Reflections from an Exponential Atmosphere

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A quantitative explanation of tropospheric radio propagation is derived without the use of arbitrary numerical factors. It is based primarily on the average value and the standard deviation of the index of refraction at the earth's surface, both of which decrease exponentially with height.

This method seems to bridge the gap between the internal reflection and scatter hypotheses. The quantitative results are in good agreement with experimental data on distance dependence, frequency dependence, climatic and seasonal variations, and effective antenna gain. One consequence is a new concept of the decrease in effective gain of narrow-beam antennas, which accounts for some hitherto unexplained experimental results.

I. INTRODUCTION

Several conflicting hypotheses have been proposed over the years in an attempt to explain the transhorizon radio field intensities, which in some cases are hundreds of decibels stronger than predicted by smooth earth diffraction theory.[1-6] The rapid fading ordinarily associated with this type of transmission led naturally to the concept of many scattered components with random phase angles.

In order to account for the average field intensity, frequency dependence and other characteristics, the scatter hypotheses must make a number of more or less arbitrary assumptions regarding the number, size and distribution of the irregularities and whether the scatterers are spherical "blobs" or flat "pancakes." The scatter concept is a statistical framework that can be adjusted by arbitrary parameters to fit almost any consistent experimental data and, in fact, the values of the parameters have been changed significantly when the need to accommodate unexpected experimental results has arisen.

On the other hand, Snell's law of reflection indicates that reflections must occur from the gradual decrease in dielectric permittivity with height. This concept focuses attention on the average received power but, of course, it is recognized that the fading must be associated with the motion of the irregularities in the atmosphere. The object of the internal reflection hypothesis is to relate the average received field to the rate of decrease in atmospheric density, and hence to eliminate some of the arbitrary parameters required by the scattering concepts. Many theorists have concluded that the internal reflections cancel, or at least that their effect is too small to be important.

If the problem of inhomogeneous atmosphere could be solved rigorously, there would be no difference of opinion, but a rigorous solution that can be readily evaluated has not yet been achieved without significant approximations, and the relative accuracy of the different approximations is in question.

The purpose of this paper is to present a new approach that to the present author seems to bridge the gap between the internal reflection concept and the scatter hypotheses. Although the method is not as rigorous as would be desired, it does provide quantitative predictions that are in excellent agreement with experimental data without the use of arbitrary numerical factors.

II. METHOD

The procedure is to find first the magnitude of the internal reflection from a smooth exponential atmosphere and then to superimpose random variations on the exponential decrease.

The difference in dielectric permittivity ϵ between two horizontal planes at heights y_1 and y_2 is assumed to be a random variable with a Gaussian distribution. The average value depends on the average index of refraction at the surface, together with the exponential decrease in atmospheric density with height; the variance at any height depends on the variance in the surface index of refraction together with the same exponential decrease with height. The assumed rate of exponential decrease or scale factor cuts in half the atmospheric density, mean value of permittivity and variance in permittivity for each increase in elevation of 7 or 8 kilometers. These quantities are based on meteorological data and are independent of transhorizon radio experiments.

It is further assumed that the reflection coefficient is proportional to the change in dielectric permittivity and hence is also Gaussian distributed. The average reflection coefficient is derived from electromagnetic theory as a function of the average exponential decrease in permittivity.

Finally, it is assumed that the variance of the reflection coefficient is related to the variance of the dielectric permittivity by the same function that relates the average reflection coefficient to the average permittivity.

The resulting reflection coefficient is Gaussian distributed and has both positive and negative values, which are almost equally probable when the standard derivation σ is large compared with the mean value μ. For comparison with experimental data it is necessary to find the mean of a Gaussian distribution taken without regard to sign, and this is given with sufficient accuracy by either the mean value μ or $\sqrt{2/\pi}\sigma$, whichever is the larger. The former is usually controlling below 100 mc and the latter is usually controlling at UHF and above.

This paper is not intended to be a complete description of all tropospheric phenomena. Some variations exist in the horizontal plane but they are small compared with the variations in the vertical plane and have been neglected in order to simplify the presentation. The resulting modified Gaussian distribution is similar to a Rayleigh distribution; consequently, this derivation has a fading component, but it does not give quantitative results on either the speed of fading or on the associated problem of useful bandwidth. There is a hint that the speed of fading depends on the number of Fresnel zones in the common volume and that the useful bandwidth varies inversely as this quantity, but further work is needed.

In addition, this paper is concerned only with the effects of the atmosphere that have been neglected in the smooth sphere diffraction theory. The latter will be controlling at short distances and low frequencies.

