Existence of Eigenvalues of a Class of Integral Equations Arising in Laser Theory

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It is proved that the integral equation

$$\int_{-1}^{1} G(x)F(xy)H(y)f(y)\,dy = \lambda f(x)$$

has at least one nonzero eigenvalue if F is any integral function of finite order, G and H are any bounded functions on $[-1,1]$, and the trace of the kernel $G(x)F(xy)H(y)$ does not vanish. In particular, this theorem furnishes the first rigorous proof that the kernel exp $[ik(x - y)^2]$, which arises in the theory of the gas laser, has an eigenvalue for arbitrary complex k.

I. INTRODUCTION AND SUMMARY

In an idealized model of the gas laser or optical maser, as studied by Fox and Li[1,2] and others, electromagnetic radiation is reflected back and forth between two infinitely long metal strips which are mirror images of each other. A typical field quantity, such as the current density, at the surface of each reflector satisfies the integral equation

$$\int_{-1}^{1} \exp\{i[k(x - y)^2 - h(x) - h(y)]\}\, f(y)\,dy = \lambda f(x), \qquad (1)$$

where k is a dimensionless real parameter which depends on the width and spacing of the reflectors and the wavelength, and $h(x)$ is a real function specifying the departure of the reflecting surfaces from parallel planes.

The eigenfunctions of (1) represent the field distributions at the reflectors of the possible modes of oscillation of the laser, and the eigenvalue λ corresponding to a particular mode represents the complex factor by which the field strength is multiplied as a result of one reflection and transit between the reflectors. From the magnitude of λ one can deduce the amount of amplification which would have to be provided by an active medium between the reflectors in order just to sustain oscillations in the given mode, while the phase of λ determines admissible reflector spacings for oscillations at a particular frequency.

The mathematical interest of (1) centers around the fact that its kernel $K(x,y)$ is complex symmetric but not Hermitian;* that is,

$$K(x,y) = K(y,x) \qquad \text{but} \qquad K(x,y) \neq \overline{K(y,x)}. \tag{2}$$

The ordinary theory of Hermitian kernels does not even suffice to prove the existence of eigenvalues of complex symmetric kernels. Fox and Li[1] have made extensive calculations of the eigenvalues and eigenfunctions of (1) for $h(x) = 0$ by iterative numerical techniques up to about $k = 60$ (in applications k may be as large as a few hundred); but heretofore there has been no formal mathematical proof of the existence of solutions except† for $| k | \ll 1$, which is not a case of physical interest.

This paper contains a proof of the following

Theorem: Let $G(x)$ *and* $H(x)$ *be any bounded functions on the interval* $-1 \leqq x \leqq 1$, *and let* $F(z)$ *be any integral function of finite order such that*

$$\int_{-1}^{1} G(x)F(x^2)H(x)\,dx \neq 0. \tag{3}$$

Then the integral equation

$$\int_{-1}^{1} G(x)F(xy)H(y)f(y)\,dy = \lambda f(x) \tag{4}$$

has at least one nonzero eigenvalue.

As a corollary, it follows that the integral equation (1) has at least one eigenvalue for arbitrary complex k, provided only that

$$\int_{-1}^{1} e^{-2ih(x)}\,dx \neq 0. \tag{5}$$

Furthermore if $h(x)$ is an even function of x, then (1) has at least two eigenvalues for all but certain exceptional values of k, a particular exceptional value being $k = 0$.

The idea of the proof is quite simple. The assumption that $F(xy)$ in (4) is an integral function of finite order means that ultimately the coefficients of its Taylor series in powers of xy fall off with extreme rapidity.

