Interchannel Interference in FM Systems Produced by Continuous Random Mode Conversion in Circular Electric Waveguide

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Long-distance, high-capacity transmission via the TE_{01} mode in circular waveguide is an attractive goal because the theoretical attenuation caused by heat loss decreases monotonically as the frequency of operation increases. However, continuous random mode conversion caused by the manufacturing or the laying of the waveguide causes severe random fluctuations with frequency in the transfer function of present-day waveguide, a fact which apparently eliminates all but the "toughest" modulation schemes, such as pulse code modulation and angle modulation.

In this paper, we present the derivation of a technique for analyzing the effects of continuous random mode conversion on an angle-modulated wave. The random coupling coefficient is assumed to have a Gaussian probability density. The modulating signal is assumed to be a single-sideband frequency division multiplex which may be simulated by a band of random Gaussian noise. The quantity of interest is the expected value of the interchannel interference noise which appears at baseband. A self-contained section on principal results and examples is presented. The examples use typical state-of-the-art data and are significant in that they demonstrate that FM appears to be an attractive modulation scheme. They are, however, not meant to be an exhaustive study of the problem. Copper waveguide (with and without mode filters) and helix waveguide are considered.

The Fourier series, multiple-echo representation of a stochastic transfer function presented in this paper should prove useful in handling other modulation schemes as well as other random channels.

I. INTRODUCTION

Long-distance, high-capacity transmission via the TE_{01} mode in the millimeter-wave circular-electric waveguide is an attractive goal because the theoretical attenuation caused by heat loss decreases monotonically as the frequency of operation increases. However, in order to obtain sufficiently small heat loss attenuation and delay distortion, the operating frequency must be well above the TE_{01} cutoff frequency and far into the multimode region. For example, a two-inch I.D. perfect copper circular guide will have a theoretical TE_{01} heat loss of 1.54 db per mile at 55 kmc and will propagate 223 additional (spurious) modes.[1] Real waveguides are never perfect right-circular cylinders but have continuous random imperfections arising from the manufacturing or the laying of the waveguide. These random imperfections cause conversion of the TE_{01} signal mode to spurious modes.[2] We call the attenuation produced by this energy loss "conversion attenuation." Usually more important, the imperfections also cause converted energy in the spurious modes to be reconverted into the TE_{01} signal mode. Since the group velocities of the various modes will be different, the reconverted energy from each mode will be delayed or advanced in time with respect to the energy which has remained in the TE_{01} signal mode. Thus we have a signal plus a continuum of echoes. The effect causes the amplitude and phase of the TE_{01} wave to fluctuate randomly with frequency. A transfer function which is a random variable can be defined to take this fluctuation into account in a statistical sense. We call it the "reconversion transfer function."

The fluctuations in the reconversion transfer function distort any signal which might be transmitted through the waveguide. A "tough" modulation scheme is required in order to overcome this effect. In this paper we consider the use of large-index frequency modulation (FM). The recent work of Rowe and Warters[2] provides us with a sound and tractable mathematical analysis of continuous random mode conversion. We shall use the results of their work as our starting point. The baseband signal used to frequency or phase modulate the carrier wave is assumed to consist of a single-sideband frequency division multiplex of telephone channels. For analysis purposes we can approximate this baseband signal by a band of random Gaussian noise.[3] When such an angle-modulated wave is passed through a network whose attenuation and phase are nonlinear functions of frequency, interchannel interference appears in the demodulated signal. In way of explanation, imagine that the energy in the baseband modulating signal corresponding to some particular telephone channel is removed. When such a signal is impressed on a frequency modulator and the resulting wave is transmitted through the network in question, detected, and finally demodulated, the received output in the originally clear channel represents interchannel interference. Measurements of this type have been discussed by Albersheim and Schafer.[4]

To the author's knowledge, there is only one tractable technique for determining interchannel interference in angle-modulated systems which does not require that the fluctuation rate of the attenuation and phase functions be slow over the bandwidth occupied by the signal. This exception is the method of equivalent echoes.[5,6] Since the phase and attenuation of the reconversion transfer function fluctuates rapidly across the bandwidths of interest, we are necessarily confined to this technique or variations thereof. We see that some sort of multiple-echo approximation to the reconversion transfer function is required if any accuracy in our results is to be expected. How many echoes are required, and how should they be distributed and weighted? The answer is clear when we recall that a small echo produces or is equivalent to a sinusoidal variation with frequency in the transfer function, i.e., the Fourier transform of the echo. We simply expand the reconversion transfer function in a Fourier series over a frequency interval which we know to be sufficiently large to contain all of the significant energy in the FM wave. The Fourier transform of this series is the desired train of echoes and is an exact representation over the bandwidth under consideration. The echoes are regularly spaced, and their amplitudes are proportional to the magnitude of the coefficients of the Fourier series. The problem is now reduced to obtaining the Fourier series coefficients, and until recently this would have been a major stumbling block. Fortunately, however, Rowe, Warters, and Young have recently made important contributions[2,7,8] to the understanding of continuous random mode conversion. They have demonstrated that over the moderate percentage bandwidths in which we are interested, the effects of random mode conversion give rise to a transfer function which is a random function of frequency much as a noise wave is a random function of time. Using their results, we can evaluate the statistics of the Fourier series coefficients, i.e., echo amplitudes, which are, of course, random variables. Moreover, if the cross-correlation between coefficients is negligible, we can determine the expected value of the interchannel interference for each echo individually and then add the results to obtain the total expected value of the interchannel interference. The random coupling coefficient is assumed to have a Gaussian probability density. The effects of delay distortion caused by waveguide cutoff dispersion are not included in this paper. The system is assumed to be equalized.

Section II presents the principal results of this paper and contains examples demonstrating their use. The examples use typical state-of-the-art data obtained from S. E. Miller[9] and are significant in that they demonstrate that FM appears to be an attractive modulation scheme for use on the millimeter-wave circular-electric waveguide as far as continuous random mode conversion is concerned. Both helix waveguide and copper waveguide with mode filters are considered. The helix waveguide has superior performance, of course, but copper waveguide with periodically placed mode filters may be useful in some situations.

Section III presents the mathematical development of the results.

