Imperfections in Active Transmission Lines

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The effect of discrete imperfections on the behavior of active transmission lines (i.e., lines with distributed gain) is considered. Two cases are studied:

1. Lines with identical, equally spaced reflectors. The transmission and reflection gains versus frequency are studied as functions of the magnitude of the reflectors. Limits on the magnitude of the reflectors to guarantee stability are investigated.

2. Lines with random reflectors, having random position and/or magnitude. The statistics of the transmission are studied; in particular, the average value and the variance and covariance of the transmission are determined for small reflections. If the reflections become large enough, instability may occur, and these calculations may become invalid. Stability of active distributed systems is studied in a companion paper.[1]

I. INTRODUCTION

In the present paper we consider the theory of active transmission lines (i.e., lines with gain) with discrete imperfections. Both equally spaced, identical imperfections and random imperfections will be considered. This study was suggested by R. Kompfner as a rough mathematical model for the effects of imperfections in certain types of optical maser amplifiers, in which the optical signal is reflected back and forth through the active medium on essentially nonoverlapping paths by an array of mirrors. A. G. Fox has suggested that this mathematical model will also provide a description of a one-dimensional active medium (e.g., maser) with (one-dimensional) random inhomogeneities.

Consider an active transmission line that provides exponential gain to both forward and backward waves, and further provides distortionless amplification. The voltage (and current) then vary as

$$\begin{aligned} &e^{-\Gamma z} \text{ — forward wave,} \\ &e^{+\Gamma z} \text{ — backward wave,} \end{aligned} \qquad (1)$$

$$\Gamma = -\alpha + j\beta. \tag{2}$$

Since the line has gain,

$$\alpha > 0. \tag{3}$$

Since we assume distortionless transmission, the propagation constant β is related to the angular frequency ω by

$$\beta = \omega/v \tag{4}$$

where the velocity of propagation v is a constant independent of the frequency ω. Further, the gain constant α is independent of ω. We may thus interpret β either as the propagation constant or as the normalized frequency.

Consider a line with N discrete reflectors, as illustrated in Fig. 1. The wave traveling to the right at a distance z is denoted by $W_0(z)$, the wave traveling to the left by $W_1(z)$, as indicated in this figure. We take $W_0(L_k+)$ and $W_1(L_k+)$ as the right- and left-traveling waves just to the right of the kth reflector c_k, $W_0(L_k-)$ and $W_1(L_k-)$ as the right- and left-traveling waves just to the left of the kth reflector.

Each reflector is characterized by a scattering matrix relating incident and reflected waves. Thus for the typical reflector illustrated in Fig. 2 we have

$$\begin{bmatrix} W_1(L_k-) \\ W_0(L_k+) \end{bmatrix} = S_k \begin{bmatrix} W_0(L_k-) \\ W_1(L_k+) \end{bmatrix} \tag{5}$$

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}. \tag{6}$$

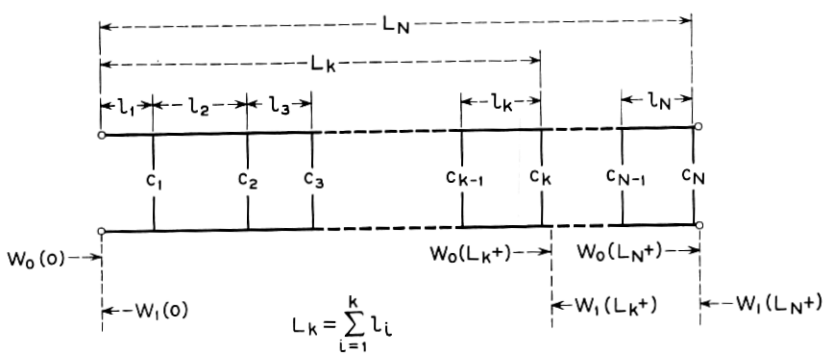

Fig. 1 — Line with N discrete reflectors.

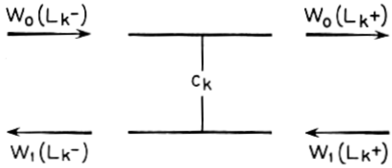

Fig. 2 — Typical reflector.

The incident and reflected wave amplitudes are assumed normalized so that the power in any wave is simply the square of its absolute magnitude. For example, if the reflected wave is absent at the left of the obstacle in Fig. 2 the power in the incident wave is $|W_0(L_k-)|^2$; similarly, if the incident wave is absent the power in the reflected wave is $|W_1(L_k-)|^2$. We make the following assumptions:

1. The powers in the forward and backward waves are additive; for example, the total power P flowing in the $+z$ direction at the left of Fig. 2 is given by

$$P = |W_0(L_k-)|^2 - |W_1(L_k-)|^2. \tag{7}$$

2. The reflectors are lossless, and consequently have unitary scattering matrices.[2] For a reflector of a given magnitude there is a single arbitrary phase parameter in the scattering matrix; this phase has been chosen in such a way as to yield a scattering matrix for the obstacle of the following form:

$$S = \begin{bmatrix} jc & \sqrt{1-c^2} \\ \sqrt{1-c^2} & jc \end{bmatrix}, \qquad 0 \leqq |c| \leqq 1. \tag{8}$$

c is a measure of the magnitude of the reflection; for $c = 0$ the reflection is zero and the guide is perfect. c is assumed to be independent of frequency, although this assumption is not compatible with physical realizability. We note that the matrix of (8) is correct only for ω (or β) > 0. For ω (or β) < 0 the signs of the diagonal terms of the matrix must be changed, so that the various responses will be real, even though unrealizable; alternately, we may change the sign of c for negative ω (or β).

Next consider the cascade connection of reflectors and ideal guide sections shown in Fig. 1. We require the wave matrix A corresponding to the scattering matrix of (8) for an obstacle. Referring to Fig. 2,

$$\begin{bmatrix} W_0(L_k-) \\ W_1(L_k-) \end{bmatrix} = A_k \begin{bmatrix} W_0(L_k+) \\ W_1(L_k+) \end{bmatrix}, \tag{9}$$

$$A_k = \frac{1}{\sqrt{1-c_k^2}} \begin{bmatrix} 1 & -jc_k \\ +jc_k & 1 \end{bmatrix}. \tag{10}$$

The wave matrix for the kth line section of length l_k between reflectors c_{k-1} and c_k is given by

$$\begin{bmatrix} W_0(L_{k-1}+) \\ W_1(L_{k-1}+) \end{bmatrix} = \begin{bmatrix} e^{\Gamma l_k} & 0 \\ 0 & e^{-\Gamma l_k} \end{bmatrix} \begin{bmatrix} W_0(L_k-) \\ W_1(L_k-) \end{bmatrix}. \tag{11}$$

