On the Discrete Spectral Densities of Markov Pulse Trains

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(Manuscript received August 12, 1963)

General formulae and existence criteria are derived for the discrete power spectral densities of first-order Markov pulse trains, viz., infinite pulse trains in which each pulse corresponds to one member of a finite set of specified waveforms and depends statistically on the previous pulse alone. These results are obtained through a distribution theoretic decomposition of the spectral formulation given for such pulse trains by Huggins and Zadeh.

I. INTRODUCTION

An important problem related to first-order Markov pulse trains is that of calculating the discrete and continuous power spectral densities of such processes. The spectral formulation first given by Huggins[1] and later extended by Zadeh[2] is perhaps the most appropriate and straightforward solution of this problem, the results being conveniently expressed in terms of the customary flow diagrams and recurrent event relations associated with Markov systems. As regards discrete spectra, however, their formulation lacks complete generality in two respects: (*i*) the limit notions of distribution theory, although essential for discrete components, are not incorporated; (*ii*) discrete components do not appear explicitly. In this paper we reformulate the Huggins-Zadeh result on a distribution theoretic basis, and derive both explicit relations and existence criteria for the discrete spectral densities. It is intended also that the analysis illustrate the distribution theoretic techniques required in cases involving more general spectral formulations.

II. BACKGROUND

The infinite pulse trains under discussion are treated as first-order Markov processes in that each pulse is assumed to correspond in waveshape to one member of a finite set (alphabet) of real time functions

$g_i(t)$, and to depend statistically on the previous pulse alone. More precisely, we consider random processes of the form

$$x(t) = \sum_{n=-\infty}^{\infty} d_n(t - t_n), \qquad t \, \varepsilon \, (-\infty, \infty) \tag{1}$$

$$t_n < t_{n+1} \tag{2}$$

where

$$d_n(t) \, \varepsilon \, \{g_i(t) \mid g_i \, \varepsilon \, L_1(-\infty, \infty); i = 1, 2, \cdots, M\} \tag{3}$$

$$P\{d_n = g_i \mid d_{n-1} = g_j ; d_{n-2} = g_k ; \cdots\} = P\{d_n = g_i \mid d_{n-1} = g_j\} \tag{4a}$$

$$P\{(t_{n+1} - t_n) \leqq \tau \mid d_n = g_i ; d_{n+1} = g_j ; \tau \geqq 0\} \equiv c_{ij}(\tau) \tag{4b}$$

with t_n signifying the nth occurrence time, and c_{ij} the cumulative transition distributions.* For fixed i and j, c_{ij} gives independently of n (i.e., the pulse position) the conditional probability of a direct transition from pulse g_i to pulse g_j within τ seconds after the occurrence of the former. As in related studies, the statistical and combinatorial structure of (1) is represented by the usual flow diagram of Fig. 1 in which nodes, or "states," symbolize pulses g_i, and directed links indicate possible transitions.†

The flow diagram in conjunction with signal flow graph techniques yields directly the more complex probability functions of general interest.‡ Most important to the development here are the cumulative distributions for first occurrences or recurrences, viz.

$$\begin{aligned} P\{(t_{n+m} - t_n) \leqq \tau \text{ for some } m \geqq 1 \mid d_{n+m} = g_j ; d_n = g_i ; \\ d_{n+\bar{m}} \neq g_j(\bar{m} = 1, \cdots, m-1); \tau \geqq 0\} \equiv q_{ij}(\tau). \end{aligned} \tag{5}$$

As indicated, q_{ij} denotes the conditional probability of a first occurrence (recurrence if $i = j$) of state j within τ seconds after an occurrence of state i. Although less basic than c_{ij}, functions q_{ij} are entirely sufficient for the calculation of spectral densities; consequently, in this paper the set $\{q_{ij}\}$ is regarded as initially specifying the Markov process in

* As applied here, the terms "cumulative distribution" and "distribution" pertain to probability theory and distribution theory, respectively.

† Zadeh[2] identifies the occurrence of state i with the generation of a unit impulse at node i, the impulse in turn functioning as the input to a linear filter with impulse response g_i; the corresponding responses due to all the nodes of the system are added directly to give the original pulse train.

‡ The expositions by Huggins[1] and Aaron[3] illustrate in detail the various flow diagram methods by which transition and recurrent event probabilities of higher order are calculated.

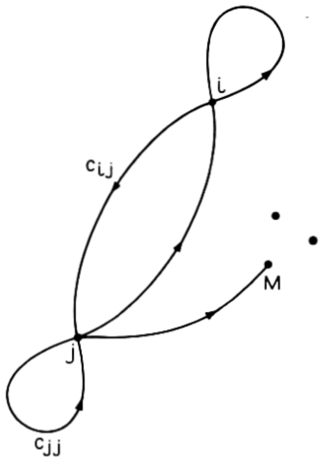

Fig. 1 — Flow diagram.

accordance with the following constraints:

(*i*) To comply with the usual probability conventions, we assume q_{ij} to be monotonically increasing, sectionally continuous, and such that

$$\begin{aligned} 0 \leqq q_{ij}(\tau) \leqq 1, \qquad & \tau \ \varepsilon \ [0, \infty) \\ q_{ij}(\tau) = 0, \qquad & \tau \ \varepsilon \ (-\infty, 0). \end{aligned} \tag{6}$$

Under these conditions both q_{ij} and the probability densities $f_{ij}(\tau) \equiv c_{ij}'(\tau)$ exist as distributions, or generalized functions.* (Earlier investigations have used f_{ij} exclusive of q_{ij}.)[1,3]

(*ii*) For pulses to occur with certainty and at distinct times ($t_n < t_{n+1}$), it is required that

$$q_{ij}(\tau) \to 1 \quad (\tau \to \infty) \tag{7}$$

$$q_{ij}(0) = q_{ij}(0^+) = 0. \tag{8}$$

Condition (7) merely asserts that every state is accessible from every other state, i.e., that the system is irreducible.

Assuming the specification of pulse trains $x(t)$ by either q_{ij} or f_{ij} and denoting the spectral density of $x(t)$ by $S_{xx}(f)$, we prove below that

* Briefly, an ordinary function $f(t)$ is an element of the space of distributions, or generalized functions, provided $[1 + t^2]^{-N} f(t) \ \varepsilon \ L_1(-\infty, \infty)$ for some $N \geqq 0$; moreover, for such functions as $f(t)$ there exist distribution derivatives of all orders and generalized Fourier transforms.[4,5,6]

$$S_{xx}(f) = \lim_{\alpha \to 0+}{}^{(D)} \left\{ \sum_i \sum_j G_i(\bar{s}) G_j(s) \left[p_i \left(\frac{F_{ij}(s)}{1 - F_{jj}(s)} + \delta_{ij} \right) + p_j \left(\frac{F_{ji}(\bar{s})}{1 - F_{ii}(\bar{s})} \right) \right] \right\} \qquad (9)^*$$

where

$$G_i(s) = \int_0^\infty g_i(\tau) e^{-s\tau} d\tau = \mathfrak{L} \cdot g_i$$

$$F_{ij}(s) = \int_0^\infty e^{-s\tau}\, dq_{ij}(\tau) \equiv \int_0^\infty e^{-s\tau} f_{ij}(\tau)\, d\tau = \mathfrak{L} \cdot f_{ij}$$

$$s = \alpha + 2\pi i f, \qquad \bar{s} = \alpha - 2\pi i f, \qquad i = \sqrt{-1}, \qquad f = \text{frequency}$$

$$p_i = \left[\int_0^\infty \tau dq_{ii}(\tau) \right]^{-1} = \lim_{\substack{s \to 0 \\ \alpha > 0}} \left[\frac{s}{1 - F_{ii}(s)} \right] = -\frac{1}{F_{ii}'(0)}$$

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases}$$

and $\lim^{(D)} \{\cdot\}$ signifies a distribution limit (cf. Ref. 4, p. 107, and Ref. 5, p. 183). The presence of $\lim^{(D)}$ and the conjugated variable $\bar{s}$ in relation (9) is especially significant, both features constituting the essential modification of the spectral density expression given by Zadeh (cf. Ref. 2, Eq. 9, and Ref. 1, Eq. 10b). These two formulations prove equivalent, however, relative to continuous spectra. Specifically, if f is such that $F_{ii}(2\pi i f) \neq 1$, then the distribution limit reduces to an ordinary limit, and S_{xx} represents the same point value of the continuous spectral density as results from Zadeh's expression. On the other hand, analyzing discrete spectra† requires a proper interpretation of functions

$$\frac{1}{1 - F_{ii}(s)}$$

in the vicinity of points $s = 2\pi i f$ for which $F_{ii}(2\pi i f) = 1$; hence, the notion of distribution limits is in general necessary. Another item to be noted in (9) is the functional form of g_i. Although it is assumed that $g_i \in L_1$, one can relax this restriction in certain cases by first considering an infinite sequence of functions $g_i^{(m)} \in L_1$ such that $g_i^{(m)} \to g_i \notin L_1$ $(m \to \infty)$, and then performing a second limit operation on the corre-

* The quantity $[F_{ij}(1 - F_{jj})^{-1} + \delta_{ij}] \equiv U_{ij}(s)$ in (9) corresponds to the Laplace transform of what Huggins terms the "expectation density" [cf. Ref. 1, Eq. (10b), p. 80].