III. EFFECT OF SMOOTH EXPONENTIAL PROFILE

The permittivity of the atmosphere is assumed to vary only in the vertical direction, and to have a mean value and a variance, both of which decrease exponentially with height. Specifically, the mean value of dielectric permittivity ϵ at any height illustrated on Fig. 1 is assumed to be

$$\epsilon = \epsilon_s + (1 - \epsilon_s)[1 - e^{-b(y-y_s)}]. \tag{1}$$

It follows that the change in dielectric permittivity in the region between $y = y_1$ and $y = y_2$ is given by

$$\Delta\epsilon_{2,1} = \epsilon_2 - \epsilon_1 = (1 - \epsilon_s)e^{-b(y_1-y_s)}[1 - e^{-b(y_2-y_1)}] \tag{2}$$

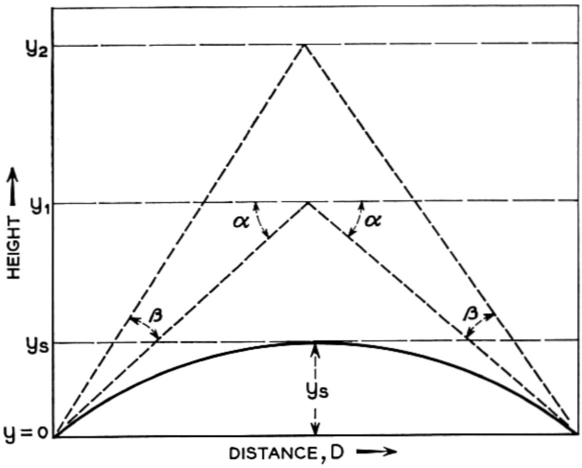

Fig. 1 — Geometry of transhorizon path.

where ϵ_s = dielectric permittivity at the surface ($y = y_s$)

b = rate of exponential decrease

$\approx$ 0.14 per kilometer.

The parameter b is primarily a measure of the rate of decrease in atmospheric density, which is in turn determined by the laws of gas diffusion and the law of gravity. The average value of b varies from about $b = 0.12$ per km near the earth's surface to about $b = 0.16$ per km at elevations greater than 7 or 8 kilometers, but it can be considered a constant in this application.*

* The parameter b is related as follows to the widely used effective earth radius factor k (which is limited to regions where $1 - e^{-by} \approx by$)

$$k = \frac{1}{1 + \frac{a}{2}\frac{d\epsilon}{dy}} = \frac{1}{1 - 2b\left(\frac{N_s}{314}\right)}$$

where

$$N_s = \frac{(\epsilon_s - 1)}{2} \times 10^6 = (n_s - 1)\,10^6$$

n = index of refraction

a = true earth radius = 6370 km

For the typical case of $N_s = 314$ and $b = 0.125$/km, the effective earth radius factor is $k = 4/3$.

When a change in dielectric permittivity occurs at a sharp boundary as shown on Fig. 2(a), the reflection coefficient q for either polarization can be derived from Snell's law with the following result

$$q = \frac{\epsilon_2 - \epsilon_1}{4 \sin^2 \alpha} \tag{3}$$

or

$$4\alpha^2 q \approx \epsilon_2 - \epsilon_1$$

where α is the angle between the ray and the boundary. This result assumes $| \epsilon_2 - \epsilon_1 | < 4 \sin^2 \alpha$ and $\sin \alpha \approx \alpha$. The corresponding reflection coefficient for an exponential decrease illustrated in Fig. 2(b) is derived in a later section with the following result

$$4\alpha^2 q = (\epsilon_2 - \epsilon_1) \frac{b}{b + i2A} F_1 \tag{4}$$

where

$$A = \frac{d\phi}{dy} = \frac{2\pi\alpha}{\lambda} \gg b$$

$$\phi = \frac{2\pi y\alpha}{\lambda}$$

$$\lambda = \text{wavelength}$$

$$F_1 = \left[\frac{1 - \exp[-(b + i2A)(y_2 - y_1)]}{1 - \exp[-b(y_2 - y_1)]}\right]$$

$$= \left[\frac{1 - \exp\left(\frac{-bD\beta}{2}\right) \exp(-i2M\pi)}{1 - \exp\left(\frac{-bD\beta}{2}\right)}\right]$$

D = path distance

β = antenna beamwidth $\approx \dfrac{2}{D}(y_2 - y_1)$

M = number of pairs of Fresnel zones in the common volume formed by the intersection of the two antenna beams

$$= \frac{2\alpha}{\lambda}(y_2 - y_1) = \alpha\beta\frac{D}{\lambda}.$$

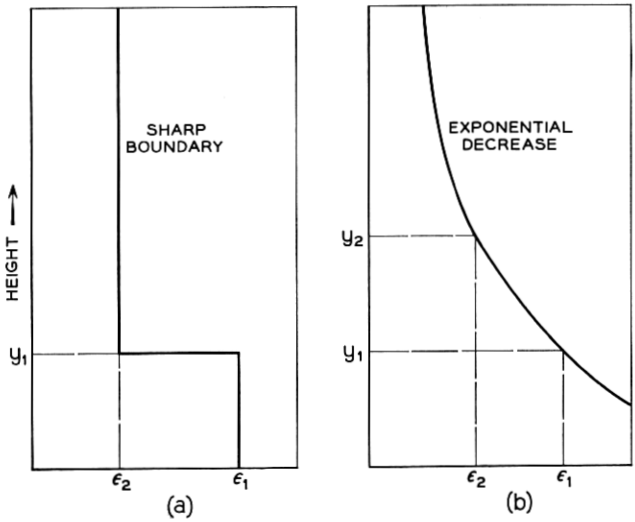

Fig. 2 — Idealized variation in dielectric permittivity.