* The kernel is normal in the special case $h(x) = kx^2$. The eigenfunctions of $\exp(-2ikxy)$ are prolate spheroidal wave functions, as pointed out in connection with lasers by Boyd and Gordon.[3]

† If $| k | \ll 1$ then $\exp[ik(x - y)^2]$ is nearly unity, and the existence of at least one eigenvalue follows from perturbation theory; see Sz.-Nagy.[4]

If we truncate the Taylor series after a finite number of terms, (4) is replaced by an integral equation with a kernel of finite rank. The eigenvalues of such a kernel are merely the latent roots of a finite matrix, and these are not all zero if their sum, which is the trace of the matrix, does not vanish. The limiting value of the trace is just the left side of (3), and does not vanish by hypothesis. By taking more and more terms of the series for $F(xy)$, we obtain a sequence of larger and larger matrices, whose elements ultimately vanish very rapidly with distance from the upper left corner. We show that it is possible to pick one eigenvalue from the set of eigenvalues of each succeeding matrix in such a way that the resulting sequence of numbers has a nonzero limit point. This limit point is an eigenvalue of the infinite matrix, and hence an eigenvalue of the original integral equation.

Details of the argument just sketched are given in a series of lemmas in the next section, followed by the proof of the main theorem. Since the existence proof makes heavy use of asymptotic inequalities, it does not generally provide a practical technique for obtaining numerical results. The important practical question of finding approximate expressions, valid for large k, for the eigenfunctions and eigenvalues of equations such as (1) is a separate problem, as is also the question whether any particular equation has a finite or infinite number of eigenvalues.

For a gas laser with finite (not strip) mirrors of arbitrary, dissimilar shape and size, the integral equation still has a complex symmetric kernel,[2] although the domain of integration is two-dimensional and the kernel is more complicated than that of (1). The existence of eigenvalues in the most general case still remains to be settled.

II. MATHEMATICAL DETAILS

We shall use the following notation referring to an $n \times n$ matrix:

$$A^{(n)} = (a_{ij}), \qquad i = 1, 2, \cdots, n; \qquad j = 1, 2, \cdots, n;$$

$$A^{(n)}(i) = \sum_{j=1}^{n} | a_{ij} |, \qquad i = 1, 2, \cdots, n; \tag{6}$$

$$S(A^{(n)}) = \sum_{i=1}^{n} A^{(n)}(i) = \sum_{i=1}^{n} \sum_{j=1}^{n} | a_{ij} |.$$

If the superscript is omitted, n is understood to be infinite.

Lemma 1:

$$| \det A^{(n)} | \leqq \prod_{i=1}^{n} A^{(n)}(i). \tag{7}$$

Proof: Using Hadamard's inequality,

$$\begin{aligned} \mid \det A^{(n)} \mid &\leqq \prod_{i=1}^{n} \left[\sum_{j=1}^{n} \mid a_{ij} \mid^2 \right]^{1/2} \\ &\leqq \prod_{i=1}^{n} \left[\left(\sum_{j=1}^{n} \mid a_{ij} \mid \right)^2 \right]^{1/2} = \prod_{i=1}^{n} A^{(n)}(i). \end{aligned} \tag{8}$$

Lemma 2:

$$\begin{aligned} &\mid det(A^{(n)} + B^{(n)}) - det\ A^{(n)} \mid \\ &\qquad \leqq \prod_{i=1}^{n} [A^{(n)}(i) + B^{(n)}(i)] - \prod_{i=1}^{n} A^{(n)}(i). \end{aligned} \tag{9}$$

Proof: The lemma is obviously true for $n = 1$. To proceed by induction, assume it is true for all determinants of order $n - 1$, and expand the determinants in (9) by minors of the first row. Let C_{1j} be the algebraic complement of $a_{1j} + b_{1j}$ in $A^{(n)} + B^{(n)}$, and let A_{1j} be the algebraic complement of a_{1j} in $A^{(n)}$. Then

$$\begin{aligned} \det(A^{(n)} + B^{(n)}) &= \sum_{j=1}^{n} (a_{1j} + b_{1j}) C_{1j} \\ &= \det A^{(n)} + \sum_{j=1}^{n} a_{1j}(C_{1j} - A_{1j}) + \sum_{j=1}^{n} b_{1j} C_{1j} . \end{aligned} \tag{10}$$

By Lemma 1,

$$\begin{aligned} \mid C_{1j} \mid &\leqq \prod_{i=2}^{n} \left[\sum_{k=1}^{n} \mid a_{ik} + b_{ik} \mid \right] \\ &\leqq \prod_{i=2}^{n} [A^{(n)}(i) + B^{(n)}(i)]. \end{aligned} \tag{11}$$