Thus the matrix X_k for the cascade connection of the kth line section of length l_k and the kth reflector is given by

$$\begin{gathered} \begin{bmatrix} W_0(L_{k-1}+) \\ W_1(L_{k-1}+) \end{bmatrix} = X_k \begin{bmatrix} W_0(L_k+) \\ W_1(L_k+) \end{bmatrix}, \\ X_k = \frac{1}{\sqrt{1-c_k^2}} \begin{bmatrix} e^{+\Gamma l_k} & -jc_k e^{+\Gamma l_k} \\ +jc_k e^{-\Gamma l_k} & e^{-\Gamma l_k} \end{bmatrix}. \end{gathered} \tag{12}$$

The over-all wave matrix $\bar{X}$ for the line consisting of N sections in Fig. 1 is

$$\begin{bmatrix} W_0(0) \\ W_1(0) \end{bmatrix} = \bar{X} \begin{bmatrix} W_0(L_N+) \\ W_1(L_N+) \end{bmatrix}, \qquad \bar{X} = X_1 X_2 \cdots X_N = \prod_{k=1}^{N} X_k. \tag{13}$$

Setting

$$\bar{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \tag{14}$$

and referring to Fig. 1, the (complex) transmission and reflection losses $\mathbf{L}_T$ and $\mathbf{L}_R$ or corresponding (complex) gains $\mathbf{G}_T$ and $\mathbf{G}_R$ are given as follows:

$$\mathbf{L}_T = \frac{1}{\mathbf{G}_T} = \frac{W_0(0)}{W_0(L_N+)} = x_{11} \tag{15}$$

$$\mathbf{L}_R = \frac{1}{\mathbf{G}_R} = \frac{W_0(0)}{W_1(0)} = \frac{x_{11}}{x_{21}}. \tag{16}$$

$W_0(0)$, $W_1(0)$ and $W_0(L_N+)$, the incident, reflected, and transmitted waves for the entire structure, are illustrated in Fig. 1.

It has been necessary to state the above analysis in terms of wave matrices that give the input as a function of the output (instead of vice versa) because the boundary conditions are known at the output. The output is assumed to be matched, so that in Fig. 1

$$W_1(L_N+) = 0. \tag{17}$$

In contrast, the reflection coefficient at the input is not known in advance, and so it is not convenient to express the output $\begin{bmatrix} W_0(L_N+) \\ W_1(L_N+) \end{bmatrix}$ as a matrix product times the input $\begin{bmatrix} W_0(0) \\ W_1(0) \end{bmatrix}$.

We consider below two cases of interest:

(a) Identical, equally spaced reflectors,

(b) Independent reflectors with random magnitude and/or position.

II. IDENTICAL, EQUALLY SPACED REFLECTORS

We now assume that all reflectors have identical magnitude and equal spacing. Setting

$$c_k = c, \qquad l_k = l$$

in (12), from (13) and (14) the over-all wave matrix becomes

$$\bar{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \frac{1}{(1 - c^2)^{N/2}} \begin{bmatrix} e^{+\Gamma l} & -jc\, e^{+\Gamma l} \\ +jc\, e^{-\Gamma l} & e^{-\Gamma l} \end{bmatrix}^N. \tag{18}$$

By the usual methods we find:

$$x_{11} = \frac{1}{(1 - c^2)^{N/2}(K_+ - K_-)} (K_+\alpha_+^N - K_-\alpha_-^N), \tag{19}$$

$$x_{21} = \frac{1}{(1 - c^2)^{N/2}(K_+ - K_-)} (\alpha_+^N - \alpha_-^N), \tag{20}$$

$$\alpha_\pm = \cosh \Gamma l \pm \sqrt{\sinh^2 \Gamma l + c^2}, \tag{21}$$

$$K_\pm = \frac{jc\, e^{+\Gamma l}}{e^{+\Gamma l} - \alpha_\pm} = \frac{\alpha_\pm - e^{-\Gamma l}}{jc\, e^{-\Gamma l}}. \tag{22}$$

With the help of (15) and (16) the transmission and reflection gains or losses may be determined.

Consider the various x_{ij} of (18), and in particular x_{11} and x_{21} of (19) and (20), to be functions of $j\beta l$, where we recall from (4) that β is proportional to the angular frequency ω. We recall from the discussion following (8) that these results are valid only for positive frequencies,

$\beta > 0$. The x_{ij} have certain general properties of interest. First, we have:

$$x_{ij}[j(\beta l + \pi)] = (-1)^N x_{ij}[j\beta l], \qquad \beta \geqq 0, \tag{23}$$

$$x_{ij}[j(\pi - \beta l)] = (-1)^{N+i+j} x_{ij}^*[j\beta l], \qquad 0 \leqq \beta l \leqq \pi. \tag{24}$$

Further,

$$x_{ij}[-j\beta l] = x_{ij}^*[j\beta l]. \tag{25}$$

Equation (23) shows that x_{ij} is periodic in the normalized frequency β, of period $2\pi/l$. Equation (25) guarantees that the over-all response to a real input is real. Taken together, (23) and (24) show that the magnitudes of the losses $|\mathbf{L}_T|$ and $|\mathbf{L}_R|$ of (15) and (16) are periodic in β of period π/l, and are symmetric about the points $\beta l = 0, \pi/2, \pi, 3\pi/2, \cdots$. Consequently in studying the magnitudes of these losses at real frequencies we need consider only the range $0 \leqq \beta l \leqq \pi/2$.

Next, from (19)–(22) it might appear that the various functions x_{ij} have branch points in the complex frequency plane because of the radicals in these equations. This is not true, however; a little study of these equations shows that the radicals really disappear for all (integral) N. Alternately, by considering the matrix multiplication of (18) it becomes clear that all the x_{ij} are single-valued functions of Γ, and that no branch points can appear.

We may thus determine the exact expression for the transmission or reflection gain via either (19)–(22) or direct matrix multiplication in (18). However, we shall most often be interested in cases where the reflection parameter c is small in some suitable sense; application of perturbation theory to (19)–(22) greatly simplifies these relations and permits a useful interpretation of these results.