The factor F_1 is unity whenever M is an integer; this is the minimum value and this is the case of interest.* The assumption of fractional Fresnel zones would lead to $|F_1| > 1$, but realizable antenna patterns cannot cut off sharply enough to make this assumption meaningful. Nonexponential models of the atmosphere lead to factors analogous to F_1 which are either zero or indeterminate, and presumably this is the basis for the widespread assumption that internal reflections cancel.

Substitution of (2) into (4) leads to the following expression for the reflection coefficient of a smooth exponential atmosphere

$$q = \left[\frac{(\epsilon_s - 1)}{4\alpha^2} e^{-bD\alpha/4}\right] \cdot \left[\frac{b\lambda}{4\pi\alpha}\right][1 - e^{-bD\beta/2}]F_1 \left[\frac{\exp\left[-b\left(\frac{y_1}{2} - y_s\right)\right]}{1 - \frac{ib\lambda}{4\pi\alpha}}\right]. \tag{5}$$

* The use of pairs of Fresnel zones is an artifice introduced by Fresnel to solve related problems in optics. By this means, a difficult vector addition problem is changed into a much easier scalar one because the phase shift in each pair is exactly 360° by definition. A small percentage change in the number of pairs of zones cannot change the sum significantly, so the exact number of pairs is not critical. If, however, the summation is taken over a region that cuts off sharply at an odd or fractional zone, the sum depends critically on the exact nature of the cutoff. In this case, the magnitude of the vector sum oscillates over a relatively wide range and mathematically the summation is very difficult or indeterminate.

The term in the first brackets represents the reflection coefficient that would have been expected if all the decrease in dielectric permittivity had occurred at a sharp boundary at height $y_1 = D\alpha/2$. The term in the second brackets is the effect of an exponential decrease for wide-beam antennas, (as long as this factor is less than unity). The third brackets show the effect of narrow-beam antennas, which depends on distance as well as beamwidth. As discussed above, the factor F_1 can be set to unity. The term in the right-hand brackets can also be neglected for a relatively smooth earth path, because $y_1 \approx 2y_s$ for this case.

Typical values of the reflection coefficient q computed from (5) are shown in Table I for wide-beam antennas; that is, only the first two bracketed terms are used.

The estimated losses given in Table I are much closer to the measured results than are those obtained from smooth earth diffraction theory but they are still 10–30 db greater than the median measured data. In addition, (5) is incomplete because it is not quite adequate in other respects.

TABLE I

Distance beyond Line of Sight		α	20 log | q |, decibels		
km	miles		300 mc	1000 mc	3000 mc
100	62	0.45°	−49	−59	−69
200	124	0.9°	−68	−78	−88
500	310	2.25°	−97	−107	−117
1000	620	4.5°	−133	−143	−153

The experimental results on the variation of the received voltage with frequency seem, on the average, to be closer to $\sqrt{\lambda}$ rather than λ as shown in (5). Seasonal and climatic effects are known to be significant, but the only possible variable in (5) is $(\epsilon_s - 1)$, and the seasonal changes in $(\epsilon_s - 1)$ are too small to account for the observed results. Finally, the fixed exponential atmosphere assumed in (5) cannot hope to explain the fading phenomena unless the variations in the atmosphere are considered.

IV. EFFECT OF ATMOSPHERIC VARIATIONS

Random fluctuations in atmospheric pressure, temperature and humidity are superimposed on the average value. The resulting variations in the index of refraction ($n = \sqrt{\epsilon}$) at the earth's surface have been measured in many experiments, but perhaps the most complete and reliable results can be obtained from long-term weather data.[7-10]

An analysis of meteorological data for 8 years from 45 U.S. weather stations indicates that the standard deviation of the surface index of refraction has a long-term average in the vicinity of Washington, D.C. of 10–15 N units, which corresponds to a variation in ϵ_s of $\sigma(\epsilon_s) \approx 20$ to 30×10^{-6}. In the northern part of the U.S., $\sigma(\epsilon_s)$ may be as low as 4×10^{-6} (2 N units) and along the Gulf and in Florida it may be as high as 52×10^{-6} (26 N units) in certain hours and seasons. The variance at the earth's surface can be defined as

$$\text{variance } (\epsilon_s) = [\sigma(\epsilon_s)]^2 = [\delta(\epsilon_s - 1)]^2. \qquad (6)$$

Since $(\epsilon_s - 1)$ is of the order of 600×10^{-6}, the average value of 10 to 15 N units corresponds to $\delta = 3$ to 5 per cent at the surface.