II. PRINCIPAL RESULTS

The present section contains a statement and discussion of the principal results of this paper. We will restrict the discussion to the case of a single spurious mode and a single polarization. If other polarizations are present, they can be considered individually and the results added on a power basis.* The same is true for our purposes for different spurious modes. It is certainly true when the spurious modes arise because of different and uncorrelated physical imperfections in the waveguide. For instance, the TE_{02} spurious mode is caused primarily by random diameter variations. Its reconversion transfer function would be uncorrelated with that of the TE_{11} spurious mode, which is caused primarily by random straightness variations, if the diameter variations are uncorrelated with the straightness variations. Even when the spurious modes arise because of the same physical imperfections, it can be shown† that the cross-correlation in the same frequency interval is negligible.

There are two cases of interest. In the first, the difference between the heat loss attenuation constant of the TE_{01} and the spurious mode, i.e., the differential attenuation constant ΔA, is uniform. An example would be the TE_{0n} family on both the helix waveguide and copper waveguide with helix mode filters. The second case of interest is when the differential attenuation is effectively zero, except at periodically located points along the waveguide where a total differential attenuation of k is introduced. An example of this type of behavior would be the TE_{1n} family on a waveguide having periodically placed helix mode filters.

2.1 *Uniform Differential Attenuation Constant*

Consider a length L of waveguide having a differential attenuation constant of ΔA for the spurious mode of interest. Assume that the random coupling coefficient between the TE_{01} and spurious mode is Gauss-

* See Ref. 2, Section 4.3.2.
† See Ref. 2, Appendix G.

ian, pure real or pure imaginary, and has a white power spectrum of density S_c. If the differential attenuation of the total length of waveguide is much larger than one, i.e., $|\Delta A| L \gg 1$, then the mean-square value of the equivalent echoes $\overline{r_n^2}$, normalized by the signal power, is given by the expression

$$\overline{r_n^2} = (T/\tau_0)(P_x/P_0)\epsilon^{-n(T/\tau_0)} \tag{1}$$

if $P_x/P_0 \ll 1$, where

$$P_x/P_0 = \text{reconversion echo-to-signal power ratio} \tag{2}$$

$$= S_c^2 L/2 |\Delta A| \tag{2}$$

$$\tau_0 = |\bar{v}_x^2 - \bar{v}_0^2| /4c |\Delta A| \tag{3}$$

and

n = positive integer designating the nth echo. (The echoes lag the signal if $\bar{v}_x^2 > \bar{v}_0^2$ and lead if $\bar{v}_0^2 > \bar{v}_x^2$. Because the interchannel interference is independent of this, we use the absolute value $|\bar{v}_x^2 - \bar{v}_0^2|$ in (3) and require n to be positive.)

P_x = reconversion echo power, i.e., the power in that portion of the signal which has been converted and reconverted once and only once.

P_0 = signal power which has suffered no conversion. This will henceforth be considered *the* signal power.

$\bar{v}_0$, $\bar{v}_x$ = mode cutoff factors* of the TE_{01} and spurious modes at the band center frequency. This give rise to the differential phase shift.

c = velocity of light in free space.

$T \equiv 1/\Delta f$ = time delay between successive echoes.

Equation (2) has also been derived independently by Miller,[9] using a different approach. Equation (1) was developed from a Fourier series expansion of the reconversion transfer function over the frequency interval Δf. Hence, Δf must satisfy certain conditions if meaningful answers are to be obtained. First, all of the significant energy in the angle-modulated signal must be contained within the bandwidth Δf. Thus we require a bandwidth of

$$\Delta f \geqq 2(f_b + 4\sigma) \tag{4}$$

where f_b is the baseband bandwidth and σ is the root-mean-square frequency deviation of the angle-modulated wave. This is simply a statement of the familiar Carson bandwidth. Second, the mathematical

* See Ref. 2, Appendix A.

derivation of (1) requires the differential phase constant to be approximately proportional to frequency over the bandwidth used in the Fourier series expansion. This sets an upper limit on Δf. In order to satisfy this requirement we require that the percentage bandwidth and the square of the mode cutoff factors of the modes involved be small compared to unity, i.e.

$$\Delta f/2f_0 \ll 1, \bar{v}_0^2 \ll 1 \quad \text{and} \quad \bar{v}_x^2 \ll 1. \tag{5}$$

This will usually be true for the bandwidth and modes of interest, but it should always be checked as a safeguard. Third, the Fourier series coefficients must be uncorrelated with one another. It is well known that Fourier series coefficients become uncorrelated as the expansion interval approaches infinity. In a practical case, "infinity" is a number which is much larger than the correlation interval of the random variable in question. In the present case the autocorrelation function of the reconversion transfer function varies as $1/[1 + (2\pi\tau_0 f')^2]$, where f' is measured about the band center frequency. Thus in order for the cross-correlation between coefficients to be negligible we require that

$$(\pi\tau_0\Delta f)^2 \gg 1. \tag{6}$$

Lastly, the expression (1) for $\overline{r_n^2}$ is limited by the accuracy of the perturbation theory from which it was developed. It should be good, however, as long as $P_0/P_x \gg 1$ as calculated from (2). A plot of the equivalent echo train is shown in Fig. 1. We see that τ_0 is the "time constant" associated with the envelope of the echo train, i.e., it is the value of time delay where the envelope of the echo train caused by reconversion has dropped to the relative value $1/\epsilon$.

If the cross-correlation between Fourier series coefficients, and hence echoes, is negligible, we may substitute the expression (1) directly

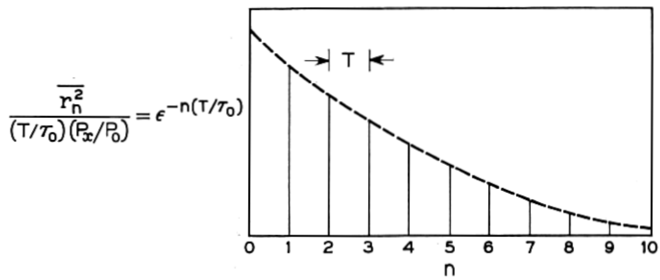

Fig. 1 — A plot of the reconversion echo train for uniform differential attenuation.

into the formulas given by Bennett et al.[5] and sum the results to determine the total baseband interchannel interference in the top channel.