By the inductive hypothesis,

$$\mid C_{1j} - A_{1j} \mid \leqq \prod_{i=2}^{n} [A^{(n)}(i) + B^{(n)}(i)] - \prod_{i=2}^{n} A^{(n)}(i). \tag{12}$$

where we have used the fact that the right-hand side is increasing as a function of the $A^{(n)}(i)$ and $B^{(n)}(i)$. Hence (10) gives

$$| \det(A^{(n)} + B^{(n)}) - \det A^{(n)} |$$

$$\leqq A^{(n)}(1) \left\{ \prod_{i=2}^{n} [A^{(n)}(i) + B^{(n)}(i)] - \prod_{i=2}^{n} A^{(n)}(i) \right\}$$

$$+ B^{(n)}(1) \prod_{i=2}^{n} [A^{(n)}(i) + B^{(n)}(i)] \tag{13}$$

$$= \prod_{i=1}^{n} [A^{(n)}(i) + B^{(n)}(i)] - \prod_{i=1}^{n} A^{(n)}(i),$$

and the induction is complete.

Now let $\mathfrak{B}$ be the Banach space* whose elements are all bounded sequences of complex numbers, e.g.,

$$x = (x_1, x_2, \cdots, x_i, \cdots) \tag{14}$$

with norm

$$\| x \| = \sup_i | x_i |. \tag{15}$$

Let A be a linear matrix operator on the space $\mathfrak{B}$, defined by

$$(Ax)_i = \sum_{j=1}^{\infty} a_{ij} x_j, \qquad i = 1, 2, \cdots. \tag{16}$$

Ax will be an element of $\mathfrak{B}$ provided that $\sup_i A(i)$ is finite. The norm of A is defined by

$$\| A \| = \sup \{ \| Ax \|; \| x \| = 1 \}, \tag{17}$$

and it is easy to show that

$$\| A \| = \sup_i A(i). \tag{18}$$

Henceforth we shall restrict our attention to matrix operators for which

$$S(A) \equiv \sum_{i=1}^{\infty} A(i) < \infty. \tag{19}$$

Such operators are completely continuous, because they can be approximated by the sequence $\{A^{(n)}\}$ of completely continuous operators which converges in norm to A. Here $A^{(n)}$ is a matrix whose elements co-

* The standard definitions and theorems which we shall require from functional analysis may be found in Kolmogorov and Fomin.[5]

incide with those of A for $1 \leqq i \leqq n$ and $1 \leqq j \leqq n$, and are zero otherwise.

A complex number λ is said to be in the *spectrum* of an operator A if the operator $A - \lambda I$ has no inverse. An *eigenvalue* of A is any value of λ for which there exists a nonzero x satisfying the homogeneous equation

$$Ax - \lambda x = 0. \tag{20}$$

If A is completely continuous and if λ ($\neq 0$) lies in the spectrum of A, then λ is an eigenvalue of A. In finite-dimensional space the eigenvalues are the latent roots of the matrix $A^{(n)}$; that is, they are the roots of the characteristic equation

$$\det (A^{(n)} - \lambda I^{(n)}) = 0. \tag{21}$$

Lemma 3: If $A^{(n)}$ has λ as an eigenvalue, then $A^{(n)} + B^{(n)}$ has λ', where

$$|\lambda - \lambda'| \leqq \left\{ \prod_{i=1}^{n} [A^{(n)}(i) + B^{(n)}(i) + |\lambda|] - \prod_{i=1}^{n} [A^{(n)}(i) + |\lambda|] \right\}^{1/n}. \tag{22}$$

Proof: Denote the eigenvalues of $A^{(n)} + B^{(n)}$ by $\lambda_1, \lambda_2, \cdots, \lambda_n$. Then

$$\begin{aligned} &|(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)| \\ &\qquad = |\det (A^{(n)} + B^{(n)} - \lambda I^{(n)}) - \det (A^{(n)} - \lambda I^{(n)})|, \end{aligned} \tag{23}$$

the second determinant being equal to zero because λ is an eigenvalue of $A^{(n)}$. Let

$$D^{(n)} = A^{(n)} - \lambda I^{(n)}, \tag{24}$$

so that

$$D^{(n)}(i) = \sum_{j=1}^{n} |a_{ij} - \lambda \delta_{ij}| \leqq A^{(n)}(i) + |\lambda|. \tag{25}$$