At very great heights the variance of ϵ must approach zero. If the dielectric permittivity followed a smooth exponential decrease and ϵ_s were the only variable, the standard deviation σ would decrease at the same rate as the mean value given in (2), but this does not agree with the experiment. Experimental data based on meteorological measurements indicate that the standard deviation σ decreases more slowly than the mean value.[10] A better fit with experimental data can be obtained by assuming that the variance (σ^2) and not the standard deviation σ decreases at the same rate as the mean value. Quantitatively this means that the standard deviation of the dielectric permittivity, which is measured to be 3 to 5 per cent of the mean value at the surface, is taken to be about 8 per cent of the mean value at an elevation of 14 kilometers, 16 per cent at 28 kilometers, 32 per cent at 42 kilometers, etc. At higher elevations this model breaks down because the standard deviation cannot be greater than the mean value in a nonionized medium. Heights greater than 50 kilometers are approaching the ionosphere and, hence, are outside the usual range of interest for tropospheric transmission. On this basis, the variance at y_1 over the interval $y_2 - y_1$ is given by

$$\text{variance } (\epsilon_2 - \epsilon_1) = [\delta(\epsilon_s - 1)]^2 e^{-b(y_1 - y_s)}[1 - e^{-b(y_2 - y_1)}] \qquad (7)$$

or

$$\sigma(\epsilon_{2,1}) = \delta(\epsilon_s - 1) \exp\left(\frac{-b(y_1 - y_s)}{2}\right) \sqrt{1 - e^{-b(y_2 - y_1)}}.$$

The dielectric permittivity of the atmosphere, as used in this paper, has now been specified to be uniform in the horizontal plane and to have an exponential variation in the vertical coordinate with a mean value given by (2) and a variance given by (7). The mean value for the reflection coefficient corresponding to (2) has been given in (4) and (5). Inasmuch as the irregularities superimposed on the exponential decrease cause the variance of the dielectric permittivity to vary as its mean, it is assumed that the variance of the reflection coefficient is also proportional to its mean value. This critical assumption leads to

$$\text{variance } [4\alpha^2 q] = \text{variance } (\epsilon_2 - \epsilon_1) \frac{b}{b + i2A} F_1$$

which by means of (7) leads to

$$\sigma(q) = \left[\frac{\delta(\epsilon_s - 1)}{4\alpha^2} e^{-bD\alpha/8}\right] \sqrt{\frac{b\lambda}{4\pi\alpha}} \sqrt{1 - e^{-bD\beta/2}} \sqrt{F_1} \sqrt{\frac{\exp\left[-b\left(\frac{y_1}{2} - y_s\right)\right]}{1 - \frac{ib\lambda}{4\pi\alpha}}}. \tag{8}$$

The interpretation of (8) is essentially the same as for (5). The first two terms to the right of the equality sign are the important ones for wide-angle antennas, the third of the five factors is the correction for narrow-beam antennas and the two factors on the right are unity in most applications and hence can ordinarily be neglected.

When the standard deviation (8) is comparable to or larger than the mean value (5), the resulting distribution of reflection coefficients contains both positive and negative values. The significance of a change in sign is to introduce an uncertainty in phase delay of one half cycle, which is unmeasurable and unimportant in most radio applications and tests. In order to compare with experimental data it is necessary to "fold over" the negative part of the distribution, add it to the positive part and then to find the mean value of the resulting distribution.

The mean value of the folded distribution approaches the mean value μ of the Gaussian distribution when $\sigma \ll \mu$ and approaches $\sqrt{2/\pi}\sigma$ when $\sigma \gg \mu$. More precisely, the mean value of the folded distribution is given by

$$\mu'(\text{folded distribution}) = \sqrt{\frac{2}{\pi}}\, \sigma\, e^{-\mu^2/2\sigma^2} + \mu \operatorname{erf}\left(\frac{\mu}{\sigma\sqrt{2}}\right)$$

where μ and σ refer to the original Gaussian distribution.

The folded distribution indicates that fading is to be expected, at least on a diurnal and seasonal basis. The use of meteorological data taken at hourly intervals does not provide quantitative information on the fast fading, but it does not preclude it. Moreover, the fading can be larger and at a faster rate than indicated by the variance in the dielectric permittivity taken over the entire vertical height of the common volume. The fading depends on the variations within each pair of Fresnel zones rather than on the standard deviation of the entire region. The standard deviation of the variation within each pair of zones is of the order of $\sqrt{M}$ times the standard deviation of their sum, where M is the number of pairs of Fresnel zones.

In addition to the reflection coefficient of the atmosphere, the transmission loss between two ground-based antennas relative to free space is also affected by ground reflections near each terminal. Ground reflections introduce image antennas which in effect double the amplitude of the received field intensity. This factor of 2 (or more precisely $1 + |r|$, where r is the reflection coefficient of the ground) appears in plane-earth, smooth-sphere and scatter theories, and it is unimportant numerically whether it is considered as twice the free-space field incident on the atmosphere with no effect near the receiving terminal or considered to be a power addition of the antenna and its image at both terminals.

Consequently, the mean value of transmission loss (relative to free space) to be compared with experimental data is the larger of $2q$ from (5) or $2\sqrt{2/\pi\sigma}$ from (8). At frequencies above 300 mc the latter is controlling, and typical values are shown in Table II for $\delta(\epsilon_s - 1) = 25 \times 10^{-6}$, which corresponds to a standard deviation at the earth's surface of 12.5 N units.