As an illustration, for a two-inch I.D. helix waveguide at 55 kmc, the dominant spurious mode is the TE_{02}, as far as reconversion echoes are concerned. Typical data are[9]

$$\begin{aligned} |\Delta A| &= 4.3 \text{ db/mile} = 0.495 \text{ neper/mile} \\ S_c &= 0.01 \text{ db/mile} = 0.00115 \text{ neper/mile} \\ \bar{v}_0^2 &= 0.0172 \ll 1 \\ \bar{v}_x^2 &= 0.058 \ll 1 \\ \tau_0 &= 105.8 \times 10^{-7} \text{ sec.} \end{aligned}$$

For a waveguide length of 4000 miles, we have from (2)

$$P_0/P_x = 22.6 \text{ db} \gg 1.$$

Let us assume that a signal is applied to this waveguide which has been frequency modulated by a rectangular band of Gaussian noise. Let the baseband width be $f_b = 10$ mc, and the rms frequency deviation be $\sigma = 2f_b = 20$ mc. The significant energy in this wave would be contained in a bandwidth of $\Delta f = 10\sigma = 20f_b = 200$ mc. Thus, if we pick $\Delta f = 200$ mc we also satisfy (5) and (6), i.e.

$$\Delta f/2f_0 = 0.002 \ll 1, \qquad (\pi\tau_0\Delta f)^2 = 4{,}450 \gg 1.$$

From Fig. 5.7 of Bennett et al.[5] we can read off the factor $s(nf_bT)$, which multiplies the mean-square value of each echo to give the interchannel interference-to-signal power ratio for that particular echo. $s(nf_bT)$ takes into account the noise suppressing property of FM as well as the variation caused by the magnitude of the delay. A plot of $s(nf_bT)$ is shown in Fig. 2 for $\sigma/f_b = 2$ and 3. The total interchannel interference-to-signal power ratio is given by the sum

$$I/S = \sum_{n=1}^{\infty} s(nf_bT)\overline{r_n^2} \tag{7}$$

if $P_x/P_0 \ll 1$. We see that since $s(nf_bT)$ is essentially constant at -15 db for $nf_bT > 1/10$, i.e., $n > 1$, we have to a good approximation

$$\begin{aligned} I/S &= s(nf_bT) \sum_{n=1}^{\infty} \overline{r_n^2} \\ &= s(nf_bT)(P_x/P_0). \end{aligned} \tag{8}$$

In decibels

$$I/S = -15 - 22.6 = -37.6 \text{ db}.$$

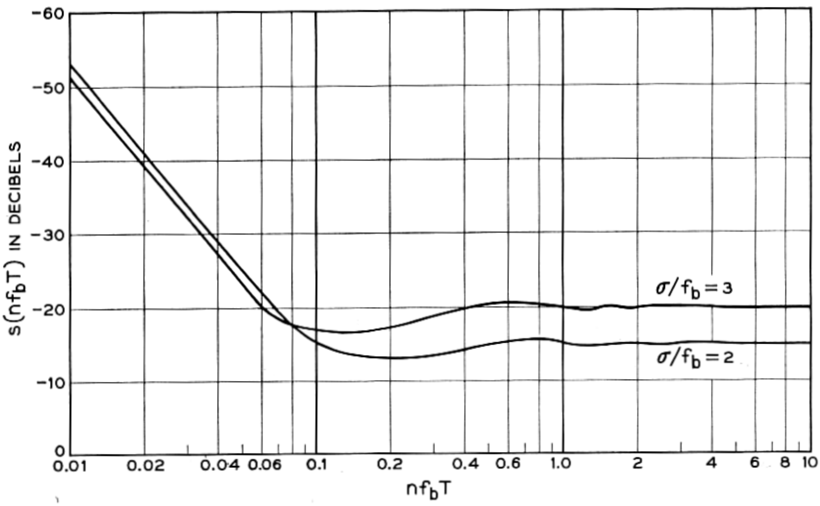

Fig. 2 — Weighting function $s(nf_bT)$ which multiplies individual echoes to give interchannel interference in top baseband channel for FM.

This corresponds to 27.4 dba at zero level. Thus, other things neglected, continuous random mode conversion does not preclude a 4000-mile two-inch I.D. helix system.

If this interchannel interference noise-to-signal ratio were too large, one could decrease it still further simply by increasing the modulation index. In other words, for the conditions of this example we are able to obtain FM improvement against reconversion echoes as well as additive thermal noise. However, from Fig. 2 we see that for $nf_bT \ll 1/10$ we would not be able to obtain this improvement. In fact, increasing the modulation index degrades the situation in that case.

2.2 *Mode Filters*

Mode filters may be placed at periodic intervals along the waveguide to provide large attenuation to the spurious modes. We assume here that each filter contributes k nepers of differential attenuation and that the waveguide between filters contributes essentially none. Let the spacing between mode filters be L_0 and the length of each mode filter be negligible in comparison. In the previous section, for uniform differential attenuation we found that the envelope of the equivalent echoes, when plotted versus delay, was an exponential. In the present case we find that the envelope consists of a piecewise approximation to an exponential using straight line segments (see Fig. 3). The exponential is ϵ^{-2ik} and the straight line segments connect the points defined by letting $i = 0, 1, 2, \cdots$, etc. Let us number the trapezoids represented by the area under each straight line segment according to the value of i at the beginning of each trapezoid. The equation for the straight line in the ith trapezoid, i.e., between ϵ^{-2ik} and $\epsilon^{-2(i+1)k}$ on the ordinate, is

$$y_i(\tau/\tau_1) = \epsilon^{-2ik}[1 + [1 - \epsilon^{-2k}][i - (\tau/\tau_1)]] \tag{9}$$

where τ_1 is the length of the base of each trapezoid. The expression for the individual echoes is

$$\overline{r_{n_i}^{\;2}} = \frac{TS_c^{\;2}LL_0}{\tau_1}\,\epsilon^{-2ik}[1 + [1 - \epsilon^{-2k}]\,[i - (nT/\tau_1)]] \tag{10}$$

where i is chosen from the inequality $i \leqq (nT/\tau_1) \leqq i + 1$. Here L_0 is the length of waveguide between mode filters, L is the total length of waveguide and $\tau_1 \equiv |\,\bar{v}_x^{\;2} - \bar{v}_0^{\;2}\,|\,L_0/2c$. All other quantities are as defined in the previous section. As in the previous section, we require that

$$\bar{v}_0^{\;2} \ll 1, \qquad \bar{v}_x^{\;2} \ll 1, \qquad \Delta f/2f_0 \ll 1 \quad \text{and} \quad \Delta f \geqq 2(f_b + 4\sigma). \tag{11}$$