Consider the radical in (21). If

$$|c| \ll |\sinh \Gamma l| \tag{26}$$

then we may expand the radical in a power series and retain only the first correction term. Since

$$|\sinh \Gamma l|^2 = \sinh^2 \alpha l + \sin^2 \beta l \geqq \sinh^2 \alpha l, \tag{27}$$

(26) will be satisfied for all β if

$$|c| \ll \sinh \alpha l. \tag{28}$$

Therefore

$$\sqrt{\sinh^2 \Gamma l + c^2} \approx \sinh \Gamma l + \frac{c^2}{2 \sinh \Gamma l}. \tag{29}$$

Then (21) and (22) become:

$$\alpha_{\pm} \approx e^{\pm\Gamma l} \pm \frac{c^2}{2 \sinh \Gamma l}, \tag{30}$$

$$K_{+} \approx -j\, 2e^{\Gamma l} \frac{\sinh \Gamma l}{c}, \tag{31a}$$

$$K_{-} \approx j\, \tfrac{1}{2}e^{\Gamma l} \frac{c}{\sinh \Gamma l}. \tag{31b}$$

Substituting (30) and (31) into (19) and (20) and neglecting various small quantities, we obtain the following approximate results:

$$x_{11} = \frac{1}{(1 - c^2)^{N/2}} e^{N\Gamma l} [1 + F], \tag{32a}$$

$$F = \left(\frac{c}{2 \sinh \Gamma l}\right)^2 (e^{-2N\Gamma l} - 1), \tag{32b}$$

$$x_{21} = \frac{jc}{(1 - c^2)^{N/2}} e^{-\Gamma l} \frac{\sinh N\Gamma l}{\sinh \Gamma l}. \tag{33}$$

We make one further assumption, often used below, that the total gain in the absence of reflectors ($c = 0$) is large; i.e., referring to (2) and (3),

$$e^{N\alpha l} \gg 1. \tag{34}$$

Then (32b) becomes

$$F = \left(\frac{c}{2 \sinh \Gamma l}\right)^2 e^{-2N\Gamma l}, \qquad e^{N\alpha l} \gg 1. \tag{35}$$

So far we have ignored the question of stability; it is clear that such an active device can oscillate under some conditions. If the device does oscillate, our present results for loss (or gain) lack physical significance, for reasons discussed below. Instability can occur only if the gain functions of (15) and (16) have poles in the right-half complex frequency plane; if all poles of $\mathbf{G}_T$ and $\mathbf{G}_R$ are in the left-half plane the device will be stable. Since from (15–16) the poles of the **G**'s are the zeros of x_{11}, we investigate the zeros of x_{11} as given by the approximate expressions of (32a) and (35).

For $c = 0$, i.e., with reflections absent, the device will be stable, and consequently the zeros of x_{11} lie in the left-half plane. It seems obvious on physical grounds that the device remains stable for small enough values of $|c|$, and will oscillate only when $|c|$ exceeds some critical value. Assuming this to be true, we determine the conditions for stability by finding the minimum value of $|c|$ for which a zero of x_{11} appears on the real frequency axis, i.e., for some value of β.

From (32a) the zeros of x_{11} occur when

$$F = -1. \tag{36}$$

Equivalently,

$$|F| = 1; \tag{37a}$$

$$\angle F = \pm\pi, \pm 3\pi, \cdots . \tag{37b}$$

Noting that

$$\sinh^2 \Gamma l = \sinh^2 (-\alpha + j\beta)l = (\sinh^2 \alpha l + \sin^2 \beta l)\, e^{-j2\varphi}, \tag{38a}$$

$$\varphi = \tan^{-1} \frac{\tan \beta l}{\tanh \alpha l}, \tag{38b}$$

where the principal value of $\tan^{-1}$ is implied, we have from (35)–(37) the following approximate relation for a zero of x_{11} lying on the real frequency axis.

$$F = \frac{c^2}{4(\sinh^2 \alpha l + \sin^2 \beta l)}\, e^{2N\alpha l}\, e^{-j(2N\beta l - 2\varphi)} = -1. \tag{39}$$

Thus

$$N\beta l = \varphi + (\pi/2) + m\pi; \qquad m = 0, \pm 1, \pm 2, \cdots \tag{40a}$$

$$\frac{c^2}{4(\sinh^2 \alpha l + \sin^2 \beta l)}\, e^{2N\alpha l} = 1. \tag{40b}$$

φ is given by (38b). We now fix αl and find the smallest value of $|c|$ for which (40) has a solution. Equation (40a), together with (38b), can be readily seen to have $2(N - 1)$ roots $(\beta l)_j$ for $0 < \beta l < 2\pi$. For each of these roots there is a corresponding solution $c = \pm|c_j|$ for (40b). It is obvious that the smallest of these $|c_j|$ corresponds to the smallest $(\beta l)_j$, which is that root lying closest to $\beta l = 0$ and which we denote $(\beta l)_1$.

For convenience we summarize the approximate results derived above in the present section.

$$x_{11} = \frac{1}{(1 - c^2)^{N/2}} e^{N\Gamma l} [1 + F] \tag{41a}$$

$$F = \left(\frac{c}{2 \sinh \Gamma l}\right)^2 e^{-2N\Gamma l}$$

$$= \frac{c^2}{4(\sinh^2 \alpha l + \sin^2 \beta l)} e^{2N\alpha l} e^{-j(2N\beta l - 2\varphi)},$$

$$\varphi = \tan^{-1} \frac{\tan \beta l}{\tanh \alpha l}. \tag{41b}$$

Conditions:

$$| c | \ll \sqrt{\sinh^2 \alpha l + \sin^2 \beta l} \tag{41c}$$

$$e^{N\alpha l} \gg 1. \tag{41d}$$

The results of (41a) and (41b) will be valid for all β if the condition of (41c) is replaced by the more restrictive condition of (42):

$$| c | \ll \sinh \alpha l. \tag{42}$$

The maximum value of the reflection coefficient magnitude $| c |$ that yields a stable amplifier is given as follows, subject to the conditions of (41d) and (42)

$$N(\beta l)_1 = \tan^{-1} \frac{\tan (\beta l)_1}{\tanh \alpha l} + \frac{\pi}{2} \qquad \text{(principal value of } \tan^{-1}) \tag{43a}$$

$$| c |_{\max} = 2e^{-N\alpha l} \sqrt{\sinh^2 \alpha l + \sin^2 (\beta l)_1}. \tag{43b}$$

In deriving (43) we required that the results of (41a) and (41b) be valid for all β. Consequently the more restrictive condition of (42) must hold; however, it is not obvious in advance that (42) will end up being satisfied in all cases. However, it is easy to show that this is indeed so, so that the approximate limits on $| c |$ imposed by the requirement of stability are indeed given by (43), so long as (41d) is satisfied (i.e., the high-gain case). From (43a) we have

$$(\beta l)_1 < \pi/N. \tag{44}$$

From (41d) and (44)

$$(\beta l)_1 \ll \alpha l \tag{45}$$

and consequently

$$\sin^2 (\beta l)_1 \ll \sinh^2 \alpha l. \tag{46}$$

Equation (43b) thus guarantees that the more restrictive bound of (42) will always be satisfied in the high-gain case.