For comparison with the scatter hypothesis, the first two terms in (8) lead to the following ratio for the average received power relative to free space for wide-beam antennas

$$\frac{P}{P_0} = [2\sqrt{2/\pi\sigma}]^2 = \frac{b\lambda[\sigma(\epsilon_s)]^2 e^{-bD\alpha/4}}{8\pi^2\alpha^5}. \tag{9}$$

V. INTERPRETATION OF RESULTS

The computed transmission losses shown in the above table for wide-beam antennas are compared on Fig. 3 with the average experimental

TABLE II

Distance		α	Transmission Loss Relative to Free Space, decibels		
km	miles		300 mc	1000 mc	3000 mc
100	62	0.45°	44	49	54
200	124	0.9°	60	65	70
500	310	2.25°	82	87	92
1000	620	4.5°	106	111	116

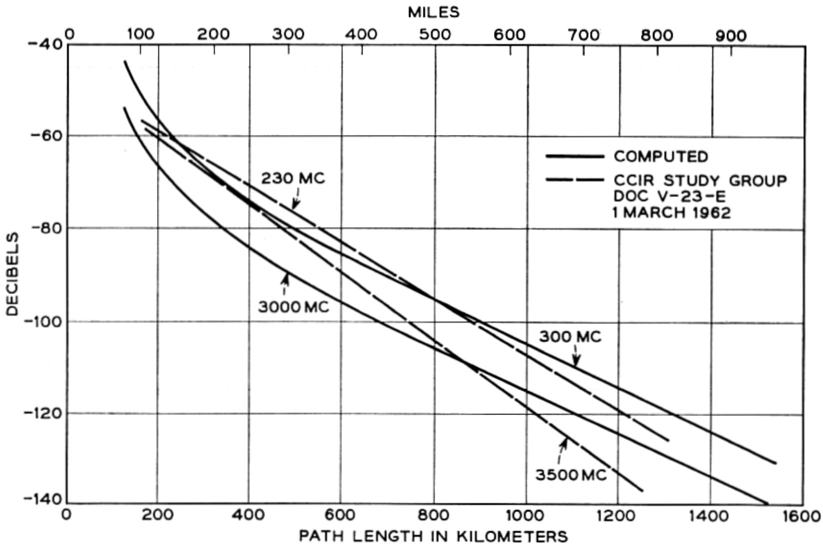

Fig. 3 — Transhorizon transmission loss for widebeam antenna.

data as given in a recent CCIR Study Group report. Since the computed values are based on the distance between the two horizons, while the experimental data are based on total path length, the computed values have been plotted on Fig. 3 at a distance of 30 km greater than given in the above table in order to allow for the line-of-sight distance between each terminal and its horizon. This assumption corresponds to an antenna height of about 50 feet over smooth earth. Any other reasonable assumption is not likely to change the comparison significantly.

The computed values are somewhat lower than the average experimental data as given in the CCIR Study Group Report, but at least part of the difference (in decibels) may be due to the assumed value of antenna to medium coupling loss used in the analysis of the original experimetal data. As shown in the following paragraph, the effect of narrow beam antennas derived herein is ordinarily less than obtained from the method used in the CCIR Study Group Report.[11,12]

The antenna degradation factor given in (8) is

$$\sqrt{1 - e^{-bD\beta/2}} = \sqrt{1 - e^{-b(y_2 - y_1)}},$$

and is evaluated in Table III.

The db loss shown in the second column of Table III depends on the height of the common volume $(y_2 - y_1)$. When the path is symmetrical and the two antennas have equal beamwidth, the corresponding path length for a given loss and beamwidth is shown by the right-hand col-

TABLE III

$(y_2 - y_1)$	Decrease in Effective Antenna Gain (Both Antennas)	Distances for Various Beamwidths		
		$\beta = 2°$	$\beta = 1°$	$\beta = 0.5°$
300 meters	13.9 db	17 km	34 km	69 km
1000	8.9	58	115	230
3000	4.7	172	345	690
10,000	1.2	575	1150	2300
20,000	0.27	1150	—	—

umns. These results are compared on Fig. 4 with typical values of the antenna-to-medium coupling based on scatter concepts for a 1° antenna. Although most of the experimental data are near the crossover region, the opposing trends provide a good basis for a critical experiment.