The requirement on Δf to ensure negligible cross-correlation between the Fourier coefficients of the reconversion transfer function, i.e., between individual echoes, is changed, however. The autocorrelation function is given by (42). The series converges quite rapidly for large

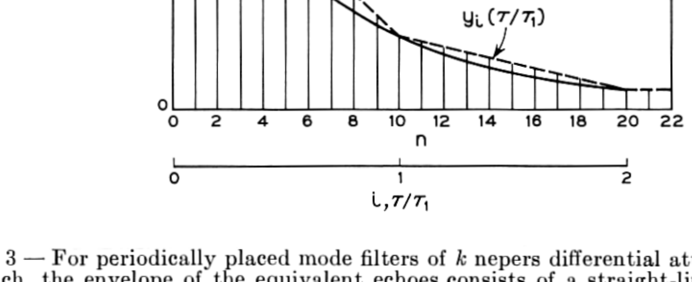

Fig. 3 — For periodically placed mode filters of k nepers differential attenuation each, the envelope of the equivalent echoes consists of a straight-line approximation $y_i\,(\tau/\tau_1)$ to the exponential ϵ^{-2ik}.

differential attenuation per mode filter k. In most cases of interest only the first term or two is required. In order for Δf to be much larger than the correlation interval we require

$$(\pi \tau_1 \Delta f/2)^2 \gg 1 \tag{12}$$

since $|\zeta| L_0 = f\tau_1$.

Equation (10) requires the number of mode filters N to be large, i.e., $N \gg 1$, and that the square of the *total* differential loss of the N filters be large, i.e., $\epsilon^{-2kN} \ll 1$. Under these conditions, the reconversion echo-to-signal power ratio is given by

$$P_x/P_0 = (LL_0 S_c^2/2) \coth k \tag{13}$$

which must be small compared to unity in order for (10) to be accurate. As a check, notice that if $k \ll 1$, we have

$$P_x/P_0 = LS_c^2/2 \mid \Delta A \mid \tag{14}$$

where $|\Delta A| \equiv k/L_0$. This is precisely the expression presented in the previous section for a constant differential attenuation.

It is frequently necessary to determine the ratio of the power in the ith trapezoid to the total. From simple geometric considerations this is given by

$$P_{x_i}/P_x = \epsilon^{-2ik} [1 - \epsilon^{-2k}] \tag{15}$$

Equations (13) and (15) have also been derived independently by S. E. Miller[9] using a different approach.

As an illustration, consider a two-inch I.D. copper waveguide system at 55 kmc having 15-db mode filters every 200 feet. The TE_{11}, TE_{12}, and TM_{11} spurious modes would all contribute significantly because of random straightness deviations. The energy in all of these modes will be lumped together and considered as if it were in the TE_{12} mode. This mode has the largest delay and consequently would produce the most interchannel interference. Thus we are assured that our calculations will be pessimistic. Numerical values were obtained from Miller[9]

$$\begin{aligned}
k &= 15 \text{ db} = 1.73 \text{ nepers} \\
L_0 &= 200 \text{ feet} = 1/26.4 \text{ miles} \\
S_c &= 2 \text{ db/mile} = 0.23 \text{ neper/mile} \\
\bar{v}_0^2 &= 0.0172 \ll 1 \\
\bar{v}_x^2 &= 0.0334 \ll 1 \\
\tau_1 &= 16.08 \times 10^{-10} \text{ sec.}
\end{aligned}$$

Equation (10) requires that the signal-to-echo ratio be large compared to unity. If we arbitrarily select $P_0/P_x = 17$ db, we find from (13) that the permissible length of waveguide is $L = 18.7$ miles. If we now demodulate and pass the resultant signal through a baseband filter of bandwidth f_b to obtain the FM advantage, we obtain a new baseband signal not unlike the original. Thus we can remodulate and proceed for another 18.7 miles without analytical complications, as long as the baseband signal-to-interchannel interference ratio remains large. It should be emphasized that placing a demodulate-remodulate repeater every 18.7 miles is done to keep out of analytical difficulties. We have not shown and are not implying that performance requires this crutch.

As in the previous section, assume an FM system with $f_b = 10$ mc and $\sigma/f_b = 2$. We choose Δf to satisfy the inequalities (11) and (12). Picking $\Delta f = 2$ kmc yields $\Delta f/2f_0 = 0.018 \ll 1$, $\Delta f > 2\ (f_b + 4\sigma) = 200$ mc and $(\pi\tau_1\Delta f/2)^2 = 25 \gg 1$. In order to calculate interchannel interference, we might again multiply each echo by the factor $s(nf_bT)$ given by Fig. 5.7 of Bennett et al.[5] and sum the result. However, that paper gives the analytical expression

$$s(\tau) = 2(\sigma/f_b)^2\ (\pi f_b \tau)^4 \tag{16}$$

which is valid for the small delays of this illustration, i.e., $2\pi f_b nT \ll 1$ and $(2\pi f_b nT)^2\ (\sigma/f_b)^2 \ll 1$. We can integrate rather than sum. Thus in (10) we replace nT by τ and T by $d\tau$. For the interchannel interference-to-signal ratio contributed by the ith trapezoid, we have

$$\begin{aligned} I_i/S &= \left[\frac{2LL_0(S_c\sigma f_b)^2\pi^4}{\tau_1}\right]\epsilon^{-2ik}\int_{i\tau_1}^{(i+1)\tau_1}\tau^4 \\ &\qquad \{1 + (1 - \epsilon^{-2k})[i - (\tau/\tau_1)]\}\ d\tau \\ &= \left[\frac{2LL_0(S_c\sigma f_b)^2\pi^4\tau_1^4\epsilon^{-2ik}}{30}\right][\{6 + (i - 5)(1 - \epsilon^{-2k})\}\ (1 + i)^5 \\ &\qquad - \{6 + i\ (1 - \epsilon^{-2k})\}\ i^5]. \end{aligned}$$

It turns out, in this example, that the $i = 0$, 1 and 2 trapezoids all contribute significantly, with the $i = 1$ trapezoid producing the most interchannel interference. In physical terms, the value of i tells us how many mode filter spacings the converted energy traveled in the spurious mode before reconverting. The ith trapezoid represents energy which traveled a distance larger than i mode filters spacings but less than $i + 1$ spacings. In this example the $i = 1$ trapezoid has substantially less power in it than the $i = 0$ trapezoid, but its larger delay more than makes up for this difference (see Fig. 2). The total signal-to-interchannel interference ratio in the top channel is $S/I = 66.4$ db. For a total system length of 4000 miles, we would need $4000/18.7 = 214$ repeaters and would obtain a signal-to-interchannel interference ratio of $66.4 - 10 \log_{10} 214 = 43.1$ db if the only contribution were due to continuous random mode conversion. This corresponds to 22 dba at zero level.