The general behavior of the gain-vs-frequency (or βl) curve is readily seen from (41a) and (41b). In the second line of (41a) the first factor and φ vary slowly with βl, while the factor $e^{-j2N\beta l}$ varies rapidly. The angle of F increases steadily as βl increases from 0 to 2π; the magnitude of F is largest at $\beta l = 0, \pi, 2\pi, \cdots$, and decreases rapidly away from these points. Therefore the gain $\mathbf{G}_T$ of (15) plotted vs βl (or frequency) will have an oscillatory behavior, with the magnitude of oscillation greatest near $\beta l = 0, \pi, 2\pi, \cdots$, and quite small elsewhere. The larger N, the more rapid will be the rate of oscillation.

It is instructive to consider a few numerical examples. We consider the following two cases:

$$20 \log_{10} e^{N\alpha l} \equiv 20 \log_{10} e^{\alpha L_N}$$
$$= 30 \text{ db, total gain in } (i) \text{ and } (ii) \text{ below}$$

(i)
$$20 \log_{10} e^{\alpha l} = 1 \text{ db, gain per section}$$
$$N = 30, \text{ number of sections}$$
$$(180/\pi)\cdot(\beta l)_1 = 4.05°, \text{ phase shift per section at oscillation}$$
$$|c|_{\max} = 0.00860, \text{ maximum value of reflection coefficient for stability}$$

(ii)
$$20 \log_{10} e^{\alpha l} = 0.1 \text{ db, gain per section}$$
$$N = 300, \text{ number of sections}$$
$$(180/\pi)\cdot(\beta l)_1 = 0.405°, \text{ phase shift per section at oscillation}$$
$$|c|_{\max} = 0.000860, \text{ maximum value of reflection coefficient for stability.}$$

The total gain in both cases is large, and hence $|c|_{\max}$ has been computed by (43). The transmission gain $\mathbf{G}_T$ plotted versus the normalized frequency βl for these two cases is shown in Figs. 3 and 4 respectively for several values of c. These results are computed by direct matrix multiplication [see (18)] rather than via (19)–(22) or via the approximate results of (41). Figs. 3(a) and 4(a) show the gain vs normalized frequency for three values of $|c|$ less than $|c|_{\max}$ as well as for $c = |c|_{\max}$ [computed via the approximate results of (43)], which corresponds to the limiting case of stability. It is readily seen how the device approaches instability as c approaches $|c|_{\max}$. Figs. 3(b) and 4(b) show computed curves of the "gain" versus frequency for a value of c greater than $|c|_{\max}$. Under these conditions the device is unstable, so that these curves have little direct physical significance; however, these curves do not look too different from the stable ones of Figs. 3(a) and 4(a). This should provide explicit warning against taking any such computed curve seriously without first investigating stability.

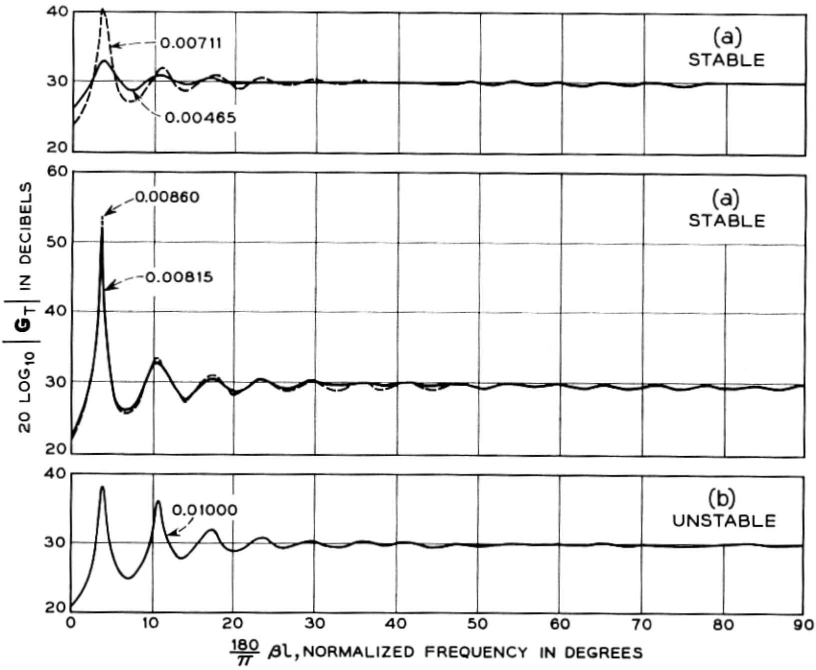

Fig. 3 — Transmission gain vs normalized frequency for one-dimensional active medium with identical, equally-spaced reflectors. $N = 30$, number of sections; $20 \log_{10} e^{\alpha l} = 1$ db, gain per section; total gain = 30 db; c = magnitude of reflectors, parameter indicated on curves.

A detailed picture of the behavior of these devices could be worked out in terms of the poles of the gain function in the complex plane. For small $|c|$ the poles lie in the left-half plane. As $|c|$ is increased the poles move toward the j-axis, causing greater oscillation in the gain-frequency curve. As $|c| \to |c|_{\max}$ the closest pole touches the j-axis, causing the gain to approach infinity at one frequency. Finally, as $|c|$ becomes greater than $|c|_{\max}$ this pole moves to the right-half plane and the "gain"-frequency curve becomes finite. As $|c|$ increases further the first peak decreases, but the next pole approaches the j-axis, so that the second peak increases, approaches infinity, and eventually decreases. The different peaks in the gain-frequency curve behave in a similar manner as the various poles cross the j-axis in succession.

Figs. 5 and 6 show similar curves for the reflection gain $\mathbf{G}_R$. $\mathbf{G}_R$ approaches infinity for the same values of $|c|$ and βl as does $\mathbf{G}_T$; this must be so, since for the limiting case of stability, power must emerge from both ends of the device in the absence of any incident wave. As in