A previously unexplained experimental result at 400 mc on a 1000-km path is that antennas up to 120 feet in diameter (beamwidths greater than $1\frac{1}{4}°$) showed less than 1 db loss in effective antenna gain when measured at one terminal only.[13] This result agrees with the present

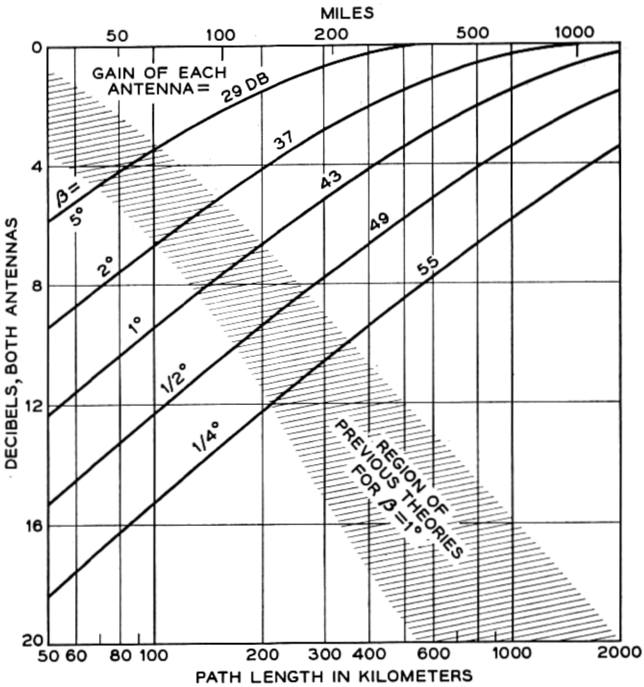

Fig. 4 — Loss in effective antenna gain.

method but is in sharp disagreement with earlier predictions. At the other extreme, measurements at 9000 mc on a 46-mile path with 4-foot and 1-foot antennas (about 1.6° and 6.6°) showed a loss of about 9 db when a change in antenna was made at both terminals.[14] The data on Fig. 4 would indicate no more than 6 db for a 4-to-1 change at short distances, but is closer to the measured result than the earlier methods which indicate 2 to 4 db.

Long-term comparisons between 8-foot and 60-foot antennas at 4000 mc (1.9° and 0.25°) on a 171-mile path indicated about 12 db loss in antenna effectiveness.[15] On the average the signal received on the 60-foot antenna was 5.7 db greater than on the 8-foot antenna, rather than 17.5 db, as would be expected in free space. The resulting loss of 11.8 db compares with an expected value of 10 log 6 = 7.8 db, since the path profile given in the reference indicates that the height of the common volume was reduced by a factor of 6.

The conclusion on the loss in effective antenna gain is that although the data on Fig. 4 seem a few db too low, the general shape of decreasing loss with increasing distance is to be preferred to the opposite trend derived from scatter theory.

It will be noted that the voltage ratio given in (8) varies as $\sqrt{\lambda}$, which is in general agreement with experiment, and that variations in $\delta(\epsilon_s - 1)$ can be 20 db or more with variations in climate, season and time of day. At UHF and above, the standard deviation (8) is large compared with the mean value (5), so considerable variation with time is to be expected; in the lower VHF and below, the standard deviation for the usual distances is less than the mean value, and this is consistent with the reduced amplitude of fading at the lower frequencies.

The remainder of this paper outlines the derivation of the reflection coefficient of a smooth exponential atmosphere, which was stated without proof in (4) and (5).

VI. DERIVATION OF REFLECTION FROM A SMOOTH EXPONENTIAL ATMOSPHERE

Some reflections must occur whenever the dielectric permittivity changes. When the change occurs at a sharp boundary, the reflection coefficient is proportional to the difference in dielectric permittivity; when the dielectric permittivity is constant with height, the reflection is zero. The problem is to find the reflection coefficient for the intermediate case of a gradual change in permittivity.

A useful formulation of this problem was published by Schelkunoff in 1951.[16] The geometry and three possible variations are indicated on Fig

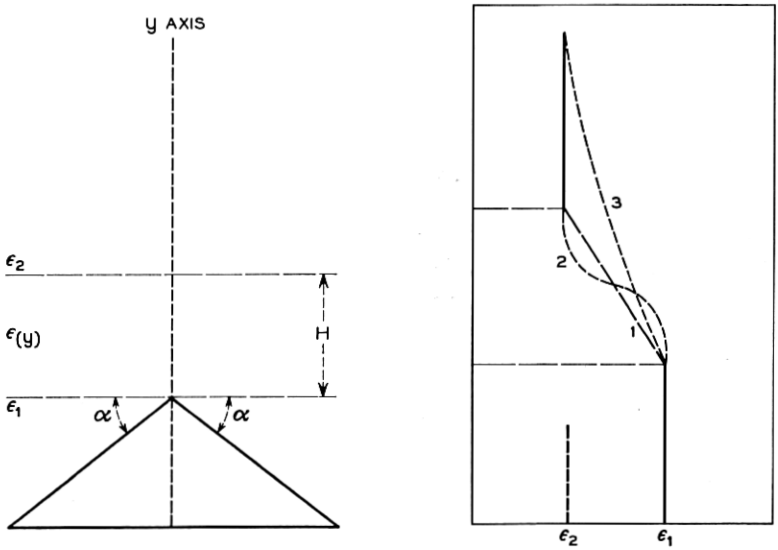

Fig. 5 — Idealized geometry for reflection from stratified media.