This is only part of the story. The TE_{0n} spurious modes are not damped by helix mode filters. If it is conjectured[9] that the coupling coefficients, differential attenuation, etc., for these modes are the same for copper and helix waveguides, then the calculations and results of the previous section apply directly. The interchannel interference produced by the TE_{02} spurious mode predominates over that from all other modes. Consequently, it looks as though the performance of this copper waveguide with mode filters is comparable with that of the helix waveguide. The imposed requirement of demodulate-remodulate repeaters, however, is certainly a disadvantage. One must determine whether or not they are really necessary before any definite comparison can be made.

III. MATHEMATICAL DEVELOPMENT

Our approach is to represent the effects of continuous random mode conversion by an infinite number of uncorrelated echoes. These echoes follow directly from the expansion of the reconversion transfer function in a Fourier series over the frequency interval of interest. The coupled line equations serve as our starting point. They are

$$dI_0(z)/dz = -\Gamma_0 I_0(z) + jc(z)I_x(z)$$

$$dI_x(z)/dz = +jc(z)I_0(z) - \Gamma_x I_x(z)$$

where

$I_0(z), I_x(z)$ = complex wave amplitude of the TE_{01} mode and spurious mode, respectively,

Γ_0, Γ_x = deterministic propagation constants of the TE_{01} and spurious modes, respectively,

$c(z)$ = random coupling coefficient, assumed Gaussian and pure real or pure imaginary, and

z = distance along the waveguide.

If one imposes the initial conditions $I_0(0) = 1$ and $I_x(0) = 0$, then $I_0(z)$ is the transfer function of the waveguide. This may be factored into two components. One component, $\epsilon^{-\Gamma_0 z}$, is deterministic and represents the attenuation and phase which would be present in the absence of random mode conversion. Since in theory it is completely predictable and equalizable, we shall not consider this component further. The other component is a random variable and represents the attenuation and phase which is contributed by random mode conversion. This is by definition the reconversion transfer function and is given by

$$G_0(z) \equiv \epsilon^{+\Gamma_0 z} I_0(z). \tag{19}$$

Rowe[7] shows that it may be approximated directly by use of Picard's method of successive approximations, or with improved accuracy it may be obtained from the reconversion propagation function $\Lambda(z)$ defined by the relation $G_0(z) \equiv \epsilon^{-\Lambda(z)}$, where $\Lambda(z)$ is approximated by Picard's method. We use Rowe's second approximation throughout this paper

$$\Lambda(z) = \int_0^z c(s)\epsilon^{+\Delta\Gamma s}\, ds \int_0^s c(t)\epsilon^{-\Delta\Gamma t}\, dt.$$

The reconversion propagation function $\Lambda(z)$ may be written as $\Lambda(z, \xi, \Delta A)$, where $\xi \equiv \Delta\beta/2\pi \equiv (\beta_0 - \beta_x)/2\pi$, $\Delta A \equiv A_0 - A_x$ and $\Gamma_0 \equiv A_0 + j\beta_0$, $\Gamma_x \equiv A_x + j\beta_x$. Notice that ΔA will always be negative since A_x is positive and larger than A_0. Λ is nonstationary when considered as a function of z. Under certain conditions, however, Λ is approximately stationary, when considered as a function of ξ, over the small percentage bandwidths of interest.[2] The requirement is that the random coupling coefficient $c(z)$ have a white power spectrum; that is, the autocorrelation function $R_c(u) \equiv \overline{c(z)c(z+u)}$ be an impulse function or a close approximation thereof. The bar denotes an ensemble average. This seems to be true or at least approximately true for good waveguides over the moderate bandwidths required in our problem. Fortunately, it also is true that over the small percentage bandwidths in which we are interested, ξ is approximately proportional to frequency;* that is,

$$\xi \equiv \xi_0 + \xi' \doteq m\,(f_0 - f') \tag{20}$$

where ξ_0 and f_0 are measured at the band center, and $\xi \equiv \xi_0 + \xi'$ and $f \equiv f_0 + f'$. $m \equiv (\bar{v}_x^2 - \bar{v}_0^2)/2c$ where $\bar{v}_0$ and $\bar{v}_x$ are the mode cutoff factors of the TE_{01} and spurious modes, respectively, evaluated at the band center frequency f_0, and c is the velocity of light in free space.

* From Ref. 2, Appendix A, $\xi = (f/2c)[(1 - v_0^2)^{\frac{1}{2}} - (1 - v_x^2)^{\frac{1}{2}}] \doteq (f/4c)(v_x^2 - v_0^2) =$ constant/f if $v_0^2 \ll 1$ and $v_x^2 \ll 1$. Letting $f = f_0 + f'$ and $\xi = \xi_0 + \xi'$ we have $\xi' + \xi_0 \doteq (\text{constant}/f_0)\,(1 - f'/f_0) \equiv m\,(f_0 - f')$.

Thus we may determine the mean-square value of the coefficients of the Fourier series expansion of the reconversion propagation function Λ, when Λ is considered as a function of frequency, as a limiting case of the power spectrum of Λ, when Λ is considered as a function of ξ. The functional form of Λ depends upon whether ξ or f is the variable. For this reason we shall use a subscript 1 when ξ is the variable, and a subscript 2 when f is the variable.

Formulas for the power spectrum $S_1(u)$ of the random variable $\Lambda_1(\xi)$ are developed for the case of $\Delta A = 0$ by Rowe and Warters,[2] for $\Delta A =$ constant by Young,[8] and for the case of periodically placed mode filters later in this paper.