Then, using Lemma 2,

$$\begin{aligned} \prod_{k=1}^{n} |\lambda - \lambda_k| &\leqq \prod_{i=1}^{n} [D^{(n)}(i) + B^{(n)}(i)] - \prod_{i=1}^{n} D^{(n)}(i) \\ &\leqq \prod_{i=1}^{n} [A^{(n)}(i) + B^{(n)}(i) + |\lambda|] \\ &\qquad - \prod_{i=1}^{n} [A^{(n)}(i) + |\lambda|], \end{aligned} \tag{26}$$

since the right side of the first line is an increasing function of $D^{(n)}(i)$. It follows from (26) that for at least one of the factors $|\lambda - \lambda_k|$ the inequality (22) holds.

Lemma 4: Let A be an infinite matrix with $S(A) < \infty$. Suppose that from the eigenvalues of the sequence of finite matrices $\{A^{(n)}\}$ we can pick a sequence $\{\lambda^{(n)}\}$ such that $\lambda^{(n)}$ does not approach zero as $n \to \infty$. Then A has a nonzero eigenvalue.

Proof: The $\lambda^{(n)}$ are bounded, since in fact

$$|\lambda^{(n)}| \leqq \| A^{(n)} \| = \max_i A^{(n)}(i) \leqq S(A). \tag{27}$$

Also for sufficiently large n we can pick a subsequence which is bounded away from zero, and which therefore has at least one nonzero limit point. Suppose that the subsequence $\lambda^{(p)}$ converges to the limit point $\lambda \neq 0$, as p runs through some increasing sequence of integers. We assert that λ is an eigenvalue of A. If it were not so, then $(A - \lambda I)^{-1}$ would exist and therefore be bounded. Suppose $(A - \lambda I)^{-1}$ were bounded, and let $x^{(p)}$ be the characteristic vector of $A^{(p)}$ corresponding to $\lambda^{(p)}$. Then we would have

$$\begin{aligned} x^{(p)} &= (A - \lambda I)^{-1}(A - \lambda I)x^{(p)} \\ &= (A - \lambda I)^{-1}[A^{(p)}x^{(p)} - \lambda^{(p)}x^{(p)} \\ &\qquad + (A - A^{(p)})x^{(p)} - (\lambda - \lambda^{(p)})x^{(p)}] \\ &= (A - \lambda I)^{-1}[(A - A^{(p)})x^{(p)} - (\lambda - \lambda^{(p)})x^{(p)}], \end{aligned} \tag{28}$$

where in the last equation $A^{(p)}$ represents an infinite matrix which coincides with A in a square of side p in the upper left corner, and has zeros elsewhere. Taking norms, we have

$$\begin{aligned} \| x^{(p)} \| &\leqq \|(A - \lambda I)^{-1}\| \, \|(A - A^{(p)})x^{(p)} - (\lambda - \lambda^{(p)})x^{(p)}\| \\ &\leqq \|(A - \lambda I)^{-1}\| \, [\| A - A^{(p)} \| + |\lambda - \lambda^{(p)}|] \, \| x^{(p)} \|, \end{aligned} \tag{29}$$

or

$$\| (A - \lambda I)^{-1} \| \geqq \frac{1}{\| A - A^{(p)} \| + |\lambda - \lambda^{(p)}|}. \tag{30}$$

But since both $\| A - A^{(p)} \|$ and $|\lambda - \lambda^{(p)}|$ go to zero as $p \to \infty$, we derive a contradiction.

Theorem: Let A be an infinite matrix with $S(A) < \infty$ and with $Tr(A) \neq 0$. If

$$S(A) - S(A^{(n)}) < (c/n^{\epsilon})^n, \tag{31}$$

for some $c, \epsilon > 0$, then A has a nonzero eigenvalue.