† The term "discrete" relates to both the discrete power spectrum and the line spectral density composed of Dirac delta functions.

sponding density functions $S_{xx}^{(m)}$. An example illustrating this approach appears in Appendix A.

The following development deals primarily with the distribution theoretic formulation of (9) and its decomposition into discrete and continuous components. A detailed proof of this formulation and an analysis of the two types of components are given in Sections III and IV, respectively. Discrete spectral density expressions for the basic classes of first-order Markov pulse trains are derived in Sections 4.3, 4.4, 4.5, and 4.6 (cf. Theorems II–VI).

III. THE HUGGINS-ZADEH SPECTRAL DENSITY FORMULATION

In deriving S_{xx}, we find it convenient first to decompose $x(t)$ into M separate pulse trains which consist individually of identical pulses; i.e., we set

$$x(t) = \sum_{n=-\infty}^{\infty} d_n(t - t_n) = \sum_{i=1}^{M} x_i(t) \tag{10}$$

where

$$\begin{aligned} x_i(t) &= \sum_{m=-\infty}^{\infty} g_i(t - t_m^{(i)}) \\ t_m^{(i)} &\in \{t_n \mid d_n = g_i\} \\ t_m^{(i)} &< t_{m+1}^{(i)} \\ t_m^{(i)} &< 0 \qquad (m < 0) \\ t_m^{(i)} &\geqq 0 \qquad (m \geqq 0). \end{aligned}$$

Therefore, by standard spectral theory[7] S_{xx} can be written as

$$S_{xx}(f) = \sum_i \sum_j S_{x_i x_j}(f) \tag{11}$$

where

$$\begin{aligned} S_{x_i x_j}(f) &= \lim_{T\to\infty}{}^{(\mathrm{D})} \frac{1}{2T} E\{[\overline{\mathfrak{F}\cdot x_{iT}}][\mathfrak{F}\cdot x_{jT}]\} \\ x_{iT}(t) &= \sum_{m=M_i}^{N_i} g_i(t - t_m^{(i)}) \\ N_i &= \sup\{m \mid t_m^{(i)} \in [-T, T]\} \\ M_i &= \inf\{m \mid t_m^{(i)} \in [-T, T]\} \\ \mathfrak{F}\cdot &\equiv \int_{-\infty}^{\infty} dt\, e^{-2\pi i f t} \qquad (i = \sqrt{-1}). \end{aligned}$$

It is noted here that $S_{x_i x_j}$, the cross-spectral density of x_i and x_j, holds for both stationary and nonstationary processes.

Combined with the relation

$$\mathfrak{F} \cdot x_{iT} = G_i(2\pi if) \sum_{M_i}^{N_i} \exp\left(-2\pi i f t_m^{(i)}\right) \tag{12}$$

(11) reduces to

$$S_{xx}(f) = \sum_i \sum_j G_i(-2\pi if) G_j(2\pi if) S_{ij}(f) \tag{13}$$

where

$$S_{ij}(f) = \lim_T{}^{(\mathrm{D})} \frac{1}{2T} E\left\{\sum_{M_i}^{N_i} \sum_{M_j}^{N_j} \exp\left[-2\pi if(t_n^{(j)} - t_m^{(i)})\right]\right\}. \tag{14}$$

To transform the summation indices in (14), we let

$$t_n^{(j)} - t_m^{(i)} = \tau_{m,k}^{(ij)} > 0 \tag{15}$$

where integer $k \geqq 1$ indicates the number of occurrences of state j in the interval $(t_m^{(i)}, t_n^{(j)}]$; further, to eliminate the variation of summation indices across the ensemble, we define a weighting factor $\eta_{m,k}^{(ij)}$ such that

$$\eta_{m,k}^{(ij)} = \begin{cases} 1; & t_m^{(i)} \text{ and } t_n^{(j)} \in [-T,T], \quad t_m^{(i)} < t_n^{(j)} \\ 0; & t_m^{(i)} \text{ or } t_n^{(j)} \notin [-T,T], \quad t_m^{(i)} < t_n^{(j)}. \end{cases} \tag{16}$$

These definitions along with condition (8) relating to distinct occurrence times yield

$$\begin{aligned} \sum_{M_i}^{N_i} \sum_{M_j}^{N_j} \exp\left[-2\pi if(t_n^{(j)} - t_m^{(i)})\right] &= \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} \eta_{m,k}^{(ij)} \exp\left(-2\pi i f \tau_{m,k}^{(ij)}\right) \\ &+ \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} \eta_{m,k}^{(ji)} \exp\left(2\pi i f \tau_{m,k}^{(ji)}\right) + \delta_{ij} N_{iT} \end{aligned} \tag{17}$$

with N_{iT} equal to the number of occurrences of state i in the interval $[-T,T]$.

As random variables for the time difference between occurrences, $\tau_{m,k}^{(ij)}$ are characterized statistically by the cumulative distributions q_{ij}. In particular, (15) and (5) imply that

$$P\{\tau_{m,1}^{(ij)} \leqq \tau\} = q_{ij}(\tau). \tag{18}$$

Moreover, since the quantity

$$q_{ij}(\tau - \tau')[q_{jj}(\tau' + \Delta\tau) - q_{jj}(\tau')]$$

gives the approximate probability of two specific occurrences of state j within τ seconds after that of state i, it follows that the total probability of all such mutually exclusive events is expressed as

$$P\left\{\tau_{m,2}^{(ij)} \leqq \tau\right\} = \int_0^\tau q_{ij}(\tau - \tau')dq_{jj}(\tau') \equiv q_{ij}^{(2)}(\tau). \tag{19}$$

Generally

$$\begin{aligned} P\left\{\tau_{m,k}^{(ij)} \leqq \tau\right\} &= \int_0^\tau q_{ij}^{(k-1)}(\tau - \tau')\, dq_{jj}(\tau') \equiv q_{ij}^{(k)}(\tau) \quad (k \geqq 2) \\ q_{ij}^{(1)}(\tau) &\equiv q_{ij}(\tau). \end{aligned} \tag{20}$$

At this point we introduce a basic device with which to simplify the summations in (17) as well as justify the interchange of various limit operations employed below. If functions q_{ij} are specified so as to vanish not only for $\tau \leqq 0$ [cf. (6)] but also in an arbitrarily small neighborhood $(-\epsilon, \epsilon)$, then there can be only a finite number of states in any finite time interval (i.e., $P\{-T \leqq t_m^{(i)} \leqq T\} = 0$ for all $|m|$ sufficiently large), and the summations in (17) remain finite. Despite this initial restriction on q_{ij}, the spectral density proves continuous in ϵ; consequently, the resultant spectral formulation is viewed as having a final, nonexplicit limit corresponding to $\epsilon \to 0$. Such a limiting procedure is entirely sufficient for physical pulse trains.