3.1 *Transfer Function and Equivalent Echoes*

The complex reconversion propagation function $\Lambda_2(f)$ may be expanded in a Fourier series over the interval Δf centered at f_0

$$\Lambda_{2s}(f) = \sum_{n=-\infty}^{+\infty} a_n \epsilon^{j2\pi nfT} \tag{21}$$

where $\Lambda_{2s}(f) = \Lambda_2(f)$ for $f_0 - \Delta f/2 \leqq f \leqq f_0 + \Delta f/2$. $T = 1/\Delta f$ and the subscript s reminds us that we have a periodic series approximation. If all of the significant energy in the FM wave, or any other wave for that matter, lies within Δf, then the periodic series (21) may be used for calculations. The $n \neq 0$ terms represent fluctuation components. If their sum is always small compared to unity with large probability, then the reconversion transfer function $G_0(f) \equiv \exp[-\Lambda_2(f)]$ can be approximated by the series

$$G_{0s}(f) = \exp\left[-\sum_{n=-\infty}^{+\infty} a_n \epsilon^{j2\pi fnT}\right] \doteq \epsilon^{-a_0}\left[1 - \sum_{n=-\infty}^{+\infty}{}' a_n \epsilon^{j2\pi fnT}\right] \tag{22}$$

where the prime on the summation indicates that the $n = 0$ term is omitted. The impulse response of this periodic transfer function, i.e., its inverse Fourier transform, is

$$h_{0s}(t) = \epsilon^{-a_0}\left[\delta(t) - \sum_{n=-\infty}^{+\infty}{}' a_n \delta(t + nT)\right]. \tag{23}$$

Thus we have an undistorted signal plus a series of small echoes.

The coefficients a_n are given by

$$a_n = (1/\Delta f) \int_{f_0-\Delta f/2}^{f_0+\Delta f/2} \Lambda_2(f) \epsilon^{-j2\pi nf/\Delta f}\, df. \tag{24}$$

Replacing f by $f' + f_0$ we have

$$a_n = (1/\Delta f)\epsilon^{-j2\pi n f_0/\Delta f} \int_{-\Delta f/2}^{+\Delta f/2} \Lambda_2(f' + f_0)\epsilon^{-j2\pi n f'/\Delta f}\, df'. \qquad (25)$$

Now, over the small percentage bandwidths in which we are interested, the differential phase constant is proportional to frequency: i.e., $\xi \equiv \xi' + \xi_0 \doteq m(f_0 - f')$. Since $\Lambda_2(f) = \Lambda_1(\xi)$, we may rewrite (25) as

$$\begin{aligned} a_n &= \epsilon^{-j2\pi n f_0/\Delta f} \left\{ \frac{1}{m\Delta f} \int_{-m\Delta f/2}^{+m\Delta f/2} \Lambda_1(\xi' + \xi_0)\epsilon^{-j2\pi n \xi/(-m\Delta f)}\, d\xi' \right. \\ &= \epsilon^{-j2\pi n f_0 T} b_{(-\mathrm{sgn}\, m)n} \end{aligned} \qquad (26)$$

where b_n is the coefficient of the nth term of the Fourier series expansion of $\Lambda_1(\xi' + \xi_0)$ over the interval $-\Delta\xi/2 \leqq \xi' \leqq +\Delta\xi/2$. For $\Delta\xi$ much larger than the correlation interval of $\Lambda_1(\xi)$, we may approximate the expected value of $| b_n^2 |$, and hence $| a_n^2 |$, in terms of the power spectrum of $\Lambda_1(\xi)$. The result is

$$\overline{| a_n^2 |} = (T/| m |)S_{\Lambda_1}(-nT/m). \qquad (27)$$

The imaginary part of the complex random variable $\Lambda_1(\xi)$ is the Hilbert transform[2,8] of the real part, so the power spectrum $S_{\Lambda_1}(u)$ vanishes for negative u. Thus, the coefficients a_n are nonzero only when the sign of n opposes that of m. Negative n indicates that the echoes lag the signal and vice versa. Since the interchannel interference is independent of whether the echoes lead or lag, we may drop the minus sign in (27), use the absolute value of m, and restrict n to positive values.

3.2 *Uniform Differential Attenuation*

The power spectrum of the real part of the reconversion propagation constant $\Lambda_1(\xi)$ is given by Rowe and Warters,[2] for the case of zero differential attenuation and a Gaussian random coupling coefficient, as

$$S(u) = \begin{cases} (\tfrac{1}{4})[\delta(u)L^2 S_c^2 + S_c^2[L - | u |]], & | u | \leqq L \\ 0, \text{ otherwise.} \end{cases} \qquad (28)$$

Since the real and imaginary parts of $\Lambda_1(\xi)$ are Hilbert transforms, the power spectrum of $\Lambda_1(\xi)$ itself is given by*

$$S_{\Lambda_1}(u) = \begin{cases} (\tfrac{1}{4})L^2 S_c^2 \delta(u) + S_c^2[L - u], & (0-) \leqq u \leqq L \\ 0, \text{ otherwise.} \end{cases} \qquad (29)$$

* Discussions of the autocorrelation function and power spectrum of complex random variables, whose real and imaginary parts are Hilbert transforms, are given by Dugundji,[10] Zakai,[11] and Deutsch.[12]

Young[8] demonstrates that if the differential attenuation is nonzero but is a constant equal to ΔA, the real and imaginary parts of $\Lambda_1(\xi)$ are again Hilbert transforms and its power spectrum is found by multiplying that for $\Delta A = 0$ by $\epsilon^{-2|\Delta A u|}$. If $|\Delta A| L \gg 1$ we have

$$S_{\Lambda_1}(u) \doteq \begin{cases} (S_c^2 L^2/4)\delta(u) + S_c^2 L \epsilon^{-2|\Delta A|u}, & (0-) \leqq u < \infty \\ 0, \text{ otherwise.} \end{cases} \tag{30}$$

Equation (30) can also be obtained as a limiting case of the results of the next section.