Proof: Since $\mathrm{Tr}(A) \neq 0$ and $\mathrm{Tr}(A^{(n)}) \to \mathrm{Tr}(A)$, it follows that for $n \geqq n_1$ (say) and some $\delta > 0$, we have $|\mathrm{Tr}(A^{(n)})| \geqq \delta$. Since the trace is the sum of the eigenvalues, $A^{(n)}$ must have at least one eigenvalue $\lambda^{(n)}$ such that

$$|\lambda^{(n)}| \geqq \delta/n. \tag{32}$$

We shall in fact show that if n_1 is a sufficiently large fixed integer, and if

$$n_j = 2^{j-1}n_1, \qquad j = 1, 2, 3, \cdots \tag{33}$$

then for each j there exists an eigenvalue which is *uniformly* bounded away from zero, i.e.,

$$\lambda^{(n_j)} \geqq \delta/2n_1. \tag{34}$$

Then by Lemma 4 the theorem will be proved.

We substitute into Lemma 3 as follows:

$$\begin{aligned} n &= n_{j+1}, \\ |\lambda| &= |\lambda^{(n_j)}| = t, \\ A^{(n)} &= A^{(n_j)}, \\ B^{(n)} &= A^{(n_{j+1})} - A^{(n_j)}, \end{aligned} \tag{35}$$

where it is understood that $A^{(n_j)}$ now represents the original matrix $A^{(n_j)}$ augmented below and to the right with enough zeros to give it dimensions $n_{j+1} \times n_{j+1}$. Then (22) becomes

$$\begin{aligned} &|\lambda^{(n_j)} - \lambda^{(n_{j+1})}| \\ &\quad \leqq \left\{ \prod_{i=1}^{n_{j+1}} [A^{(n_{j+1})}(i) + t] - t^{n_{j+1}-n_j} \prod_{i=1}^{n_j} [A^{(n_j)}(i) + t] \right\}^{1/n_{j+1}} \\ &\quad \leqq \left\{ \prod_{i=1}^{n_{j+1}} [A(i) + t] - t^{n_{j+1}-n_j} \prod_{i=1}^{n_j} [A^{(n_j)}(i) + t] \right\}^{1/n_{j+1}}. \end{aligned} \tag{36}$$

Since

$$|\lambda^{(n_j)} - \lambda^{(n_{j+1})}| \geqq t - |\lambda^{(n_{j+1})}|, \tag{37}$$

we can rearrange (36) to get

$$|\lambda^{(n_{j+1})}| \geqq t - \left\{ \prod_{i=1}^{n_{j+1}} [A(i) + t] - t^{n_{j+1}-n_j} \prod_{i=1}^{n_j} [A^{(n_j)}(i) + t] \right\}^{1/n_{j+1}}. \tag{38}$$

Hence

$$\frac{|\lambda^{(n_{j+1})}|}{|\lambda^{(n_j)}|} \geqq 1 - \left\{ \prod_{i=1}^{n_{j+1}} \left[1 + \frac{A(i)}{t} \right] - \prod_{i=1}^{n_j} \left[1 + \frac{A^{(n_j)}(i)}{t} \right] \right\}^{1/n_{j+1}}$$

$$\geqq 1 - \left\{ \prod_{i=1}^{n_{j+1}} \left[1 + \frac{n_j A(i)}{\delta} \right] - \prod_{i=1}^{n_j} \left[1 + \frac{n_j A^{(n_j)}(i)}{\delta} \right] \right\}^{1/n_{j+1}}, \quad (39)$$

since we already know that $t \geqq \delta/n_j$.

Now consider

$$\prod_{i=1}^{n_j} \left[1 + \frac{n_j A(i)}{\delta} \right] \leqq \prod_{i=1}^{n_j} \exp \left[\frac{n_j A(i)}{\delta} \right] \leqq \exp \left[\frac{n_j S(A)}{\delta} \right]. \quad (40)$$