5. It is assumed that the region in which the dielectric permittivity is changing is sandwiched between two uniform media of dielectric permittivity ϵ_1 and ϵ_2, respectively. The reflection coefficient q of the stratified medium is derived from wave equations with the following result

$$q = \frac{1}{2}\int_0^\theta \frac{d \log K(\phi)}{d\phi}\, e^{-i2\phi}\, d\phi$$

$$= \frac{1}{2}\int_0^\theta \frac{1}{K}\frac{dK}{d\phi}\, e^{-i2\phi}\, d\phi$$

where

$$K = \sqrt{\frac{2\pi}{\lambda}\,(\epsilon - \epsilon_1 + \sin^2 \alpha)}$$

$$dK \approx \frac{Kd\epsilon}{2 \sin^2 \alpha}$$

$$\phi = \frac{2\pi y\alpha}{\lambda} = Ay; \qquad \theta = \frac{2\pi H\alpha}{\lambda} = AH$$

λ = wavelength in same units as height y

$H = y_2 - y_1$.

It follows that for $(\epsilon_2 - \epsilon_1) \ll \sin^2 \alpha$

$$q \approx \frac{1}{4 \sin^2 \alpha} \int_0^\theta \frac{d\epsilon}{d\phi} e^{-i2\phi} \, d\phi. \tag{10}$$

For a linear variation corresponding to

$$\epsilon = \epsilon_1 + \frac{\phi}{\theta} (\epsilon_2 - \epsilon_1)$$

the reflection coefficient reduces to

$$q = R \frac{\sin \theta}{\theta} e^{-i\theta}, \qquad R = \frac{(\epsilon_2 - \epsilon_1)}{4 \sin^2 \alpha}.$$

The factor R is the reflection coefficient expected for a sharp boundary with the same total change in dielectric permittivity, and θ is the total phase change in the inhomogeneous media.

In a similar manner, the reflection coefficient for the antisymmetrical cubic illustrated by case 2 is given by

$$q = R \frac{3}{\theta^2} \left(\frac{\sin \theta}{\theta} - \cos \theta \right) e^{-i\theta}.$$

The above examples for linear and S shaped curves are given by Schelkunoff. The same general method has been used more recently by du Castel, Misme and Voge to obtain the reflection coefficient for the additional cases of parabolic, sinusoidal and hyperbolic variations.[17] Their results can be summarized as follows:

$$\text{linear } | q | = R \frac{\sin \theta}{\theta} \tag{11}$$

$$\text{sinusoidal } | q | = R \frac{\cos \theta}{1 - \theta} \tag{12}$$

$$\text{parabolic } | q | = R \left(\frac{\sin \theta}{\theta} \right)^2 \tag{13}$$

$$\text{hyperbolic } | q | = R \frac{\theta}{\sinh \theta}. \tag{14}$$

In each case, as θ approaches zero, the reflection coefficient approaches the value given by Snell's law for an abrupt change.

The effect of a gradual decrease in dielectric permittivity is to reduce the reflection coefficient below what would have been expected if all of the change occurred at a sharp boundary. The reduction in reflection coefficient depends on the slope of transition and not on the boundary conditions alone. For the linear case it varies as $1/\theta$ and for the hyperbolic case it ultimately decreases exponentially. As θ approaches infinity the reflection coefficient goes to zero for these particular cases, but before reaching the conclusion that q always goes to zero as the upper limit goes to infinity, it is instructive to formulate the general case.

In general, the dielectric permittivity ϵ can be represented by

$$\epsilon = \epsilon_1 + m(\epsilon_2 - \epsilon_1)$$

where the parameter m can be any function that varies from $m = 0$ at the lower boundary ($\epsilon = \epsilon_1$) to $m = 1$ at the upper boundary ($\epsilon = \epsilon_2$). The corresponding reflection coefficient is

$$q = R \int_0^\theta \frac{dm}{d\phi} e^{-i2\phi} \, d\phi. \tag{15}$$

An exponential decrease with height provides a better match to the average atmospheric variation than is possible with any of the above models. Consequently, it is instructive to consider the case of $m = 1 - e^{-by}$, where b is the rate of decrease in atmospheric density. The magnitude of b is about 0.14 per km and is determined by the characteristics of gases and the law of gravity. For the case of a smooth exponential decrease, the reflection coefficient given by (10) or (15) reduces to

$$q = R \int_0^\theta be^{-by} \frac{dy}{d\phi} e^{-i2\phi} \, d\phi \tag{16}$$

$$= R \int_0^H be^{-(b+i2A)y} \, dy$$

$$= R \frac{b}{b + i2A} [1 - e^{-(b+i2A)H}]$$

$$= \frac{\epsilon_2 - \epsilon_1}{4 \sin^2 \alpha} \frac{b}{b + i2A} [1 - e^{-(b+i2A)H}] \tag{17}$$

where

$$\theta = AH$$

$$\phi = Ay.$$

It will be noted that the upper limit H appears only in the exponential factor and that the term in brackets approaches unity as $H \to \infty$. The reflection coefficient for a smooth exponential atmosphere is the reflection coefficient R that would result if all the change in dielectric permittivity occurred at a sharp boundary, multiplied by $b/(b + i2A)$. In the atmosphere, $b \ll A$ for VHF and above, as long as α is greater than 0.01 degree.