It is often convenient to work in terms of the reconversion echo-to-signal ratio P_x/P_0, defined as the ratio of the power in the echoes in (23) to that in the signal. This is given by

$$\begin{aligned} P_x/P_0 &= \sum_{n=1}^{\infty} \overline{|a_n^2|} = \sum_{n=1}^{\infty} (T/|m|) S_{\Lambda_1}(nT/|m|) \\ &\rightarrow \int_{0+}^{\infty} (1/|m|) S_{\Lambda_1}(\tau/|m|)\, d\tau \\ &= S_c^2 L/2|\Delta A|. \end{aligned} \tag{31}$$

3.3 *Mode Filters*

Consider a long section of waveguide of length L. Assume ideal mode filters are placed at L_0, $2L_0$, $\cdots$, $NL_0 = L$. Let each mode filter be of zero length and produce k nepers of differential attenuation and a constant differential phase shift of θ radians, with no other side effects. If the differential attenuation of the length of waveguide L_0 between each mode filter is negligible in comparison with that produced by a mode filter, we may approximate the differential propagation function of the waveguide by the relation

$$\Delta\Gamma(z) = j2\pi\xi + (-k + j\theta) \sum_{n=1}^{N} \delta(z - nL_0) \tag{32}$$

where $2\pi\xi = \Delta\beta =$ differential phase constant of the waveguide alone.

In the coupled line equations (18) the propagation factors Γ_0 and Γ_x are constants. In the present case, however, they are functions of position along the waveguide, i.e., functions of z. Rowe has shown in unpublished work that in (18) we may replace

$$\Gamma_0 z \quad \text{by} \quad \gamma_0(z) = \int_0^z \Gamma_0(S)\, dS$$

and

$$\Gamma_x z \quad \text{by} \quad \gamma_x(z) = \int_0^z \Gamma_x(S)\, dS,$$

and use Picard's method of successive approximations to obtain the approximate solution

$$\Lambda(L) = \int_0^L c(t)\epsilon^{\Delta\gamma(t)}\, dt \int_0^t c(S)\epsilon^{-\Delta\gamma(S)}\, dS \tag{33}$$

The derivation and result are exactly the same as Rowe's published derivation[7] for constant Γ's with the exception of the substitution mentioned. Substituting (32) into (33) we have

$$\begin{aligned}\Lambda(NL_0) &= \int_0^{NL_0} c(t)\exp\left[j2\pi\xi t - (k - j\theta)\sum_{n=1}^{N} u(t - nL_0)\right] dt \\ &\quad \cdot \int_0^t c(S)\exp\left[-j2\pi\xi S + (k - j\theta)\sum_{l=1}^{N} u(S - lL_0)\right] dS,\end{aligned} \tag{34}$$

where $u(x) = \begin{cases} 1, & 0 \leqq x \leqq \infty \\ 0 & \text{otherwise.}\end{cases}$

Simplifying and rearranging, we have

$$\begin{aligned}\Lambda(NL_0) &= \sum_{n=0}^{N-1}\sum_{l=0}^{n-1} \epsilon^{(-n+l)(k-j\theta)} \int_{nL_0}^{(n+1)L_0} c(t)\epsilon^{j2\pi\xi t}\, dt \\ &\quad \cdot \int_{lL_0}^{(l+1)L_0} c(S)\epsilon^{-j2\pi\xi S}\, dS \\ &\quad + \sum_{n=0}^{N-1} \int_{nL_0}^{(n+1)L_0} c(t)\epsilon^{j2\pi\xi t}\, dt \int_{nL_0}^{t} c(S)\epsilon^{-j2\pi\xi S}\, dS\end{aligned} \tag{35}$$

which after more rearranging and simplification reduces to

$$\begin{aligned}\Lambda(NL_0) &= \sum_{n=0}^{N-1}\sum_{l=0}^{n-1} \epsilon^{(-n+l)(k-j\theta-j2\pi\xi L_0)} \int_0^{L_0} c(t + nL_0)\epsilon^{j2\pi\xi t}\, dt \\ &\quad \cdot \int_0^{L_0} c(S + lL_0)\epsilon^{-j2\pi\xi S}\, dS \\ &\quad + \sum_{n=0}^{N-1} \int_0^{L_0} c(t + nL_0)\epsilon^{j2\pi\xi t}\, dt \int_0^t c(S + nL_0)\epsilon^{-j2\pi\xi S}\, dS.\end{aligned} \tag{36}$$

If the random coupling coefficient $c(z)$ has a white power spectrum, the reconversion propagation function Λ is an approximately stationary random variable[2] when considered as a function of the differential phase variable ξ. We shall now calculate the autocorrelation function $R_{\Lambda_1}(\xi)$ of Λ. Since $c(z)$ has a white power spectrum of density S_c, it has an autocorrelation function which is an impulse, i.e.

$$R_c(\xi) = S_c\delta(\xi). \tag{37}$$

Thus, each term in the series (36) is uncorrelated with all others. The autocorrelation function of the sum is the sum of the autocorrelation functions of each individual term.

The autocorrelation function of a typical term

$$\int_0^{L_0} c(t + nL_0)\epsilon^{j2\pi\xi t}\,dt \tag{38}$$

is

$$R(\xi) = S_cL_0\left(\frac{\sin \pi L_0\xi}{\pi L_0\xi}\right). \tag{39}$$

In view of the fact that $l < n$ in the double series of (36), the two integrals in each term are uncorrelated. The autocorrelation function of the double series is then

$$\sum_{n=0}^{N-1}\sum_{l=0}^{n-1} \epsilon^{2(l-n)(k-j\pi\xi L_0)} R^2(\xi) = \sum_{n=1}^{N-1} (N - n)\epsilon^{-2n(k-j\pi\xi L_0)} R^2(\xi) \tag{40}$$

since we have $N - 1$, $n - l = 1$ terms; $N - 2$, $n - l = 2$ terms; $\cdots$; 1, $n - l = N - 1$ terms.