Also

$$\prod_{i=n_j+1}^{n_{j+1}} \left[1 + \frac{n_j A(i)}{\delta} \right] \leqq \exp \left[\frac{n_j}{\delta} \sum_{i=n_j+1}^{\infty} A(i) \right]$$

$$\leqq \exp \frac{n_j}{\delta} [S(A) - S(A^{(n_j)})] \quad (41)$$

$$\leqq \exp \left[\frac{n_j}{\delta} \left(\frac{c}{n_j^{\epsilon}} \right)^{n_j} \right] \leqq 1 + \frac{2n_j}{\delta} \left(\frac{c}{n_j^{\epsilon}} \right)^{n_j},$$

provided that n_1 and hence n_j are sufficiently large, where in the next to last step we have used (31) and in the last step we have used $e^x \leqq 1 + 2x$ for $0 \leqq x \leqq 1$, say. Finally,

$$\prod_{i=1}^{n_j} \left[1 + \frac{n_j A^{(n_j)}(i)}{\delta} \right]$$

$$= \prod_{i=1}^{n_j} \left[1 + \frac{n_j A(i)}{\delta} - \frac{n_j}{\delta} \{ A(i) - A^{(n_j)}(i) \} \right]$$

$$\geqq \prod_{i=1}^{n_j} \left[1 + \frac{n_j A(i)}{\delta} \right]$$

$$- \frac{n_j}{\delta} \sum_{k=1}^{n_j} \left\{ [A(k) - A^{(n_j)}(k)] \prod_{i=1}^{n_j} \left[1 + \frac{n_j A(i)}{\delta} \right] \right\} \quad (42)$$

$$\geqq \prod_{i=1}^{n_j} \left[1 + \frac{n_j A(i)}{\delta} \right]$$

$$- \frac{n_j^2}{\delta} [S(A) - S(A^{(n_j)})] \prod_{i=1}^{n_j} \left[1 + \frac{n_j A(i)}{\delta} \right]$$

$$\geqq \left\{ 1 - \frac{n_j^2}{\delta} \left(\frac{c}{n_j^{\epsilon}} \right)^{n_j} \right\} \prod_{i=1}^{n_j} \left[1 + \frac{n_j A(i)}{\delta} \right].$$

Substituting (40), (41), and (42) into (39) yields

$$\frac{|\lambda^{(n_{j+1})}|}{|\lambda^{(n_j)}|} \geqq 1 - \left\{\frac{n_j(2+n_j)}{\delta}\left(\frac{c}{n_j^{\epsilon}}\right)^{n_j} \exp \frac{n_j S(A)}{\delta}\right\}^{1/n_{j+1}} = 1 - \left[\frac{n_j(2+n_j)}{\delta}\right]^{1/2n_j} \frac{c_1}{n_j^{\epsilon/2}}, \tag{43}$$

where in the last step we have used the fact that $n_{j+1} = 2n_j$ and have set

$$c_1 = c^{1/2} \exp [S(A)/2\delta]. \tag{44}$$

If we assume in advance that

$$\delta \leqq 2, \qquad n_1 \geqq \max (2, 4/\epsilon), \tag{45}$$

then

$$\left[\frac{n_j(2+n_j)}{\delta}\right]^{1/2n_j} \frac{c_1}{n_j^{\epsilon/2}} \leqq \left[\frac{2n_j^2}{\delta}\right]^{1/2n_j} \frac{c_1}{n_j^{\epsilon/2}} \leqq \frac{2c_1}{\delta n_j^{\epsilon/4}} = \frac{c_2}{2^{(j-1)\epsilon/4}} = c_2 r^{j-1}, \tag{46}$$

where

$$c_2 = \frac{2c_1}{\delta n_1^{\epsilon/4}}, \qquad r = 2^{-\epsilon/4} < 1. \tag{47}$$

Hence (43) and (46) imply

$$\frac{|\lambda^{(n_{j+1})}|}{|\lambda^{(n_j)}|} \geqq 1 - c_2 r^{j-1}, \tag{48}$$

and by induction

$$\frac{|\lambda^{(n_J)}|}{|\lambda^{(n_1)}|} \geqq \prod_{j=1}^{J-1} [1 - c_2 r^{j-1}]. \tag{49}$$