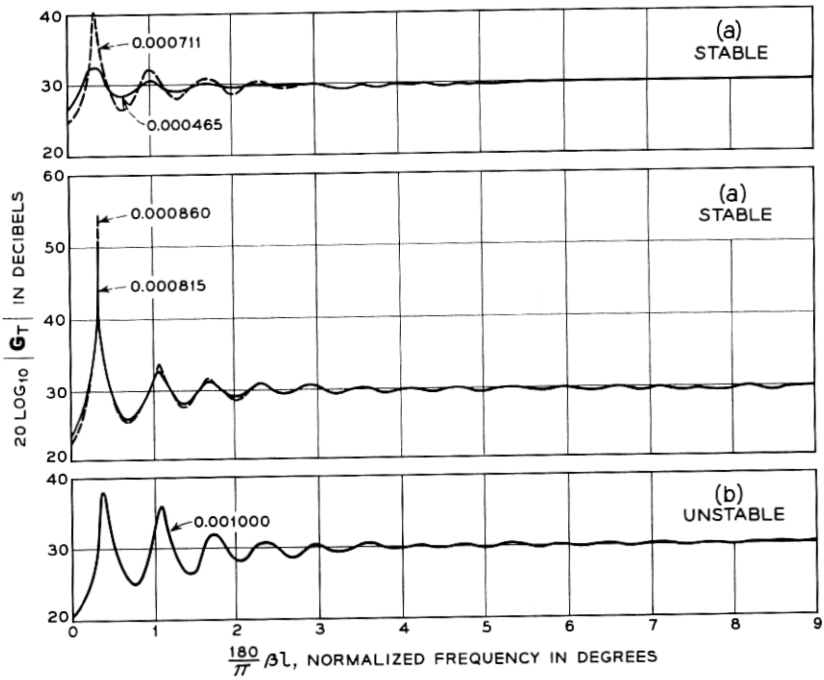

Fig. 4 — Transmission gain vs normalized frequency for one-dimensional active medium with identical, equally-spaced reflectors. $N = 300$, number of sections; $20 \log_{10} e^{\alpha l} = 0.1$ db, gain per section; total gain $= 30$ db; $c =$ magnitude of reflectors, parameter indicated on curves.

Figs. 3(b) and 4(b), the curves of Figs. 5(b) and 6(b) correspond to instability and hence lack direct physical significance.

If the total gain in the absence of reflectors is not large, then the above results of (43) are not valid, and the approximate results of (41) are not valid over the entire range of permissible values of c. It is interesting to examine the exact computer solutions for one such case.

$$
\begin{aligned}
(iii) \qquad 20 \log_{10} e^{\alpha l} &= 0.1 \text{ db, gain per section} \\
N &= 50, \text{ number of sections} \\
20 \log_{10} e^{N\alpha l} &\equiv 20 \log_{10} e^{\alpha L_N} \\
&= 5 \text{ db, total gain} \\
(180/\pi)\cdot(\beta l)_1 &= 5^\circ, \text{ phase shift per section at oscillation} \\
|\, c \,|_{\max} &= 0.065, \text{ maximum value of reflection coefficient for stability.}
\end{aligned}
$$

Gain-frequency curves for several values of c are shown in Figs. 7 and 8. The values of $(\beta l)_1$ and $|\, c \,|_{\max}$ given above have been determined

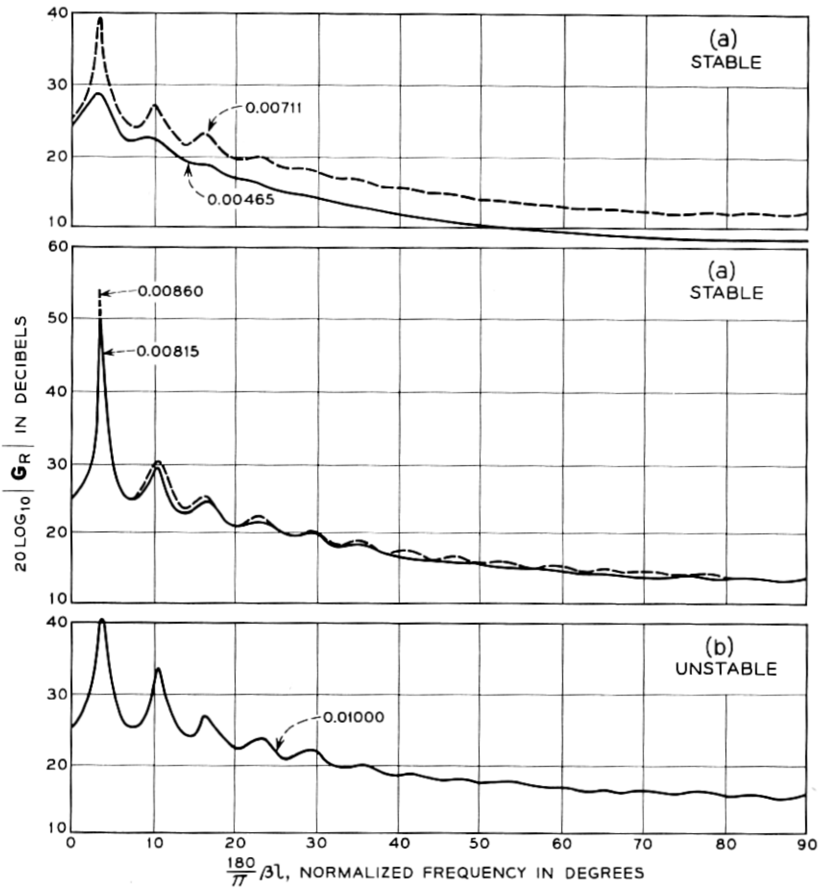

Fig. 5 — Reflection gain vs normalized frequency for one-dimensional active medium with identical, equally-spaced reflectors. $N = 30$, number of sections; $20 \log_{10} e^{\alpha l} = 1$ db, gain per section; total gain = 30 db; c = magnitude of reflectors, parameter indicated on curves.

from these curves. As above, Figs. 7(a) and 8(a) show the transmission and reflection gains for the stable case, $|c| \leqq |c|_{\max}$, while Figs. 7(b) and 8(b) show the "gains" for an unstable case. The general comments given above for examples (*i*) and (*ii*) apply also to this case. The approximation of (43), which was valid in examples (*i*) and (*ii*) above, would have predicted $(\beta l)_1 = 3.37°$, $|c|_{\max} = 0.0135$ for the oscillation conditions; this approximation is quite inaccurate in the present low-gain case, particularly for $|c|_{\max}$.