The autocorrelation function of each term in the single series of (36) is the same as the autocorrelation function for a single piece of waveguide of length L_0. Thus, for $c(z)$ Gaussian, it is the inverse Fourier transform of the power spectrum given by (29), i.e., it is

$$\tfrac{1}{4} R^2(0) + \tfrac{1}{2} [R^2(\xi) + j\widehat{R^2}(\xi)] \tag{41}$$

where R is still defined by (39) and the circumflex stands for the Hilbert transform. Keeping in mind the fact that the dc terms, i.e., $R(0)$ terms, add in phase, we conclude

$$R_{\Lambda_1}(\xi) = (N^2/4)R^2(0) + (N/2)[R^2(\xi) + j\widehat{R^2}(\xi)] + R^2(\xi) \sum_{n=1}^{N-1} (N - n)\epsilon^{-2n(k-j\pi\xi L_0)}. \tag{42}$$

In order to find the reconversion echo-to-signal ratio, we could integrate over the power spectrum of $\Lambda_1(\xi)$ as in (31). However, the alternative method of letting $\xi = 0$ in the autocorrelation function is easier. Note that the 0+ limit on the integral in (31) corresponds to dropping the constant term, i.e., the $(\frac{1}{4})R^2(0)$ term in (42). Thus the reconversion echo-to-signal ratio becomes

$$P_x/P_0 = (N/2)R^2(0) + \sum_{n=1}^{N-1} (N - n)\epsilon^{-2nk} R^2(0).$$

If $N \gg 1$ and $\epsilon^{-2Nk} \ll 1$, then

$$P_x/P_0 \doteq \frac{LL_0S_c^2}{2} \coth k \tag{43}$$

where $L = NL_0$.

The power spectrum of $\Lambda_1(\xi)$ is the inverse Fourier transform of (42) and is given by

$$\begin{aligned} S_{\Lambda_1}(u) &= (\tfrac{1}{4})N^2L_0^2S_c^2\delta(u) \\ &\quad + S_c^2 \sum_{i=0}^{N-1} (N - i)\epsilon^{-2ik}((L_0 - |u - iL_0|)) \end{aligned} \tag{44}$$

for $u \geqq (0-)$, where

$$((x)) = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ if } x \leqq 0. \end{cases}$$

If $N \gg 1$ and $\epsilon^{-2Nk} \ll 1$, then the terms diminish quite rapidly. Thus, for $i \ll N$ we have the simpler expression

$$\begin{aligned} S_{\Lambda_1}(u) &\doteq (\tfrac{1}{4})N^2L_0^2S_c^2\delta(u) \\ &\quad + NL_0S_c^2\epsilon^{-2ik} [1 + (1 - \epsilon^{-2k})(i - (u/L_0))] \end{aligned} \tag{45}$$

where $(-0) \leqq i \leqq u/L_0 \leqq i + 1$. A rough sketch of this function is shown in Fig. 3. The sketch consists of an impulse at the origin plus a straight-line approximation to an exponential.

It is often convenient to have an expression for the ratio of the reconversion echoes under the ith straight line, i.e., from $u = iL_0$ to $u = (i + 1)L_0$, to the total. This is given by

$$P_{x_i}/P_x = (P_{x_i}/P_0)/(P_x/P_0) = (1 - \epsilon^{-2k})\epsilon^{-2ik}. \tag{46}$$

If we allow the differential attenuation k per mode filter and the spacing L_0 between mode filters both to decrease without limit such that their ratio remains finite, (43) reduces to (31), and (45) reduces to (30), where $| \Delta A | \equiv k/L_0$.

3.4 *Interchannel Interference*

Let the angle-modulated wave

$$e_i(t) = \epsilon^{j\omega_0 t + j\varphi(t)} \tag{47}$$

be applied to a filter having an impulse response given by (23), i.e., by

$$h_{0s}(t) = \epsilon^{-a_0}[\delta(t) - \sum_{n=-\infty}^{+\infty}{}' a_n \delta(t + nT)]. \tag{48}$$

$\varphi(t)$ is assumed to represent stationary random Gaussian noise. The output wave is found by convolution and is

$$e_0(t) = \epsilon^{-a_0}[\epsilon^{j\omega_0 t + j\varphi(t)} - \sum_{n=-\infty}^{\infty}{}' a_n \epsilon^{j\omega_0(t+nT) + j\varphi(t+nT)}]. \tag{49}$$

From (26) we have $a_n = \epsilon^{-jn\omega_0 T} b_{\pm n} \equiv \epsilon^{j\theta_n - jn\omega_0 T} r_n$, where r_n and θ_n are real. If there are a large number of sample points[2] contained in the small percentage bandwidths of interest, then to a very good approximation the θ_n's will be uniformly distributed in the interval $(0, 2\pi)$ and the b_n's will be uncorrelated. Substituting, (49) reduces to

$$e_0(t) = \epsilon^{-b_0}[\epsilon^{j\omega_0 t + j\varphi(t)} - \sum_{n=-\infty}^{+\infty}{}' r_n \, \epsilon^{j\omega_0 t + j\theta_n + j\varphi(t+nT)}]. \tag{50}$$

The output consists of the original signal plus an infinite number of small, uncorrelated echoes. If the total power contained in the echoes is much smaller than that contained in the signal, we may follow a procedure identical to that of Bennett et al.[5] to determine the baseband interchannel interference in the top channel as a function of time which results from each of the above echoes. We then determine the autocorrelation function of the interference and average over r_n and θ_n. Since the θ_n's are uniformly distributed, the interference caused by each echo adds on a power or mean-square basis. In their work, Bennett et al. carried along a deterministic term pT which would be replaced by θ_n in our case. This term appears in the final result as $\cos \theta_n$, which has an average value of zero. The curves of Fig. 5.7 of Ref. 5 are for $\cos pT = 0$ and so apply to our case directly.

IV. CONCLUSIONS

We have presented an analysis of the effects of continuous random mode conversion on an angle modulated wave. The modulating signal was assumed to be a band of random Gaussian noise which is used to simulate a single sideband frequency division multiplex. A technique was presented for determining the expected value of the interchannel interference noise which appears in the top channel at baseband. Examples using typical state-of-the-art data demonstrated that FM appears to be an attractive modulation scheme. Copper waveguide, with and without mode filters, and helix waveguide were considered.

It should be pointed out that the techniques and the results presented here should be applicable to other multimode and multipath situations virtually unchanged. The Fourier series characterization of the transmission channel should also be of use in the study of other modulation schemes.

Our work contains two key, but reasonable, assumptions which one must keep in mind:

(*i*) The perturbation technique upon which our work is based requires that the reconverted energy consist primarily of energy which has been converted and reconverted once and only once.[2,7]

(*ii*) The random coupling coefficient is assumed to be Gaussian and to have a power spectrum which is white, or uniform, over a bandwidth which is large compared to the bandwidth "occupied" by our signal.

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