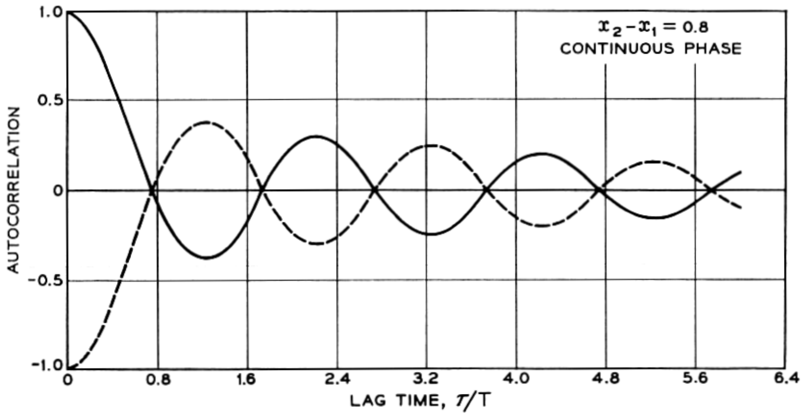

Fig. 8 — Envelope of autocorrelation function when phase is continuous at transitions and $f_2 - f_1 = 0.8f_s$. Actual autocorrelation is full line curve multiplied by $(A^2/2) \cos [(\omega_2 + \omega_1)\tau/2]$.

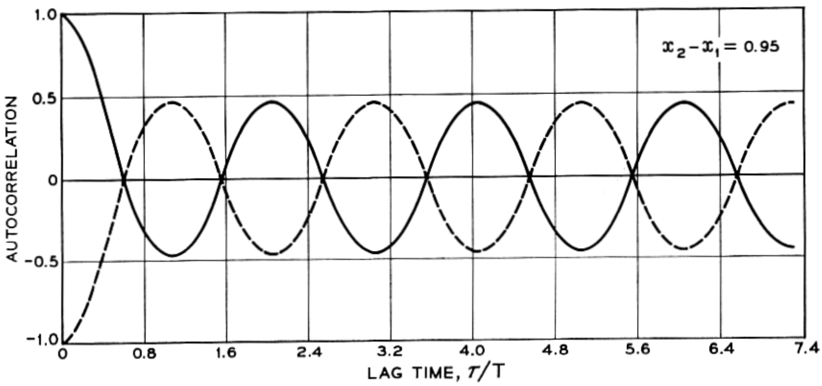

Fig. 9 — Envelope of autocorrelation function when phase is continuous at transitions and $f_2 - f_1 = 0.95f_s$. Actual autocorrelation is full line curve multiplied by $(A^2/2) \cos [(\omega_2 + \omega_1)\tau/2]$.

in each interval and the oscillations retain appreciable amplitude for very large lag times. Such behavior is emphasized in Fig. 9 for the case of $x_2 - x_1 = 0.95$, $\cos (x_2 - x_1)\pi = -0.9877$. The very slow departure from constant amplitude oscillations indicates that the signal wave contains components which are very nearly sinusoidal. The time domain analysis thus agrees with the sharp high peaks found in the frequency domain analysis, as shown in Fig. 3 for $x_2 - x_1 = 0.95$.

Fig. 10 shows the limiting case in which the phase is continuous and

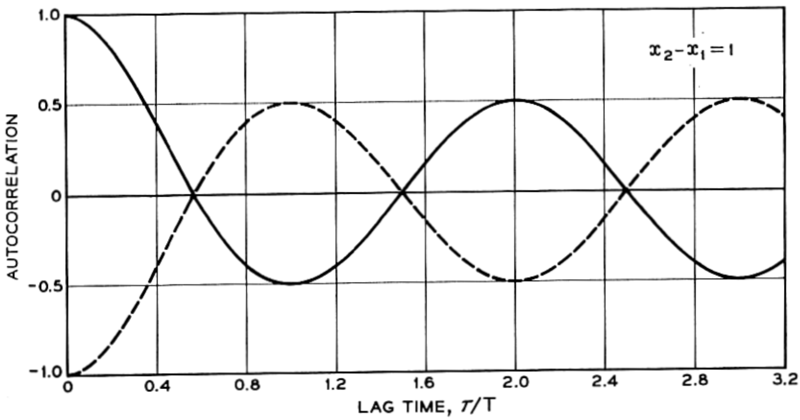

Fig. 10 — Envelope of autocorrelation function when phase is continuous at transitions and $f_2 - f_1 = f_s$. Actual autocorrelation is full line curve multiplied by $(A^2/2) \cos [(\omega_2 + \omega_1)\tau/2]$.

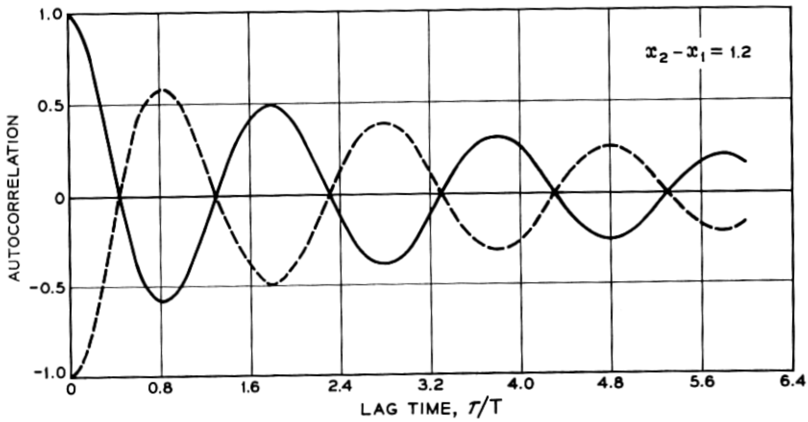

Fig. 11 — Envelope of autocorrelation function when phase is continuous at transitions and $f_2 - f_1 = 1.2 f_s$. Actual autocorrelation is full line curve multiplied by $(A^2/2) \cos [(\omega_2 + \omega_1)\tau/2]$.