For evaluating the expectation in (14), we first define

$$P_m^{(i)}(t) = P\{t_m^{(i)} \leqq t\} \tag{21}$$

$$\mu(x) = \begin{cases} 1 & (x \geqq 0) \\ 0 & (x < 0) \end{cases} \tag{22}$$

$$\delta(x) = \frac{d\mu(x)}{dx}. \tag{23}$$

Hence, for any state i

$$\begin{aligned} &\lim_T{}^{(\mathrm{D})} \frac{1}{2T} \sum_m \int_{-T}^{T-\tau} dP_m(t) \\ &= \lim_T{}^{(\mathrm{D})} \frac{1}{2T} \sum_m \int_{-\infty}^{\infty} [\mu(T - \tau - t) - \mu(-T - t)]\, dP_m(t) \\ &= \lim_T{}^{(\mathrm{D})} \frac{1}{2T} E\left\{\sum_m \int_{-T}^{T-\tau} \delta(t' - t_m)\, dt'\right\} = \lim_T \frac{1}{2T} E\{N_{iT}\} \\ &= [E\{t_m^{(i)} - t_{m-1}^{(i)}\}]^{-1} = \left[\int_0^\infty \tau dq_{ii}(\tau)\right]^{-1}. \end{aligned} \tag{24}$$

On the other hand, since

$$\frac{1 - e^{-s\tau}}{s} \to \tau \qquad (s \to 0)$$

$$\left| \frac{1 - e^{-s\tau}}{s} \right| \leqq \tau \qquad (\text{Re } s = \alpha \geqq 0)$$

the dominated convergence theorem[8] yields

$$\begin{aligned} \lim_{\substack{s\to 0 \\ \alpha>0}} \frac{s}{1 - F_{ii}(s)} \equiv p_i &= \left[\lim_{\substack{s\to 0 \\ \alpha>0}} \int_0^\infty \left(\frac{1 - e^{-s\tau}}{s} \right) dq_{ii}(\tau) \right]^{-1} \\ &= \left[\int_0^\infty \tau dq_{ii}(\tau) \right]^{-1} = \lim_T \frac{1}{2T} E\{N_{iT}\}. \end{aligned} \tag{25}$$

Thus, again by the convergence theorem, there results

$$\begin{aligned} p_i \int_0^\infty & e^{-2\pi i f\tau} \, dq_{ij}^{(k)}(\tau) \\ &= \lim_T{}^{(\mathrm{D})} \frac{1}{2T} \int_0^{2T} \left[\sum_m \int_{-T}^{T-\tau} dP_m(t) \right] e^{-2\pi i f\tau} \, dq_{ij}^{(k)}(\tau) \\ &= \lim_T{}^{(\mathrm{D})} \frac{1}{2T} \sum_m \int_0^{2T} \left[\int_{-T}^{T-\tau} dP_m(t) \right] e^{-2\pi i f\tau} \, dq_{ij}^{(k)}(\tau). \end{aligned} \tag{26}$$

Fundamental to the analysis of (26) is the following distribution theoretic identity, a detailed proof of which appears in Appendix B:

$$\lim_{N\to\infty}{}^{(\mathrm{D})} \sum_{k=1}^{N} \int_0^\infty e^{-2\pi i f\tau} \, dq_{ij}^{(k)}(\tau) = \lim_{\alpha\to 0+}{}^{(\mathrm{D})} \frac{F_{ij}(s)}{1 - F_{jj}(s)}. \tag{27}$$

From (26) and (27) it is found that

$$\begin{aligned} \lim_N{}^{(\mathrm{D})} & \sum_{k=1}^{N} p_i \int_0^\infty e^{-2\pi i f\tau} \, dq_{ii}^{(k)}(\tau) \\ &= \lim_N{}^{(\mathrm{D})} \sum_{k=1}^{N} \lim_T{}^{(\mathrm{D})} \frac{1}{2T} \sum_{m=-\infty}^{\infty} \int_0^{2T} \left[\int_{-T}^{T-\tau} dP_m(t) \right] e^{-2\pi i f\tau} \, dq_{ij}^{(k)}(\tau) \\ &= \lim_T{}^{(\mathrm{D})} \frac{1}{2T} \sum_{k=1}^{\infty} \sum_{m=-\infty}^{\infty} \int_0^{2T} \left[\int_{-T}^{T-\tau} dP_m(t) \right] e^{-2\pi i f\tau} \, dq_{ij}^{(k)}(\tau) \\ &= \lim_T{}^{(\mathrm{D})} \frac{1}{2T} E\left\{ \sum_k \sum_m \eta_{m,k}^{(ij)} \exp\left(-2\pi i f\tau_{m,k}^{(ij)}\right) \right\} \\ &= p_i \lim_{\alpha\to 0+}{}^{(\mathrm{D})} \frac{F_{ij}(s)}{1 - F_{jj}(s)}. \end{aligned} \tag{28}$$

Hence, (13), (14), (17), (25), and (28) combine to give

$$S_{xx}(f) = \lim_{\alpha\to 0+}{}^{(\mathrm{D})} \left\{ \sum_i \sum_j G_i(\bar{s}) G_j(s) \cdot \left[p_i \left(\frac{F_{ij}(s)}{1 - F_{jj}(s)} + \delta_{ij} \right) + p_j \left(\frac{F_{ji}(\bar{s})}{1 - F_{ii}(\bar{s})} \right) \right] \right\}. \tag{29}$$

IV. DISCRETE AND CONTINUOUS SPECTRA

The evaluation of the distribution limit in relation (9), as shown below, centers mainly on analyzing the asymptotic behavior of functions

$$\frac{F_{ij}(s)}{1 - F_{jj}(s)} \qquad (\mathrm{Re}\ s = \alpha \geqq 0) \tag{30}$$

as the variable s approaches singular points along the frequency axis, viz., points $s = 2\pi if$ for which $F_{jj}(2\pi if) = 1$; the results of this analysis together with certain general properties of F_{ij} serve to resolve S_{xx} into discrete and continuous components.

Considering singularities of (30) first, one notes that

$$F_{jj}(0) = \int_0^\infty dq_{jj}(\tau) = \lim_{\tau\to\infty} q_{jj}(\tau) - q_{jj}(0) = 1 \tag{31}$$

$$\begin{aligned} | F_{jj}(s) | \leqq \int_0^\infty e^{-\alpha\tau} dq_{jj}(\tau) &= \alpha \int_0^\infty e^{-\alpha\tau} q_{jj}(\tau) d\tau \\ &< \alpha \int_0^\infty e^{-\alpha\tau} d\tau = 1 \qquad (\mathrm{Re}\ s > 0) \end{aligned} \tag{32}$$

$$F_{jj}(-2\pi if) = \bar{F}_{jj}(2\pi if). \tag{33}$$

Consequently, for all processes point $s = 0$ is singular, points in the open half plane Re $s > 0$ are nonsingular, and the existing singularities on the frequency axis occur in conjugate pairs. In establishing notation, we define

$$\left.\begin{aligned} &s_{j,n}\ \varepsilon\ \{s \mid F_{jj}(s) = 1; \qquad \mathrm{Re}\ s = 0\} \\ &s_{j,n} = 2\pi if_{j,n} = \bar{s}_{j,-n} \\ &f_{j,n} < f_{j,n+1} \\ &f_{j,0} = 0 \end{aligned}\right\} \tag{34}$$

$$\left.\begin{aligned} &p_{j,n} = \left[\int_0^\infty \tau \exp(-s_{j,n}\tau)\, dq_{jj}(\tau) \right]^{-1} = -\frac{1}{F'(s_{j,n})} \\ &p_{j,n} = \bar{p}_{j,-n} \\ &p_{j,0} = p_j \end{aligned}\right\} \tag{35}$$

Then, as in (25)

$$\begin{aligned}\frac{1}{1 - F_{jj}(s)} &= \left[\int_0^\infty \exp\,(-s_{j,n}\tau)\,dq_{jj}(\tau) - \int_0^\infty e^{-s\tau}\,dq_{jj}(\tau)\right]^{-1} \\ &= \frac{1}{s - s_{j,n}}\left\{\int_0^\infty \left[\frac{1 - \exp\,[-(s - s_{j,n})\tau]}{s - s_{j,n}}\right] \cdot \exp\,(-s_{j,n}\tau)\,dq_{jj}(\tau)\right\}^{-1} \\ &\sim \frac{1}{s - s_{j,n}}\,p_{j,n} \qquad (s \to s_{j,n}\,,\ \mathrm{Re}\ s > 0)\end{aligned} \tag{36}$$

On the basis of this asymptotic result it is found convenient to rearrange (30) as

$$\frac{F_{ij}(s)}{1 - F_{jj}(s)} = Q_{ij}(s) + R_{ij}(s) \tag{37}$$

where

$$Q_{ij}(s) = \frac{F_{ij}(s)}{2} \sum_n p_{j,n}\left[\frac{1}{\bar{s} + s_{j,n}} + \frac{1}{s - s_{j,n}}\right] \tag{38}$$

$$R_{ij}(s) = S_{ij}(s) - \sum_n p_{j,n} T_n^{(ij)}(s) \tag{39}$$

$$S_{ij}(s) = \frac{F_{ij}(s)}{1 - F_{jj}(s)} - F_{ij}(s) \sum_n \frac{p_{j,n}}{s - s_{j,n}} \tag{40}$$

$$T_n^{(ij)}(s) = \frac{F_{ij}(s)}{2}\left[\frac{1}{\bar{s} + s_{j,n}} - \frac{1}{s - s_{j,n}}\right]. \tag{41}$$

The summations in (37) are considered for the moment to be finite and to involve only those singularities present in a frequency interval $(-f_A\,, f_A)$.