The bilinear model (uniform decrease with increasing height until $\epsilon = 1$) was used extensively in earlier studies, but its usefulness was questioned because of the discontinuity at the upper boundary $H = 1/b$. In the present nomenclature the reflection coefficient for the bilinear model with a slope b is

$$q = R \frac{b}{i2A} [1 - e^{-i2AH}]. \tag{18}$$

This is equivalent to the derivation of equation (2) in Ref. 2. The similarity between (17) and (18) shows that the use of the exponential decrease removes the effect of the upper discontinuity without any significant change in the magnitude of the reflection coefficient.

The foregoing analysis has been based on a rectangular coordinate system centered on the lower boundary to facilitate comparison with earlier papers. It is now desirable to shift to the geometry illustrated in Fig. 1. In this case

$$\begin{aligned} m &= 1 - e^{-b(y-y_s)} \\ \epsilon &= \epsilon_s + [1 - e^{-b(y-y_s)}][1 - \epsilon_s] \qquad (19) \\ &= 1 + (\epsilon_s - 1)e^{-b(y-y_s)} \\ d\epsilon &= -(\epsilon_s - 1)be^{-b(y-y_s)}dy \qquad (20) \end{aligned}$$

where ϵ_s is the dielectric permittivity at the surface.

Substituting (20) into (10),

$$\begin{aligned} q &= -\frac{(\epsilon_s - 1)}{4 \sin^2 \alpha} e^{by_s} b \int_{Ay_1}^{Ay_2} e^{-(b+i2A)y} \frac{dy}{d\phi} d\phi \\ &= -\frac{(\epsilon_s - 1)}{4 \sin^2 \alpha} e^{by_s} b \int_{y_1}^{y_2} e^{-(b+i2A)y} dy \end{aligned}$$

which reduces to

$$q = -\frac{(\epsilon_s - 1)}{4 \sin^2 \alpha} e^{-b(y_1-y_s)} e^{-i2Ay_1} \frac{b}{b + i2A} [1 - e^{-(b+i2A)(y_2-y_1)}]. \tag{21}$$

From the geometry in Fig. 1 it can be seen that for the symmetrical case the grazing angle $\alpha \approx 2y_1/D$ and the antenna beamwidth $\beta \approx$

$(2/D)(y_2 - y_1)$. In addition, each successive Fresnel zone corresponds to a change in phase of π radians, so $y_2 - y_1 = M\pi/A$, where M is the number of pairs of Fresnel zones. With these substitutions (21) can be rearranged as follows

$$q = \frac{(\epsilon_s - 1)}{4 \sin^2 \alpha} e^{-bD\alpha/4} \frac{b}{2A} [1 - e^{-bD\beta/2}] \left[\frac{1 - e^{-bD\beta/2} e^{-i2M\pi}}{1 - e^{-bD\beta/2}}\right] \left[\frac{\exp\left(-b\left(\frac{y_1}{2} - y_s\right)\right)}{1 - \frac{ib}{2A}}\right]. \quad (22)$$

This equation has already been given by (5). The first bracket to the right of the equality sign is equivalent to equation (27) in Ref. 18 except that the latter is 2.24 times larger to account for the effect of the earth and for a gradual cutoff near the horizon.

This completes the derivation of (5), but a better understanding of the physical problem can be obtained by introducing the factor $(1 - e^{-b\pi/A})$ in both numerator and denominator as follows

$$q = \left[\frac{(\epsilon_s - 1)}{4 \sin^2 \alpha} (1 - e^{-b\pi/A}) e^{-b(y_1 - y_s)}\right]\left[\frac{b}{b + i2A}\right] \left[\frac{1 - e^{-b\pi M/A} e^{-i2M\pi}}{1 - e^{-b\pi/A}}\right]. \quad (23)$$

The terms inside the first brackets represent the magnitude of the reflection coefficient that would result if the change in dielectric permittivity within one pair of Fresnel zones (phase change of 360°) were concentrated at a sharp boundary. The middle bracket corrects for a gradual change instead of a sharp boundary. The right-hand brackets represent the sum of M pairs of Fresnel zones and will be recognized as the sum of a geometric progression of M terms whose common ratio is $e^{-b\pi/A}e^{-i2\pi}$. The factor $(1 - e^{-b\pi/A})$ is the fraction of the reflected energy that remains in each pair of Fresnel zones after the almost complete cancellation caused by opposite phase in two adjacent Fresnel zones.

The concept of Fresnel zones is used to simplify the problem of adding a large number of components with a wide range of phases. By grouping into blocks of 2π radians, the contribution from each block adds in phase, and it is easy to see that the inclusion of weak components beyond nominal boundaries does not cause the answer to disappear. The cutoff merely neglects this contribution and the true answer is slightly greater than the computations indicate. In other words, the effect of the "discontinuity" at the lower limit of integration, which has proved troublesome in some formulations of the problem, seems to have a negligible effect on the magnitude of the signal returned from the atmosphere.

Although the Fresnel zone concept provides a clearer understanding of the problem, it is easier for computation purposes to use (5).

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