Straightforward calculation based on (18) or (19)–(22) in the peri-

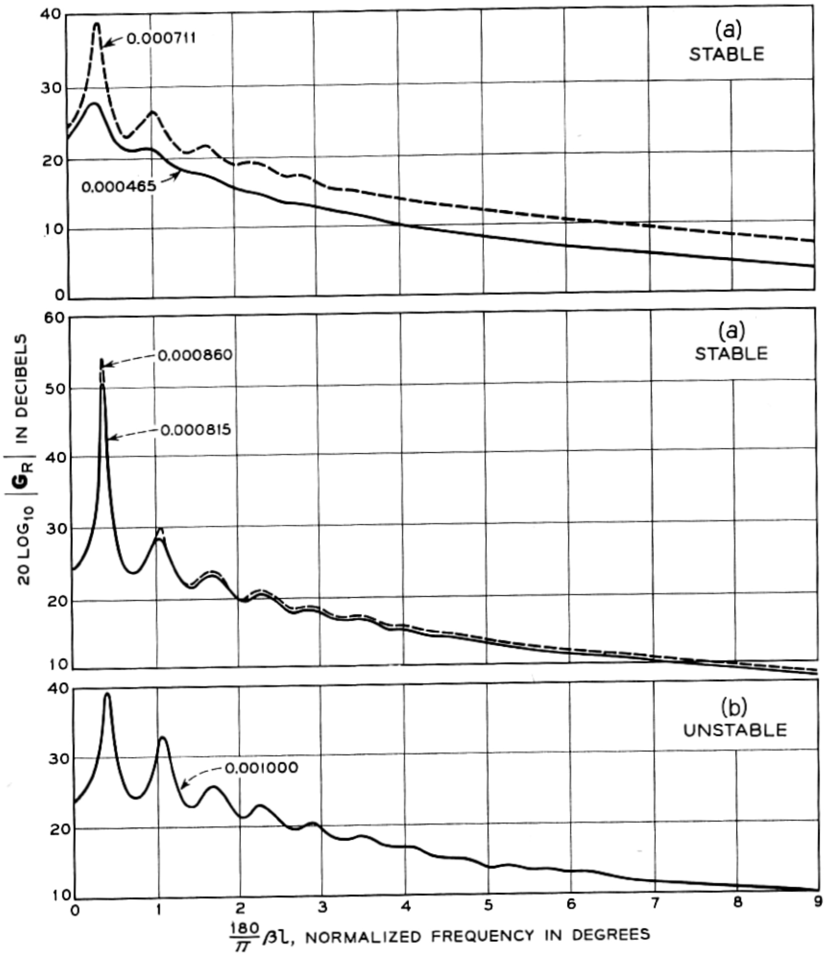

Fig. 6 — Reflection gain vs normalized frequency for one-dimensional active medium with identical, equally-spaced reflectors. $N = 300$, number of sections; $20 \log_{10} e^{\alpha l} = 0.1$ db, gain per section; total gain = 30 db; c = magnitude of reflectors, parameter indicated on curve.

odic case, or (12) and (13) in the general case, will of course always lead to some definite result for x_{11} as a function of frequency, whether or not the device is stable. However, only if we are assured that the device is stable will x_{11} have the desired physical significance of the steady-state loss function $\mathbf{L}_T$. If the device is unstable it will of course oscillate, and ultimately the linear behavior assumed here must break down. However,

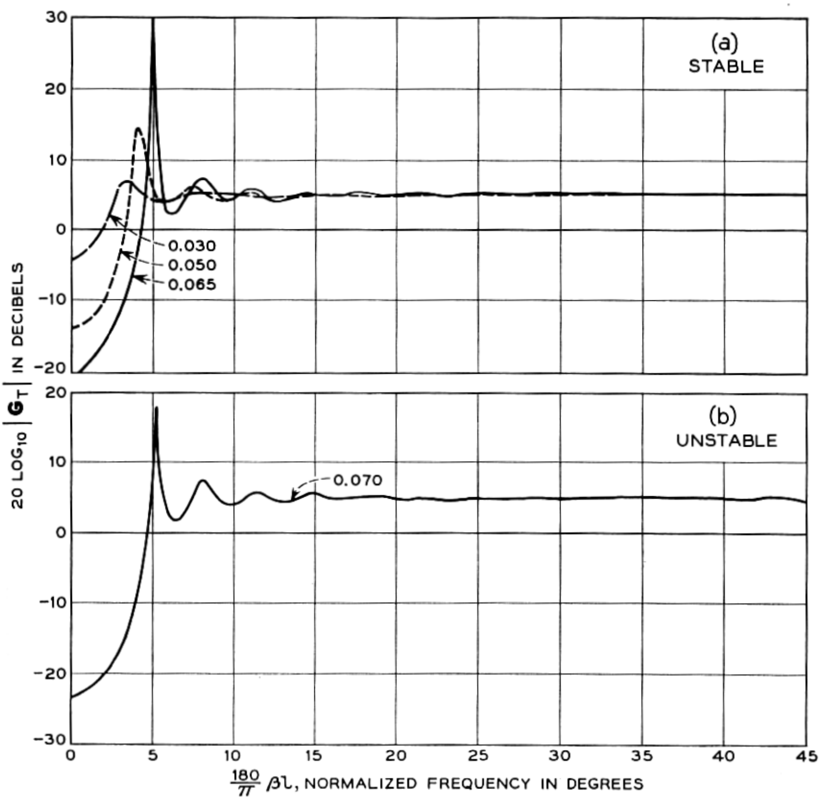

Fig. 7 — Transmission gain vs normalized frequency for one-dimensional active medium with identical, equally-spaced reflectors. $N = 50$, number of sections; $20 \log_{10} e^{\alpha l} = 0.1$ db, gain per section; total gain = 5 db; c = magnitude of reflectors, parameter indicated on curves.

by demanding that the device be at rest at $t = 0$ and examining the initial build-up of oscillation, the mathematical significance of x_{11} may be examined in the unstable case. Suppose the device is initially at rest, and a sinusoidal input is applied at $t = 0$. The total response may be divided into a steady-state response, whose envelope is constant with time, and a transient response, whose envelope ultimately grows or decays exponentially with time in the unstable and stable cases respectively. The steady-state response is given by x_{11} in both cases. In the stable case, since the transients ultimately decay with time, only the steady-state response remains. In the unstable case the steady-state response retains the same mathematical meaning, but since the tran-