4.1 *Functions Q_{ij} and R_{ij}*

It is shown next that for $f \in (-f_A\,, f_A)$ functions Q_{ij} and R_{ij} can be identified as contributing respectively to the discrete and continuous spectra:

(*i*) That functions Q_{ij} give rise to only discrete components follows immediately from the relation

$$\underset{\alpha\to 0^+}{\mathrm{Lim}}^{(\mathrm{D})}\, G_i(\bar{s})G_j(s)Q_{ij}(s)$$

$$= \tfrac{1}{2}[G_i(-2\pi if)G_j(2\pi if)F_{ij}(2\pi if)]$$

$$\cdot \sum_n p_{j,n} \lim_{\alpha}{}^{(\mathrm{D})} \frac{2\alpha}{[\alpha^2 + 4\pi^2(f - f_{j,n})^2]}$$

$$= \tfrac{1}{2}[\bar{G}_i G_j F_{ij}]_f \sum_n p_{j,n} \lim_{\alpha}{}^{(\mathrm{D})} \cdot \mathfrak{F} \cdot \exp\,[-(\alpha\,|\,t\,|) + 2\pi i f_{j,n} t] \tag{42}$$

$$= \tfrac{1}{2}[\bar{G}_i G_j F_{ij}]_f \sum_n p_{j,n} \mathfrak{F} \cdot \lim_{\alpha}{}^{(\mathrm{D})} \cdot \exp\,[-(\alpha\,|\,t\,|) + 2\pi i f_{j,n} t]$$

$$= \tfrac{1}{2}[\bar{G}_i G_j F_{ij}]_f \sum_n p_{j,n} \mathfrak{F} \cdot \exp\,(2\pi i f_{j,n} t)$$

$$= \tfrac{1}{2}[\bar{G}_i G_j F_{ij}]_f \sum_n p_{j,n} \delta(f - f_{j,n}).$$

(*ii*) As regards functions R_{ij}, we first determine the behavior of functions S_{ij} in the neighborhood of points $s_{j,n}$. Substituting definition (35) into (40) yields

$$s_{ij}(s) \sim F_{ij}\left[\frac{1}{1 - F_{jj}} - \frac{p_{j,n}}{s - s_{j,n}}\right]$$

$$= \frac{p_{j,n}F_{ij}}{1 - F_{jj}} \left\{ \int_0^\infty \left[\tau - \frac{1 - \exp\,[-(s - s_{j,n})\tau]}{s - s_{j,n}}\right] \cdot \exp\,(-s_{j,n}\tau)\, dq_{jj}(\tau) \right\} \tag{43}$$

$$\to \frac{p_{j,n}{}^2 F_{ij}(s_{j,n})}{2} \int_0^\infty \tau^2 \exp\,(-s_{j,n}\tau)\, dq_{jj}(\tau)$$

$$(s \to s_{j,n},\ \mathrm{Re}\ s > 0)$$

which implies that functions S_{ij} are both bounded and integrable in $(-f_A, f_A)$, and that points $s_{j,n}$ correspond to simple poles with residues $p_{j,n}F_{ij}(s_{j,n})$. Since functions S_{ij} are integrable, they can contribute to only the continuous portion of the power spectrum. Regarding functions $T_n^{(ij)}$ next, we note that

$$
\begin{aligned}
&\lim_{\alpha\to 0^+}{}^{(D)}\, G_i(\bar{s})G_j(s)T_n^{\ (ij)}(s) \\
&\qquad = \tfrac{1}{2}[\bar{G}_iG_jF_{ij}]_f \lim_{\alpha}{}^{(D)} \left[\frac{4\pi i(f - f_{j,n})}{\alpha^2 + 4\pi^2(f - f_{j,n})^2}\right] \\
&\qquad = \tfrac{1}{2}[\bar{G}_iG_jF_{ij}]_f \lim_{\alpha}{}^{(D)} \cdot \mathfrak{F}\cdot[(\mu(-t) - \mu(t)) \\
&\qquad\qquad \cdot \exp(-\alpha\,|\,t\,|\, + 2\pi i f_{j,n}t)] \\
&\qquad = \tfrac{1}{2}[\bar{G}_iG_jF_{ij}]_f\mathfrak{F}\cdot[(\mu(-t) - \mu(t))\exp(2\pi i f_{j,n}t)] \\
&\qquad = \tfrac{1}{2}[\bar{G}_iG_jF_{ij}]_f \left[-\frac{1}{2\pi i(f - f_{j,n})}\right] \to \infty \qquad (f \to f_{j,n}).
\end{aligned} \tag{44}
$$

Hence, in a deleted neighborhood of $s_{j,n}$, functions $T_n^{(ij)}$ appear to predominate all other terms of S_{xx}. For showing that functions $T_n^{(ij)}$ in fact sum so as to remain bounded, we set all pulses equal to zero except one, viz., g_i. If under this condition S_{xx} becomes unbounded as $f \to f_{j,n}$, then (44) and (9) give

$$
S_{xx}(f) \sim p_j\{|\,G_i\,|^2[p_{j,n}F_{jj} - \bar{p}_{j,n}\bar{F}_{jj}]\}_f \left[\frac{-1}{2\pi i(f - f_{j,n})}\right], \qquad (f \to f_{j,n}) \tag{45}
$$

However, since the factor in braces is continuous at $f_{j,n}$, the sign reversal of the unbounded factor indicates that S_{xx} assumes, contrary to definition, arbitrarily large negative values; therefore,

$$
p_{j,n}F_{jj}(2\pi i f_{j,n}) - \bar{p}_{j,n}F_{jj}(-2\pi i f_{j,n}) = 0
$$

which by (34) becomes

$$
p_{j,n} = \bar{p}_{j,n} = p_{j,-n} \tag{46}
$$

[The trivial case $p_{j,n} = 0$ need not be considered inasmuch as the associated terms in (37)–(41) vanish identically under this condition]. Condition (46) is sufficient as well as necessary for the ratio

$$
\begin{aligned}
\frac{F_{jj}(2\pi i f) - F_{jj}(-2\pi i f)}{2\pi i(f - f_{j,n})} &= [F_{jj}'(s_{j,n}) - F_{jj}'(\bar{s}_{j,n})] + 0(f - f_{j,n}) \\
&= \left[\frac{1}{p_{j,n}} - \frac{1}{\bar{p}_{j,n}}\right] + 0(f - f_{j,n}) \\
&= 0(f - f_{j,n}) \quad (f \to f_{j,n})
\end{aligned} \tag{47}
$$

to be bounded in a neighborhood of point $f_{j,n}$. Similarly, allowing two pulses to be nonzero and arbitrary yields

$$S_{xx}(f) \sim p_i p_{j,n}[\bar{G}_i G_j F_{ij} - \bar{G}_j G_i F_{ij}]_f \left[\frac{-1}{2\pi i(f - f_{j,n})}\right] + p_j\,, p_{i,m}[\bar{G}_j G_i F_{ji} - \bar{G}_i G_j \bar{F}_{ji}]_f \left[\frac{-1}{2\pi i(f - f_{i,m})}\right] \qquad (f \to f_{j,n}) \tag{48}$$

where the second term is present provided $f_{j,n} = f_{i,m}$. It is evident that with the second term absent and both g_i and g_j arbitrary the first term cannot be made to vanish identically at $f_{j,n}$; thus

$$s_{j,n} = s_{i,m} = s_{i,n} \equiv s_n \tag{49}$$

and

$$\{\bar{G}_i G_j[p_i p_{j,n} F_{ij} - p_j p_{i,n} \bar{F}_{ji}] + \bar{G}_j G_i[p_j p_{i,n} F_{ji} - p_i p_{j,n} \bar{F}_{ij}]\}_{f_n} = 0. \tag{50}$$

Again because of arbitrary g_i and g_j there results

$$p_i p_{j,n} F_{ij}(2\pi i f_n) = p_j p_{i,n} F_{ji}(-2\pi i f_n). \tag{51}$$

As in (47), this is a necessary and sufficient condition that (48) be bounded in a neighborhood of point $f = f_{j,n} = f_n$; thus, for $f \,\varepsilon\, (-f_A\,, f_A)$ functions $T_n^{(ij)}$, S_{ij}, and sums R_{ij} contribute to only the continuous spectrum. It is important to note that although the use of R_{ij} is necessary for an appropriate decomposition of S_{xx}, the complete continuous spectrum can be obtained directly from relation (9) with $f \neq f_n$ [cf. (9) et seq.]. Nevertheless, from a computational standpoint functions R_{ij} might be more suitable.