$x_2 - x_1 = 1$. The appearance of the line spectral terms is indicated by the constancy of the amplitude of oscillations for $\tau > T$. Fig. 11 for $x_2 - x_1 = 1.2$ corresponds to Fig. 5. The decay rate is the same as in Fig. 8, since $\cos 1.2\pi = \cos 0.8\pi$. The period of the oscillations is decreased.

Fig. 12 shows the singular case in which the sum and difference frequencies are both odd multiples of half the signaling frequency. The values chosen are the same as those of Fig. 6. The autocorrelation func-

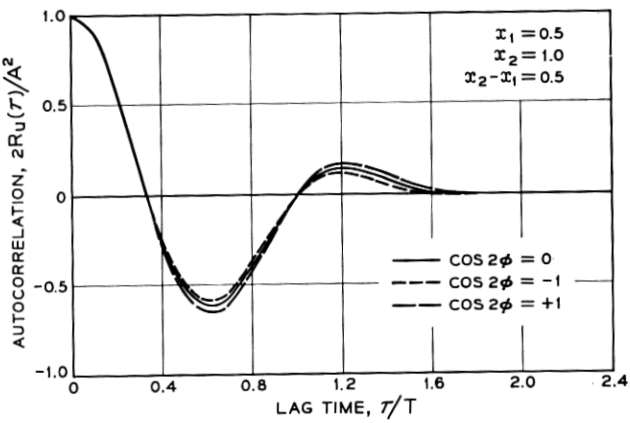

Fig. 12 — Normalized autocorrelation function when phase is continuous, $f_1 = f_s/2$, $f_2 = f_s$, and $f_2 - f_1 = f_s/2$.

tion is time limited, vanishing for all values of τ greater than $2T$. The dependence on initial phase is indicated by the three curves, which are drawn for the cases of $\cos 2\phi = 0$, 1, and -1 as in Fig. 6. The total variation in the height of the negative peak with ϕ is 0.07.

Fig. 7 differs from Figs. 8–12 in that the discontinuity in phase of the signal wave produces a discontinuity in the slope of the autocorrelation curve at $\tau = T$. This is consistent with the decay of the spectral density of Fig. 1 with the inverse square of frequency at high frequencies. The spectral densities of Figs. 2 to 6 vary ultimately as the inverse fourth power of frequency, which requires not only the slope but the second derivative of the corresponding autocorrelation functions, Figs. 8 to 12, to be continuous at all values of τ.

VI. SUMMARY OF RESULTS FOR SPECTRAL DENSITY AND AUTOCORRELATION

Table II lists the equation numbers of the expressions giving $w_u(f)$ and $R_u(\tau)$ for the various cases which can arise. Let $f_s = 1/T$ be the signaling frequency and f_1, f_2 be the marking and spacing frequencies. Also, let l, r denote integers.

TABLE II — LIST OF EQUATIONS FOR SPECTRAL DENSITY AND AUTOCORRELATION OF FSK WAVE

Case	Equation Numbers	
	$w_u(f)$	$R_u(\tau)$
Discontinuous phase:		
(a) general case	(15), (18)	(19)
(b) degenerate cases	(17), (18)	(19)
Continuous phase:		
(c) $f_2 - f_1 = rf_s$, $f_2 + f_1 \neq lf_s$	(52), (53), (54)	(19)
(d) $f_2 - f_1 = rf_s$, $f_2 + f_1 = lf_s$	(52), (56)	(19)
(e) $\left.\begin{aligned} f_2 - f_1 &= (r + \tfrac{1}{2})f_s \\ f_2 + f_1 &= (l + \tfrac{1}{2})f_s \end{aligned}\right\}$	(67)	(65), (66)
(f) all other continuous phase cases	(48)	(36), (37)

VII. OTHER RELATED PUBLICATIONS

Jenks and Hannon[2] have given spectral density curves for the case of frequency shift equal to bit rate which they state have been taken from a forthcoming paper by Pushman in the Journal of the British Institute of Radio Engineers. The curves shown are in agreement with ours for the same case. We have not seen the complete work. Our interest in the problem was initially stimulated by discussions with I. Dorros, who has made use of some of our results in a study[3] of the transmission of binary data by FM over a band-limited channel. Since completing the work, we have become aware of a publication by Postl,[4] who has calculated the spectral density for binary continuous phase narrow-band FSK in which the midband frequency is large compared with both the frequency shift and the signaling rate.

REFERENCES

1. Sunde, E. D., Ideal Binary Pulse Transmission by AM and FM, B.S.T.J., **38,** November, 1959, pp. 1357–1426.
2. Jenks, F. G. and Hannon, D. C., Comparison of the Merits of Phase and Frequency Modulation for Medium Speed Serial Binary Digital Data Transmission Over Telephone Lines, J. Brit. I.R.E., **24,** July, 1962, pp. 21–36.
3. Dorros, I., Performance of a Binary FM System as a Function of the Channel, Columbia University Doctoral Dissertation, November, 1962.
4. Postl, W., Die Spektrale Leistungsdichte bei Frequenzmodulation eines Trägers mit einem Stochastischen Telegraphiesignal, Frequenz, **17,** March, 1963, pp. 107–110.