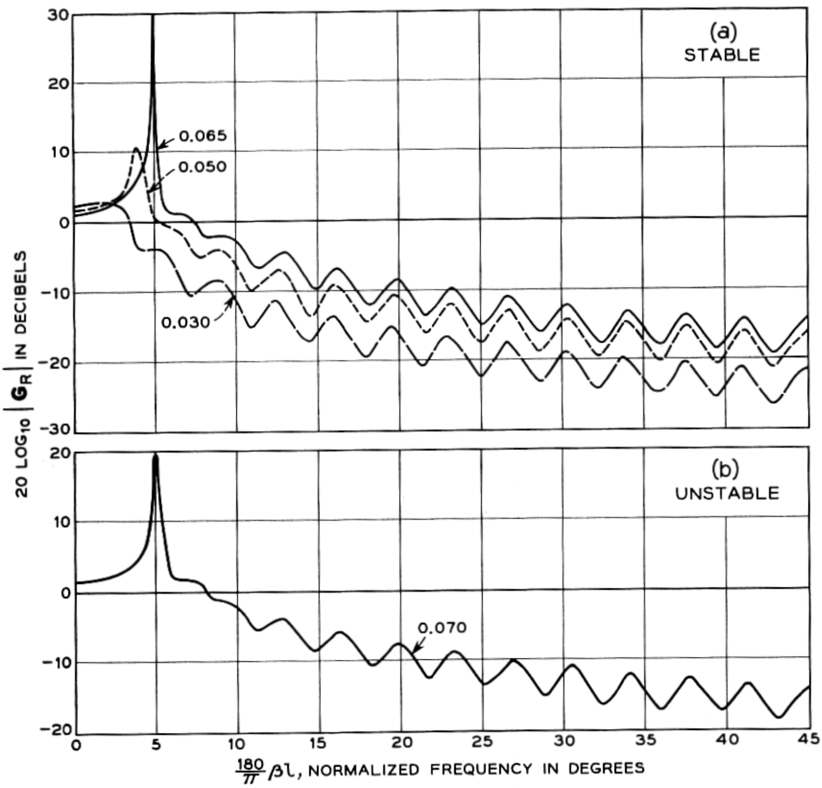

Fig. 8 — Reflection gain vs normalized frequency for one-dimensional active medium with identical, equally-spaced reflectors. $N = 50$, number of sections; $20 \log_{10} e^{\alpha l} = 0.1$ db, gain per section; total gain = 5 db; c = magnitude of reflectors, parameter indicated on curves.

sient response grows exponentially with time, the steady-state response loses much of its physical significance.

III. RANDOM REFLECTORS

In the present section we consider active devices with reflectors having random position and/or magnitude; different reflectors are assumed statistically independent. Since the imperfections are random, the loss (or gain) is also a random variable, and we seek various statistics of the loss-frequency curve. The loss $\mathbf{L}_T$ is determined from (12)–(15); we study the average loss and the second-order statistics of the fluctuations about the average, i.e., the variance and covariance of the loss fluctuations. The form of (12)–(15) requires us to study the loss statistics rather than the gain statistics, which are of more direct interest. However, if the loss fluctuations about the average are small, then the loss and gain fluctuations will be almost identical (except for a change in sign), and their statistics will thus also be approximately identical.

As discussed above, (12)–(15) yield the transmission loss $\mathbf{L}_T$ only if the device is stable. If the device is unstable so that oscillation occurs, then the steady-state response $\mathbf{L}_T$ given by (12)–(15) loses much of its physical significance, as discussed in the previous section. The statistics of $\mathbf{L}_T$ computed below are effectively averaged over all cases, so that these results will not be meaningful unless the probability of oscillation is so small that for practical purposes it may be ignored. Thus the results below are valid in the limit of very small reflections, in analogy to the perturbation case of the previous section. In a companion paper[1] useful sufficient conditions guaranteeing stability are obtained; these stability conditions extend the range of validity of the present calculations to finite reflections.

Three different statistical models of an active device with random reflectors are considered in the present paper:

(*i*) random magnitude and spacing
(*ii*) equal magnitude, random spacing
(*iii*) random magnitude, equal spacing.

Thus for case (*i*) in (12)–(15), c_k and l_k will be random variables with appropriate distributions; we assume that the different c_k and l_k are independent random variables. In case (*ii*) the c_k are all equal to the same constant c_0, the l_k are independent random variables. In case (*iii*) the c_k are independent random variables, the l_k equal to the same constant l_0. Case (*ii*) has been suggested by R. Kompfner as being applicable to certain optical maser amplifiers.

In cases (*i*) and (*iii*) we will assume that c_k is symmetrically distributed about 0, with a distribution narrow compared to 1.

We assume in the present paper that l_k is always a large number of wavelengths, so that

$$\beta l_k \gg 2\pi. \tag{47}$$

We further assume in cases (*i*) and (*ii*) that the distribution of l_k about its mean is very narrow with respect to the mean, but wide compared to $2\pi/\beta$. These assumptions are compatible with conditions existing in certain optical amplifiers to which these results might be applied. For certain calculations we need assume in addition only a smooth, symmetrical distribution for l_k about its mean. However, for certain other calculations we must be more specific; here we will assume a Gaussian distribution for l_k, as follows:

$$p(l_k) = \frac{1}{\sqrt{2\pi}\sigma_l} e^{-(l_k - l_0)/2\sigma_l^2}, \tag{48}$$

where l_0 is the expected value and σ_l^2 the variance of l_k,

$$\begin{aligned} l_0 &= \langle l_k \rangle, \\ \sigma_l^2 &= \langle l_k^2 \rangle - \langle l_k \rangle^2. \end{aligned} \tag{49}$$

In accord with (47) and the discussion immediately following, we assume that

$$2\pi/\beta \ll \sigma_l \ll l_0 ; \qquad \text{cases } (i) \text{ and } (ii). \tag{50}$$

Note that in case (iii) $l_k = l_0$, as stated above, and $\sigma_l = 0$.

In the following work we make use of the Kronecker matrix product.[3] For convenience we define this product and summarize some of its properties.

Consider two matrices A and B with elements a_{ij} and b_{ij}. The matrices A and B need not be square, have the same dimensions, or be conformable; their dimensions are completely arbitrary, so that the ordinary matrix products AB or BA may not exist. The Kronecker product, written as $A \times B$, (as opposed to the ordinary matrix product, written as AB) is defined as follows:[3]

$$A \times B = \begin{bmatrix} a_{11}B & a_{12}B & a_{13}B & \cdots \\ a_{21}B & a_{22}B & a_{23}B & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{51}$$

$A \times B$ has been written in (51) in partitioned form, with each submatrix consisting of a scalar element of A, a_{ij}, multiplied by the entire matrix B.

Kronecker products have the following useful properties:[3]

$$A \times B \times C = (A \times B) \times C = A \times (B \times C) \tag{52}$$

$$(A + B) \times (C + D) = A \times C + A \times D + B \times C + B \times D \tag{53}$$

$$(A \times B)\,(C \times D) = (AC) \times (BD). \tag{54}$$

As stated above, products without $\times$'s in (52) indicate ordinary matrix products, and the two matrices to be so multiplied must be conformable. Equation (54) may be extended to yield

$$\begin{aligned} (A_1 \times B_1)\,(A_2 \times B_2) \cdots (A_N \times B_N) \\ = (A_1A_2 \cdots A_N) \times (B_1B_2 \cdots B_N). \end{aligned} \tag{55}$$

We now return to the results of Section I for the transmission of a general active device. From (13) we have (see Fig. 1)

$$\begin{bmatrix} W_0(0) \\ W_1(0) \end{bmatrix} = X_1 X_2 \cdots X_N \begin{bmatrix} W_0(L_N+) \\ W_1(L_N+) \end{bmatrix}. \tag{56}$$