4.2 *General Formulation for Discrete Spectra*

At this point we consider in detail both formulae and existence criteria for the discrete spectral density. With respect to the complete spectral density, the substitution of definition (37) into (9) gives at once the decomposition

$$S_{xx}(f) = \lim_{\alpha\to0^+}{}^{(\mathrm{D})} \left\{\sum_i \sum_j G_i(\bar{s}) G_j(s)[p_i Q_{ij}(s) + p_j Q_{ji}(\bar{s})]\right\} + \lim_{\alpha\to0^+}{}^{(\mathrm{D})} \left\{\sum_i p_i \mid G_i(s) \mid^2 + \sum_i \sum_j [p_i R_{ij}(s) + p_j R_{ji}(\bar{s})]\right\} \tag{52}$$

where according to the properties of functions Q_{ij} and R_{ij} [cf., (42), (51) et seq.] the first term in braces consists of discrete components only, and the second is bounded for $f \,\varepsilon\, (-f_A\,, f_A)$. Consequently, on letting

$S_{xx}^{(d)}(f)$ denote the discrete spectral density in the interval $(-f_A, f_A)$, we obtain

$$S_{xx}^{(d)}(f) = \lim_{\alpha \to 0^+}{}^{(D)} \left\{ \sum_i \sum_j G_i(\bar{s}) G_j(s) [p_i Q_{ij}(s) + p_j Q_{ji}(\bar{s}) \right\} \quad (53)$$

which by (42), (46), (49), and (51) becomes

$$\begin{aligned} S_{xx}^{(d)}(f) &= \tfrac{1}{2} \sum_i \sum_j \bar{G}_i G_j \Big[p_i F_{ij} \sum_n p_{j,n} \delta(f - f_n) \\ &\qquad + p_j \bar{F}_{ji} \sum_n p_{i,n} \delta(f + f_n) \Big] \\ &= \tfrac{1}{2} \sum_i \sum_j \bar{G}_i G_j \Big[p_i F_{ij} \sum_n p_{j,n} \delta(f - f_n) \\ &\qquad + p_j \bar{F}_{ji} \sum_n p_{i,-n} \delta(f + f_{-n}) \Big] \\ &= \sum_i \sum_j \bar{G}_i G_j F_{ij} \sum_n p_i p_{j,n} \delta(f - f_n) \\ &= \sum_n \Big[\sum_i \sum_j p_i p_{j,n} G_i(-2\pi i f) \\ &\qquad \cdot G_j(2\pi i f) F_{ij}(2\pi i f) \Big] \delta(f - f_n). \end{aligned} \quad (54)$$

Since the interval $(-f_A, f_A)$ is arbitrary, the sum over n in (54) can be extended as a distribution limit to include all the singular points along the frequency axis; hence, this expression represents the general formula for the discrete spectral density. In the sections immediately following, formula (54) is applied to the two fundamental classes of first-order Markov pulse trains: entirely random and stochastically uniform pulse trains.

4.3 *Discrete Spectra of Entirely Random Pulse Trains*

We define the processes under discussion to be entirely random if for at least one state i

$$\begin{gathered} q_{ii}(\tau) = \hat{q}_{ii}(\tau) + \sum_k \alpha_k^{(ii)} \mu(\tau - \tau_k^{(ii)}) \\ f_{ii}(\tau) = q_{ii}'(\tau) = \hat{q}_{ii}'(\tau) + \sum_k \alpha_k^{(ii)} \delta(\tau - \tau_k^{(ii)}) \\ 0 \leqq \alpha_k^{(ii)} \leqq 1 \\ \hat{q}_{ii}(\infty) + \sum_k \alpha_k^{(ii)} = 1 \end{gathered} \quad (55)$$

where $\hat{q}_{ii}$ is either continuous and strictly increasing in some interval (τ_A, τ_B), i.e.

$$\hat{q}_{ii}'(\tau) > 0 \qquad \tau \in (\tau_A, \tau_B) \tag{56}$$

or $\hat{q}_{ii}$ vanishes identically and the set of parameters $\tau_k^{(ii)}$ consists of two or more incommensurate elements. Processes of this class are characterized more completely by the following theorem:

Theorem I: A pulse train is entirely random if and only if for any state i

$$\begin{aligned} F_{ii}(2\pi if) &\neq 1 \qquad (f \neq 0) \\ F_{ii}(0) &= 1. \end{aligned} \tag{57}$$

For such processes all first recurrence distributions q_{ii} have the same form.

Proof: The second condition of (57) is merely a restatement of the general result given by (31). To establish the sufficiency of the first condition, we consider the only possible form for q_{ii} not representable by (55), viz.

$$\begin{aligned} q_{ii}(\tau) &= \sum_{k-1}^{\infty} \alpha_k^{(ii)} \mu(\tau - kT_i) \\ f_{ii}(\tau) &= \sum_k \alpha_k^{(ii)} \delta(\tau - kT_i). \end{aligned} \tag{58}$$

This yields

$$F_{ii}(2\pi if) = \sum_k \alpha_k^{(ii)} e(-2\pi ifkT_i) \tag{59}$$

whence

$$F_{ii}\left(2\pi i \frac{n}{T_i}\right) = 1 \qquad (n = 0, \pm 1, \cdots). \tag{60}$$

Therefore, any q_{ii} satisfying (57) must be representable by (55), and the process entirely random. To establish necessity, we consider (55) to be satisfied for at least one state i. Under condition (56)

$$\left| \int_{\tau_A}^{\tau_B} e^{-2\pi if\tau}\, d\hat{q}_{ii}(\tau) \right| = \left| \int_{\tau_A}^{\tau_B} e^{-2\pi if\tau} \hat{q}_{ii}'\, d\tau \right|$$

$$< \int_{\tau_A}^{\tau_B} \hat{q}_{ii}'\, d\tau = \int_{\tau_A}^{\tau_B} d\hat{q}_{ii}(\tau) \qquad (f \neq 0)$$

whence

$$| F_{ii}(2\pi if) | < \int_0^{\infty} d\hat{q}_{ii}(\tau) + \sum_k \alpha_k^{(ii)} = \int_0^{\infty} dq_{ii}(\tau) = 1 \qquad (f \neq 0).$$

On the other hand, with $\hat{q}_{ii} \equiv 0$ and $\tau_k^{(ii)}$ incommensurate

$$| F_{ii}(2\pi if) | = | \sum_k \alpha_k^{(ii)} \exp(-2\pi if\tau_k^{(ii)}) | < 1 \qquad (f \neq 0).$$

Thus, (57) is necessary for state i. Finally, since $F_{ii}(2\pi if_{i,n}) = 1$ and $f_{i,n} = f_n$ for all i [cf., (31), (34), and (49)], the realization of (57) for any q_{ii} necessarily implies the same realization and consequently the same form for all q_{ii}.

Theorem I, although essential to the treatment of discrete spectra, is not the only test for identifying entirely random processes; a somewhat more direct test is afforded by the cumulative distributions c_{ij}. In particular, functions q_{ij} have form (55) provided at least one of the functions c_{ij} does also. This fact follows from a basic property of irreducible processes, viz., the property that each density $f_{ij} \equiv q_{ij}'(\tau)$ equals a specific combination of positive sums and convolutions of all the densities $c_{ij}'(\tau)$.[1,3]

As regards singular points s_n and discrete spectra, it is clear from Theorem I and (34) that the point $s = s_0 = 0$ constitutes the only singularity of entirely random processes; therefore, the formulation given by (54) becomes

$$\begin{aligned} S_{xx}^{(d)}(f) &= \Big[\sum_i \sum_j p_i p_{j,0} G_i(0) G_j(0) F_{ij}(0)\Big]\delta(f) \qquad (61)\\ &= \Big[\sum_i \sum_j p_i p_j G_i(0) G_j(0)\Big]\delta(f). \end{aligned}$$

This expression leads immediately to the following result:
Theorem II: The discrete spectral density of entirely random pulse trains is given by

$$S_{xx}^{(d)}(f) = \left\{\int_{-\infty}^{\infty} \Big[\sum_i p_i g_i(t)\Big] dt\right\}^2 \delta(f) \qquad (62)$$

which vanishes if and only if

$$\int_{-\infty}^{\infty} \Big[\sum_i p_i g_i(t)\Big] dt = 0. \qquad (63)$$

Comparing (62) with (54), we note that Theorem II applies to the $\delta(f)$, or dc, component of all the processes treated in this paper.