But if $c_2 \leqq \frac{1}{2}$, say, then

$$\prod_{j=1}^{\infty} (1 - c_2 r^{j-1}) = \exp\left[\sum_{j=1}^{\infty} \log (1 - c_2 r^{j-1})\right] \geqq \exp\left[-2\sum_{j=1}^{\infty} c_2 r^{j-1}\right] = \exp\left[-\frac{2c_2}{1-r}\right] > \tfrac{1}{2}, \tag{50}$$

where the last step requires

$$c_2 < \tfrac{1}{2}(1-r)\log 2 = \tfrac{1}{2}(1 - 2^{-\epsilon/4}) \log 2, \tag{51}$$

and by (47) this inequality can always be satisfied for large enough n_1. But (49) and (50) imply

$$\lambda^{(n_j)} \geqq \tfrac{1}{2}\lambda^{(n_1)} \geqq \delta/(2n_1) > 0 \tag{52}$$

for all j, and so the theorem follows from Lemma 4. Q.E.D.

An integral function of finite order ρ is a function $F(z)$ which has no singularities in any finite region of the z-plane, and whose maximum modulus $M(r)$ on the circle $|z| = r$ satisfies

$$\log M(r) < r^k \tag{53}$$

for all sufficiently large r when $k > \rho$, but not when $k < \rho$. Such a function may be expanded in a Taylor series,

$$F(z) = \sum_{n=0}^{\infty} a_n z^n, \tag{54}$$

which converges for all z, and whose coefficients satisfy[6]

$$|a_n| < 1/n^{n\epsilon} \tag{55}$$

for all sufficiently large n, where ϵ is any fixed number less than $1/\rho$. Alternatively, for any fixed $\epsilon < 1/\rho$, there exists a constant c such that for all $n > 0$

$$|a_n| \leqq \left[\frac{c}{(n+1)^\epsilon}\right]^{n+1}. \tag{56}$$

We are now ready to prove the result stated in Section I.

Theorem: Let $G(x)$ and $H(x)$ be any bounded functions on the interval $-1 \leqq x \leqq 1$, and let $F(z)$ be any integral function of finite order such that

$$\int_{-1}^{1} G(x)F(x^2)H(x)dx \neq 0. \tag{57}$$

Then the integral equation

$$\int_{-1}^{1} G(x)F(xy)H(y)f(y)dy = \lambda f(x) \tag{58}$$

has at least one nonzero eigenvalue.

Proof: Expand $F(xy)$ in a Taylor series, so that the integral equation becomes

$$\int_{-1}^{1} \sum_{n=1}^{\infty} [a_{n-1}^{1/2} G(x) x^{n-1}][a_{n-1}^{1/2} H(y) y^{n-1}] f(y)dy = \lambda f(x). \tag{59}$$

Let

$$f(x) = G(x) \sum_{n=1}^{\infty} f_n a_{n-1} x^{n-1}, \tag{60}$$

where $\{f_n\}$ is a bounded sequence of complex numbers; the a_n's tend to zero fast enough so that $f(z)/G(z)$ will be an integral function of finite order.

Since the powers of x are linearly independent, (59) is equivalent to the matrix equation

$$Af = \lambda f, \tag{61}$$

where

$$a_{ij} = a_{ji} = (a_{i-1}a_{j-1})^{1/2} \int_{-1}^{1} G(t)H(t)t^{i+j-2}dt, \tag{62}$$

$$i = 1, 2, \cdots; \qquad j = 1, 2, \cdots .$$

Since $G(x)$ and $H(x)$ are bounded in $-1 \leqq x \leqq 1$ and the Taylor coefficients of $F(z)$ satisfy (56), it is clear that

$$| a_{ij} | \leqq \frac{M}{i+j-1} \left(\frac{c}{i^\epsilon}\right)^{i/2} \left(\frac{c}{j^\epsilon}\right)^{j/2}. \tag{63}$$

In preparation for an application of the preceding theorem, consider

$$\begin{aligned} S(A) - S(A^{(n)}) &\leqq 2 \sum_{i=n+1}^{\infty} \sum_{j=1}^{i} \frac{M}{i+j-1} \left(\frac{c}{i^\epsilon}\right)^{i/2} \left(\frac{c}{j^\epsilon}\right)^{j/2} \\ &= 2M \sum_{i=n+1}^{\infty} \left[\left(\frac{c}{i^\epsilon}\right)^{i/2} \sum_{j=1}^{i} \frac{1}{i+j-1} \left(\frac{c}{j^\epsilon}\right)^{j/2}\right]. \end{aligned} \tag{64}$$