4.4 *Discrete Spectra of Stochastically Uniform Pulse Trains*

Processes not classified as entirely random are defined here to be stochastically uniform. It is evident that the only first recurrence distributions representing the uniform process, i.e., satisfying neither definition (55) nor the criteria of Theorem I, must be of the form

$$q_{ii}(\tau) = \sum_{k=1}^{\infty} \alpha_k^{(ii)} \mu(\tau - kT_i)$$
$$0 \leqq \alpha_k^{(ii)} \leqq 1 \tag{64}$$
$$\sum_k \alpha_k^{(ii)} = 1$$

where parameters T_i are assumed to have the largest values possible. Under this specification

$$F_{ii}(2\pi if) = \sum_{k=1}^{\infty} \alpha_k^{(ii)} \exp(-2\pi ifkT_i) \tag{65}$$

Hence, on letting i_0 denote the state for which

$$T_i \leqq T_{i_0} \qquad (i = 1, \cdots, M) \tag{66}$$

we find that all the singular values f_n satisfying

$$F_{i_0 i_0}(2\pi if_n) = 1 \tag{67}$$

are given by

$$f_n = \frac{n}{T_{i_0}} \qquad (n = 0, \pm 1, \cdots). \tag{68}$$

Furthermore, since

$$F_{ii}(2\pi if_n) = 1 \tag{69}$$

for all states [cf. (34) and (49)], then

$$T_{i0} = T_i \equiv T \qquad (i = 1, \cdots, M) \tag{70}$$

which in turn implies that all F_{ii} are periodic over an interval of length T^{-1}, and all functions q_{ii} have the basic form

$$q_{ii}(\tau) = \sum_{k=1}^{\infty} \alpha_k^{(ii)} \mu(\tau - kT). \tag{71}$$

Considering also relations (65), (68), and (35) it is seen that

$$p_{i,n} = \left[\sum_k \tau\alpha_k^{(ii)}\right]^{-1} = p_{i,0} = p_i . \tag{72}$$

Finally, results (68), (70), and (72) combine with (54) to give the following theorem:

Theorem III: The discrete spectral density of stochastically uniform pulse trains is given by

$$S_{xx}^{(d)}(f) = \left[\sum_i \sum_j p_i p_j G_i(-2\pi i f) G_j(2\pi i f) F_{ij}(2\pi i f)\right] \sum_{n=-\infty}^{\infty} \delta(f - n/T) \quad (73)$$

$$T = n/f_n$$

$$F_{ii}(2\pi i f_n) = 1$$

which vanishes if and only if

$$\left[\sum_i \sum_j p_i p_j \bar{G}_i G_j F_{ij}\right]_{n/T} = 0 \qquad (n = 0, \pm 1, \cdots) \quad (74)$$

or if

$$\left[\sum_i \sum_j p_i p_j \bar{G}_i G_j F_{ij}\right]_f = 0 \qquad (-\infty < f < \infty). \quad (75)$$

At this point we consider a special but very important subclass of uniform pulse trains, namely, that of uniformly positioned pulses.

4.5. *Discrete Spectra of Uniformly Positioned Pulse Trains*

Pulse trains are defined to be uniformly positioned over a reference interval of length T_0 if the time intervals between successive pulses can assume only the discrete values $kT_0 (k = 1, 2, \cdots)$, i.e., if function q_{ij} take the form

$$q_{ij}(\tau) = \sum_{k=1}^{\infty} \alpha_k^{(ij)} \mu(\tau - kT_0) \qquad (i, j = 1, \cdots, M)$$

$$0 \leqq \alpha_k^{(ij)} \leqq 1 \quad (76)$$

$$\sum_k \alpha_k^{(ij)} = 1$$

where T_0 constitutes the maximum value for which this representation is valid. With q_{ij} so specified there results

$$F_{ij}(2\pi i f) = \sum_k \alpha_k^{(ij)} \exp(-2\pi i f k T_0) \quad (77)$$

Consequently, for a particular state i the condition

$$\alpha_{Kk'}^{(ii)} \geqq 0 \qquad (k' = 1, 2, \cdots)$$

$$\alpha_k^{(ii)} = 0 \qquad (k \neq Kk') \tag{78}$$

holds for some maximum $K \geqq 1$, the corresponding function F_{ii} is periodic over an interval of length $(KT_0)^{-1}$, and the singular values f_n satisfying (69) are given by

$$f_n = \frac{n}{KT_0} = \frac{n}{T}. \tag{79}$$

In addition, as values f_n are independent of i, condition (78) must for all states hold for the same value of K, the specific value in any particular case being determined either from one set of coefficients $\alpha_k^{(ii)}$, from (79), or from the recurrence pattern associated with one node of the flow graph. For all $K \geqq 1$, relations (77) and (79) yield the general conditions

$$\left.\begin{aligned} F_{ij}\left(2\pi i \frac{n}{T_0}\right) &= 1 \\ F_{ii}\left(2\pi i \frac{n}{KT_0}\right) &= F_{ii}(2\pi i f_n) = 1 \\ F_{ij}\left(2\pi i \frac{n+K}{KT_0}\right) &= F_{ij}\left(2\pi i \frac{n}{KT_0}\right) \end{aligned}\right\} \begin{aligned} &(K \geqq 1; \quad i,j = 1, \cdots, M; \\ &\qquad n = 0, \pm 1, \cdots). \end{aligned} \tag{80}$$

Combining these conditions with (79) and Theorem III, we obtain

$$\begin{aligned} S_{xx}^{(d)}(f) &= \left[\sum_i \sum_j p_i p_j \bar{G}_i G_j F_{ij}\right]_f \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{KT_0}\right) \\ &= \left[\sum_i \sum_j p_i p_j \bar{G}_i G_j F_{ij}\right]_f \sum_{k=0}^{K-1} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0} - \frac{k}{KT_0}\right) \\ &= \left[\sum_i \sum_j p_i p_j \bar{G}_i G_j\right]_f \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right) \\ &\quad + \sum_{k=1}^{K-1} \left\{\left[\sum_i \sum_j p_i p_j G_i(-2\pi i f) G_j(2\pi i f)\right.\right. \\ &\qquad \left.\left. \cdot F_{ij}\left(2\pi i \frac{k}{KT_0}\right)\right] \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0} - \frac{k}{KT_0}\right)\right\}. \end{aligned} \tag{81}$$

The following theorem is based on this last expression:

Theorem IV: The discrete spectral density of pulse trains uniformly positioned over a reference interval of length T_0 *is given by*

$$S_{xx}^{(d)}(f) = \left| \sum_i p_i G_i(2\pi i f) \right|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$$
$$+ \sum_{k=1}^{K-1} \left\{ \left[\sum_i \sum_j p_i p_j G_i(-2\pi i f) G_j(2\pi i f) F_{ij}\left(2\pi i \frac{k}{KT_0}\right)\right] \cdot \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0} - \frac{k}{KT_0}\right)\right\}$$

$$\left.\begin{aligned} K &= \frac{n}{T_0 f_n} \\ F_{ii}(2\pi i f_n) &= 1 \end{aligned}\right\} (K \geqq 1;\ i,j = 1, \cdots, M;\ n = 0, \pm 1, \cdots) \quad (82)$$

which vanishes if

$$\sum_i p_i g_i(t) = 0 \quad (83)$$

$$\sum_i \sum_j p_i p_j F_{ij}\left(2\pi i \frac{k}{KT_0}\right) \int_{-\infty}^{\infty} g_i(\tau) g_j(\tau + t)\, d\tau = 0 \quad (84)$$
$$(k = 1, \cdots, K - 1).$$

A special case of Theorem IV is noted as follows:

Theorem V: The discrete spectral density of uniformly positioned pulse trains corresponding to $K = 1$ is given by

$$S_{xx}^{(d)}(f) = \left| \sum_i p_i G_i(2\pi i f) \right|^2 \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T_0}\right)$$
$$f_n = \frac{n}{T_0} \quad (85)$$

which vanishes if

$$\sum_i p_i g_i(t) = 0. \quad (86)$$

Titsworth and Welch[9] have proved Theorem V for special pulse trains in which pulses are nonoverlapping and transitions occur every T_0 seconds. This theorem is also implicit in the classic work of Bennett on synchronous pulse trains [cf. Ref. 10, Eq. (35), p. 1509].

4.6. *Aaron's Discrete Spectral Formulation for Special Classes of Pulse Trains*

The analysis in Sections 4.3 and 4.5 yields the following theorem, a result first obtained by M. R. Aaron:[3]

Theorem VI: The discrete spectral density of entirely random pulse trains *and uniformly positioned pulse trains for which* $K = 1$ [*cf.* (78) *et seq.*] *is given by*

$$S_{xx}^{(d)}(f) = \sum_n \left\{ \operatorname*{Res}_{s_n} \left[\sum_i G_i(s) U_{ji}(s) \right] \right\}^2 \delta(f - f_n) \qquad (87)^*$$

where

$$U_{ji} = F_{ji}[1 - F_{ii}]^{-1} + \delta_{ji} \qquad (88)$$

and $\operatorname*{Res}_{s_n}[\cdot]$ *denotes the residue of the quantity in brackets at* $s = s_n = 2\pi i f_n$.

Proof: From relations (36), (72) and Theorem I we find that

$$\operatorname*{Res}_{s_n}\left[\frac{G_i(s)F_{ji}(s)}{1 - F_{ii}(s)}\right] = p_i G_i(2\pi i f_n) F_{ji}(2\pi i f_n) \qquad (89)$$

for either the entirely random or $K = 1$ case. On the other hand

$$F_{ji}(2\pi i f_n) = 1 \qquad (i, j = 1, \cdots, M) \qquad (90)$$

in both cases [cf., (79) and (80)]; thus,

$$\operatorname*{Res}_{s_n}\left[\sum_i G_i U_{ji}\right] = \sum_i p_i G_i(2\pi i f_n). \qquad (91)$$

Inserting this expression into either (61) or (85) gives formula (87).

V. SUMMARY

Theorems I through VI, which constitute the principal results of the preceding sections, give explicitly the discrete spectra of first-order Markov pulse trains. As presented, these theorems provide fundamental existence criteria for not only the analysis but also the synthesis of such processes. It is important to emphasize again that the distribution theoretic techniques employed in extracting discrete components from the Huggins-Zadeh formulation are applicable also to more general spectral formulations.

VI. ACKNOWLEDGMENTS

The author wishes to express his appreciation to M. R. Aaron, B. K. Kinariwala, and M. Karnaugh for several fruitful discussions and comments during the course of this work.

* Huggins has shown that the sum $\sum_i G_i U_{ji}$ represents the Laplace transform of the average signal following the occurrence of state j [cf., Ref. 1, Eq. (23a), p. 82].

APPENDIX A

Entirely Random Square Waves

For illustrating the techniques that often apply to cases in which $g_i \not\in L_1$, we consider a random square wave process of the form

$$x(t) = a \sum_n (-1)^{n-1}[\mu(t - t_{n-1}) - \mu(t - t_n)] \quad (92)$$

$$x'(t) \equiv y(t) = 2a \sum_n (-1)^n \delta(t - t_n) \quad (93)$$

where y represents a two-state pulse train with pulses related by

$$\begin{aligned} g_1 &= -g_2 = 2a\delta(t) \not\in L_1 \\ a &= \text{constant} > 0 \end{aligned} \quad (94)$$

and an entirely random statistical structure (cf. Section 4.3) specified by c_{12}, c_{21}, and

$$c_{11} = c_{22} = 0. \quad (95)$$

(Note that states 1 and 2 can be identified with the $+a$ and $-a$ portions of the square wave x.) Thus, in accordance with definitions (4b) and (5)

$$\begin{aligned} q_{12} &= c_{12}, \qquad q_{21} = c_{21} \\ q_{11} &= \int_0^\infty c_{12}(\tau - \tau')dc_{21}(\tau') = q_{22} \end{aligned} \quad (96)$$

whence

$$\begin{aligned} F_{11} &= F_{22} = F_{12}F_{21} \\ p_1 &= \int_0^\infty \tau \, dq_{11}(\tau) = -\frac{1}{F_{11}'(0)} = p_2 \equiv p. \end{aligned} \quad (97)$$

We next construct a set of "smooth" approximations to x; i.e., we smooth out the corners and discontinuities of each of the pulse trains x into a sequence $\{x_m(t)\}$ of continuous waveforms such that

$$\begin{aligned} S_{xx}(f) &= \lim_{m\to\infty}{}^{(D)} S_{x_m x_m}(f) \qquad (m = 1, 2, \cdots) \\ x_m'(t) &\equiv y_m(t) = \sum_n (-1)^n g^{(m)}(t - t_n) \end{aligned} \quad (98)$$

where

$$
\begin{aligned}
g^{(m)} &\ \varepsilon\ L_1 \\
\lim_m{}^{(D)}\, g^{(m)} &= 2a\delta(t) \\
g^{(m)} &= g_1{}^{(m)} = -g_2{}^{(m)}.
\end{aligned}
\tag{99}
$$

Since pulse trains y_m and y have the same transition properties and therefore the same statistical specification c_{ij}, the former process is classified as entirely random; it then follows from the condition

$$
\sum_i p_i g_i{}^{(m)} = p(g_1{}^{(m)} + g_2{}^{(m)}) = 0
$$

and from Theorem II [cf. (62)] relating to entirely random pulse trains that $S_{y_m y_m}$ has no discrete components. Consequently, relations (9), (97), (98), and (99) yield

$$
\begin{aligned}
4\pi^2 f^2 S_{xx}(f) &= \lim_m{}^{(D)}\, [4\pi^2 f^2 S_{x_m x_m}(f)] = \lim_m{}^{(D)}\, S_{y_m y_m}(f) \\
&= \lim_m{}^{(D)} \left\{ 2p \mid G^{(m)}(2\pi i f) \mid^2 \cdot \mathrm{Re}\left[\frac{(1 - F_{12})(1 - F_{21})}{1 - F_{12}F_{21}}\right]_f \right\} \\
&= 8pa^2\, \mathrm{Re}\left[\frac{(1 - F_{12})(1 - F_{21})}{1 - F_{12}F_{21}}\right]_f .
\end{aligned}
\tag{100}
$$

The most general function S_{xx} satisfying this last expression is given by

$$
S_{xx}(f) = \frac{2pa^2}{\pi^2 f^2}\, \mathrm{Re}\left[\frac{(1 - F_{12})(1 - F_{21})}{1 - F_{12}F_{21}}\right]_f + K_1\delta(t) = K_2\delta'(f) \tag{101}
$$

where the first term on the right represents a continuous component, and constants K_1 and K_2 are to be determined. As spectral densities must be even functions, $K_2 = 0$. Regarding the discrete term, constant K_1 is the square of the dc, or average, component of x; hence, with

$$
\begin{aligned}
\text{ave}\,[x(t)] &= \frac{a\int_0^\infty \tau\, dc_{12}(\tau) - a\int_0^\infty \tau\, dc_{21}(\tau)}{\int_0^\infty \tau\, dq_{11}(\tau)} \\
&= ap\left\{\int_0^\infty \tau\, d[q_{12}(\tau) - q_{21}(\tau)]\right\} \\
&= ap\,[F_{21}'(0) - F_{12}'(0)]
\end{aligned}
\tag{102}
$$

(101) becomes

$$S_{xx}(f) = \frac{2pa^2}{\pi^2 f^2} \operatorname{Re}\left[\frac{(1 - F_{12})(1 - F_{21})}{1 - F_{12}F_{21}}\right]_f + a^2p^2[F_{21}'(0) - F_{12}'(0)]^2\delta(f). \tag{103}$$

It is important to note here that the discrete component in (103) arises from the pulse structure of x and not from the singularities of $[1 - F_{ii}]^{-1}$. A more extensive treatment of this particular pulse train has been given by Aaron.[11]