Now $(c/j^\epsilon)^{j/2}$ is bounded as $j \to \infty$, and

$$\sum_{j=1}^{i} \frac{1}{i+j-1} \leqq \int_{i-1}^{2i-1} \frac{dx}{x} = \log\left[\frac{2i-1}{i-1}\right], \tag{65}$$

which is bounded for $i \geqq n + 1 \geqq 2$. Hence with a new bounding constant we have

$$S(A) - S(A^{(n)}) \leqq M_1 \sum_{i=n+1}^{\infty} \left(\frac{c^{1/2}}{i^{\epsilon/2}}\right)^i. \tag{66}$$

Choose $\log n \geqq (2 + \log c)/\epsilon$, so that $n^\epsilon \geqq ce^2$; then

$$\begin{aligned} \sum_{i=n+1}^{\infty} \left(\frac{c^{1/2}}{i^{\epsilon/2}}\right)^i &\leqq \int_n^\infty \left(\frac{c^{1/2}}{x^{\epsilon/2}}\right)^x dx \leqq \int_n^\infty \left(\frac{c^{1/2}}{n^{\epsilon/2}}\right)^x dx \\ &= -\frac{(c/n^\epsilon)^{n/2}}{(\log c - \epsilon \log n)/2} \leqq \left(\frac{c^{1/2}}{n^{\epsilon/2}}\right)^n, \end{aligned} \tag{67}$$

and so from (66)

$$S(A) - S(A^{(n)}) \leqq \left(\frac{c_1}{n^{\epsilon_1}}\right)^n, \tag{68}$$

where c_1 is a new bounding constant and $\epsilon_1 = \epsilon/2$.

Finally we have

$$\begin{aligned} \operatorname{Tr}(A) &= \sum_{i=1}^{\infty} a_{ii} = \sum_{i=1}^{\infty} a_{i-1} \int_{-1}^{1} G(t)H(t)t^{2i-2}\,dt \\ &= \int_{-1}^{1} G(t)H(t)F(t^2)\,dt, \end{aligned} \tag{69}$$

and this does not vanish by hypothesis. Hence all the conditions of the previous theorem are satisfied, and the integral equation has a nonzero eigenvalue. Q.E.D.

Since $\exp(-2ikz)$ is an integral function of finite order 1, it is an obvious corollary that the kernel $\exp i[k(x-y)^2 - h(x) - h(y)]$ has a nonzero eigenvalue for arbitrary complex k, provided only that $h(x)$ is bounded and that

$$\int_{-1}^{1} e^{-2ih(x)}\,dx \neq 0. \tag{70}$$

Furthermore if $h(x)$ is an even function of x and if $f(x)$ is an even function which satisfies

$$\int_0^1 \exp\{i[k(x^2+y^2) - h(x) - h(y)]\}\cos(2kxy)f(y)dy = \tfrac{1}{2}\lambda f(x), \tag{71}$$

then $f(x)$ also satisfies (1). But the theorem just proved obviously holds for arbitrary finite limits of integration and applies to the kernel of (71), so (71) has at least one nonzero eigenvalue if

$$\int_0^1 \exp\{2i[kx^2 - h(x)]\}\cos(2kx^2)dx \neq 0. \tag{72}$$

Similarly if $h(x)$ is even and if $f(x)$ is an odd function which satisfies

$$\int_0^1 \exp\{i[k(x^2+y^2) - h(x) - h(y)]\}\sin(2kxy)f(y)dy = \tfrac{1}{2}i\lambda f(x), \tag{73}$$

then $f(x)$ also satisfies (1), and (73) has at least one nonzero eigenvalue if

$$\int_0^1 \exp\{2i[kx^2 - h(x)]\}\sin(2kx^2)dx \neq 0. \tag{74}$$

At least one of (72) and (74) will be satisfied whenever (70) holds. Except for certain particular values of k, one of which is evidently $k = 0$, both (72) and (74) will be satisfied, and (1) will have at least two distinct eigenfunctions corresponding to nonzero eigenvalues.

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