APPENDIX B

A Distribution Identity

Essential to the formulation of the spectral density is the relationship between functions F_{ij} and the limit of

$$\sum_{k=1}^{N} q_{ij}^{(k)}(\tau) \equiv y_N(\tau) \tag{104}$$

as $N \to \infty$ [cf. (11) and (18)]. It is convenient to consider initially the integral

$$\int_0^\tau y_N(\tau)\, d\tau \equiv z_N(\tau). \tag{105}$$

Inasmuch as functions $q_{ij}^{(k)}$ and, consequently, y_N are sectionally continuous, then

$$z_N'(\tau) = y_N(\tau) \tag{106}$$

almost everywhere in the classical sense or identically in the distribution sense. Also, with $q_{ij}^{(k)} \geqq 0$ [cf. (20)] function $y_N \geqq 0$, and

$$0 \leqq z_N(\tau) \leqq z_N(\tau + \Delta\tau) \qquad (\Delta\tau > 0)$$

$$0 \leqq z_N(\tau) \leqq z_{N+1}(\tau). \tag{108}$$

Considering the limit conditions on sequence $\{z_N\}$, we note first from definition (20) and the properties of Stieltjes convolution[12] that

$$\begin{aligned}\int_0^\infty e^{-s\tau}\, dz_N(\tau) &= \sum_{k=1}^{N} \int_0^\infty e^{-s\tau}\, d\left[\int_0^\tau q_{ij}^{(k)}(\tau)\, d\tau\right] \\ &= \sum_{k=1}^{N} \frac{1}{s} F_{ij}(s) F_{jj}^{k-1}(s) \\ &= \sum_{k=1}^{N} \frac{F_{ij}}{s}\left[\frac{1 - F_{jj}^{N}}{1 - F_{jj}}\right] \qquad (\operatorname{Re} s = \alpha > 0).\end{aligned} \tag{109}$$

Therefore, the inverse Stieltjes transform[12] yields

$$z_N(\tau) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s^2}\left[\frac{F_{ij}}{1-F_{jj}}\right]e^{s\tau}\,ds - \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s^2}\left[\frac{F_{ij}F_{jj}^{\ N}}{1-F_{jj}}\right]e^{s\tau}\,ds. \tag{110}$$

Finally, since (6), (8) and (9) imply

$$| F_{ij}(s) | \leqq \int_0^\infty e^{-\alpha\tau}\,dq_{ij}(\tau) = \alpha\int_0^\infty e^{-\alpha\tau}q_{ij}(\tau)\,d\tau < \alpha\int_0^\infty e^{-\alpha\tau}\,d\tau = 1 \qquad (\alpha > 0;\ \ i,j = 1,\cdots,M) \tag{111}$$

then

$$\left|\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s^2}\left[\frac{F_{ij}}{1-F_{jj}}\right]e^{s\tau}\,ds\right| \leqq \sup_f\left|\frac{F_{ij}(s)}{1-F_{jj}(s)}\right| \cdot\int_{-\infty}^{\infty}\frac{df}{\alpha^2+4\pi^2f^2} < \infty \qquad (\alpha > 0) \tag{112}$$

$$\left|\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s^2}\left[\frac{F_{ij}F_{jj}^{\ N}}{1-F_{jj}}\right]e^{s\tau}\,ds\right| \leqq \sup_f\left|\frac{F_{ij}(s)F_{jj}^{\ N}(s)}{1-F_{jj}(s)}\right|\int_{-\infty}^{\infty}\frac{df}{\alpha^2+4\pi^2f^2} \xrightarrow[N\to\infty]{} 0 \qquad (\alpha > 0) \tag{113}$$

and, hence, the limit

$$\lim_{N\to\infty} z_N(\tau) = \frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty}\frac{1}{s^2}\left[\frac{F_{ij}}{1-F_{jj}}\right]e^{s\tau}\,ds \equiv z(\tau) \qquad (\alpha > 0) \tag{114}$$

exists. Relative to the asymptotic properties of function z we obtain from (25), (114), and (107) the conditions

$$\int_0^\infty e^{-s\tau}\,dz(\tau) = \frac{1}{s}\frac{F_{ij}(s)}{1-F_{jj}(s)} \sim \frac{p_j}{s^2} \qquad (s\to 0,\ \alpha > 0) \tag{115}$$

$$z(\tau) \leqq z(\tau+\Delta\tau) \qquad (\Delta\tau > 0) \tag{116}$$

which by Karamata's Tauberian Theorem[12] give

$$z(\tau) \sim \frac{p_j}{2}\tau^2 \qquad (\tau\to\infty). \tag{117}$$

This asymptotic result together with (112) and (114) implies that

$$[1+\tau^2]^{-2}z(\tau)\ \varepsilon\ L_1(-\infty,\infty). \tag{118}$$

Thus, function z is a proper distribution, or generalized function (cf. footnote, Section II and Ref. 6, pp. 21–23). In addition, since

$$0 \leqq z_N(\tau) \leqq z_{N+1}(\tau) \leqq z(\tau) \tag{119}$$

then

$$\lim_{N\to\infty}{}^{(\mathrm{D})} z_N(\tau) = z(\tau). \tag{120}$$

The functional properties of z as given by (112) and (117) imply also that

$$\lim_{\alpha\to 0^+}{}^{(\mathrm{D})} e^{-\alpha\tau} z(\tau) = z(\tau) \qquad (\alpha > 0). \tag{121}$$

In combining (104), (105), (106), and (120), there results

$$\begin{aligned} \mathfrak{F}\cdot z''(\tau) &= \lim_N{}^{(\mathrm{D})}\cdot\mathfrak{F}\cdot z_N''(\tau) = \lim_N{}^{(\mathrm{D})}\cdot\mathfrak{F}\cdot y_N' \\ &= \lim_N{}^{(\mathrm{D})} \sum_{k=1}^{N} \int_0^\infty e^{-2\pi i f\tau}\, dq_{ij}{}^{(k)}(\tau). \end{aligned} \tag{122}$$

On the other hand, (114) and (121) give

$$\begin{aligned} \mathfrak{F}\cdot z''(\tau) &= \mathfrak{F}\cdot\frac{d^2}{d\tau^2}\cdot\lim_\alpha{}^{(\mathrm{D})}\,[e^{-\alpha\tau}z(\tau)] \\ &= \mathfrak{F}\cdot\lim_\alpha{}^{(\mathrm{D})}\left\{\left(\frac{d^2}{d\tau^2} + 2\alpha\frac{d}{d\tau} + \alpha^2\right)[e^{-\alpha\tau}z(\tau)]\right\} \\ &= \lim_\alpha{}^{(\mathrm{D})}\,\{[(2\pi i f^2 + 2\alpha(2\pi i f) + \alpha^2]\mathfrak{F}\cdot[e^{-\alpha\tau}z(\tau)]\} \\ &= \lim_\alpha{}^{(\mathrm{D})}\,\{s^2\mathfrak{F}\cdot[e^{-\alpha\tau}z(\tau)]\} \\ &= \lim_\alpha{}^{(\mathrm{D})}\,\frac{F_{ij}(s)}{1 - F_{jj}(s)}\,. \end{aligned} \tag{123}$$

We finally obtain from (122) and (123) the following identity

$$\begin{aligned} \lim_{N\to\infty}{}^{(\mathrm{D})}\sum_{k=1}^{N}\int_0^\infty e^{-2\pi i f\tau}\, dq_{ij}{}^{(k)}(\tau) &= \lim_N{}^{(\mathrm{D})}\, F_{ij}(2\pi i f)\left[\frac{1 - F_{jj}{}^N(2\pi i f)}{1 - F_{jj}(2\pi i f}\right] \\ &= \lim_{\alpha\to 0+}{}^{(\mathrm{D})}\,\frac{F_{ij}(s)}{1 - F_{jj}(s)}\,. \end{aligned} \tag{124}$$

APPENDIX C

Definitions of symbols

$x(t)$	— cf. equation (1)	$S_{xx}^{(d)}(f)$	— (53)
$x_i(t)$	— (10)	p_i	— (9)
$d_n(t)$	— (1)	$p_{i,n}$	— (35)
t_n	— (1)	$\mathfrak{F}$	— (11)
$t_m^{(i)}$	— (10)	$\mathfrak{L}$	— (9)
$g_i(t)$	— (3)	$\mu(x)$	— (22)
$G_i(s)$	— (9)	$\delta(x) = \mu'(x)$	— (23)
$s,\bar{s}$	— (9)	δ_{ij}	— (9)
$s_{j,n} = s_n$	— (34), (49)	$Q_{ij}(s)$	— (38)
α	— (9)	$R_{ij}(s)$	— (39)
f	— (9)	$S_{ij}(s)$	— (40)
$f_{j,n} = f_n$	— (34), (49)	$T_n^{(ij)}(s)$	— (41)
$c_{ij}(\tau)$	— (4b)	T	— (73)
$q_{ij}(\tau)$	— (5)	T_0	— (76)
$q_{ij}^{(k)}(\tau)$	— (20)	K	— (78), (82)
$F_{ij}(s)$	— (9)	$U_{ij}(s)$	— (88).
$S_{xx}(f)$	— (9), (11)		

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