The output is assumed matched [see (17)], so that

$$W_1(L_N+) = 0. \tag{57}$$

In computing the loss $\mathbf{L}_T$ of (15) we might as well set

$$W_0(L_N+) = 1, \tag{58}$$

so that by (15) $\mathbf{L}_T = W_0(0)$; (56) then becomes

$$\begin{bmatrix} \mathbf{L}_T \\ W_1(0) \end{bmatrix} = X_1 X_2 \cdots X_N \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{59}$$

Now, in determining the average loss and the loss fluctuations about the average we are not particularly interested in the phase variations caused by the variation in total length, which may be large compared to the optical wavelength but is small compared to the average total length. Further, the variations in gain per section will also be small compared to the average gain per section. These considerations suggest the following transformations of (59), which remove these more or less irrelevant contributions to the loss and phase variations. From Fig. 1, the total length L_N is

$$L_N = \sum_{k=1}^{N} l_k \,. \tag{60}$$

Next define $\mathfrak{L}_T$ and $\mathfrak{R}$ as follows:

$$\mathbf{L}_T = e^{+\Gamma L_N} \cdot \mathfrak{L}_T \,, \qquad \mathfrak{L}_T = e^{-\Gamma L_N} \cdot \mathbf{L}_T \tag{61}$$

$$W_1(0) = e^{+\Gamma L_N} \cdot \mathfrak{R}, \qquad \mathfrak{R} = e^{-\Gamma L_N} \cdot W_1(0). \tag{62}$$

From (12) we define a new matrix Y_k in terms of X_k as follows:

$$X_k = e^{+\Gamma l_k} \cdot Y_k \,, \tag{63}$$

where

$$Y_k = \frac{1}{\sqrt{1 - c_k^2}} \begin{bmatrix} 1 & -jc_k \\ +jc_k \, e^{-2\Gamma l_k} & e^{-2\Gamma l_k} \end{bmatrix}. \tag{64}$$

Then from (60)–(64), (59) may be written

$$e^{+\Gamma L_N}\begin{bmatrix}\mathfrak{L}_T\\ \mathfrak{R}\end{bmatrix} = e^{+\Gamma l_1}\cdot Y_1 e^{+\Gamma l_2}\cdot Y_2 \cdots e^{+\Gamma l_N}\cdot Y_N\begin{bmatrix}1\\0\end{bmatrix}$$

$$= e^{+\Gamma L_N}\cdot Y_1 Y_2 \cdots Y_N\begin{bmatrix}1\\0\end{bmatrix}. \tag{65}$$

Cancelling out the $e^{+\Gamma L_N}$ factor on both sides of (65),

$$\begin{bmatrix}\mathfrak{L}_T\\ \mathfrak{R}\end{bmatrix} = Y_1 Y_2 \cdots Y_N\begin{bmatrix}1\\0\end{bmatrix}, \tag{66}$$

where $\mathfrak{L}_T$ is defined in (61), Y_k in (64).

Equation (66) is suitable for studying the statistics of the normalized loss $\mathfrak{L}_T$, which contains the essential information regarding the loss fluctuations of the device. The quantity $\mathfrak{R}$ has to do with the reflected wave at the input corresponding to a unit output wave, and will not be of further interest here. The factor $e^{+\Gamma L_N} = e^{-\alpha L_N}e^{j\beta L_N}$ removed from the unnormalized loss $\mathbf{L}_T$ in (61) is of course a random variable, but for a given amplifier it has constant magnitude and delay.

We now compute $\langle\mathfrak{L}_T\rangle$, the expected value of the normalized loss $\mathfrak{L}_T$. Since the c_k and l_k are assumed independent random variables, the different Y_k of (66) are independent random matrices in all three cases discussed above. Taking the expected value of both sides of (66), and noting that the different Y_k have the same distribution, we have

$$\begin{bmatrix}\langle\mathfrak{L}_T\rangle\\ \langle\mathfrak{R}\rangle\end{bmatrix} = \langle Y\rangle^N\begin{bmatrix}1\\0\end{bmatrix}, \tag{67}$$

where $\langle Y\rangle$ is obtained from (64) as

$$\langle Y\rangle = \begin{bmatrix}\left\langle\dfrac{1}{\sqrt{1-c^2}}\right\rangle & -j\left\langle\dfrac{c}{\sqrt{1-c^2}}\right\rangle\\ +j\left\langle\dfrac{c}{\sqrt{1-c^2}}\right\rangle\langle e^{-2\Gamma l}\rangle & \left\langle\dfrac{1}{\sqrt{1-c^2}}\right\rangle\langle e^{-2\Gamma l}\rangle\end{bmatrix}. \tag{68}$$

Note that the independence of c_k and l_k for a given k has been used in obtaining (68); the subscript k has been omitted in the above relations, since the statistics of the different c_k's and of the different l_k's are identical. Finally, since we neglect the small variations in the gain per section, we may set

$$\langle e^{-2\Gamma l}\rangle \approx e^{2\alpha l_0}\langle e^{-j2\beta l}\rangle, \tag{69}$$

where l_0 is given in (49) as the average length of the sections. Then (68) becomes

$$\langle Y \rangle = \begin{bmatrix} \left\langle \frac{1}{\sqrt{1-c^2}} \right\rangle & -j \left\langle \frac{c}{\sqrt{1-c^2}} \right\rangle \\ +j\, e^{2\alpha l_0} \left\langle \frac{c}{\sqrt{1-c^2}} \right\rangle \langle e^{-j2\beta l} \rangle & e^{2\alpha l_0} \left\langle \frac{1}{\sqrt{1-c^2}} \right\rangle \langle e^{-j2\beta l} \rangle \end{bmatrix}. \tag{70}$$

Now in cases (*i*) and (*iii*) above we have

$$\left\langle \frac{c}{\sqrt{1-c^2}} \right\rangle = 0, \tag{71}$$

since the distribution of c is assumed symmetric about 0. In cases (*i*) and (*ii*) we have

$$\langle e^{-j2\beta l} \rangle \approx 0, \tag{72}$$

in view of the assumptions about the distribution of l. Consequently (70) becomes in the three cases:

$$\langle Y \rangle = \begin{cases} \left\langle \frac{1}{\sqrt{1-c^2}} \right\rangle \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, & \text{case } (i) \\ \frac{1}{\sqrt{1-c_0^2}} \begin{bmatrix} 1 & -jc_0 \\ 0 & 0 \end{bmatrix}, & \text{case } (ii) \\ \left\langle \frac{1}{\sqrt{1-c^2}} \right\rangle \begin{bmatrix} 1 & 0 \\ 0 & e^{2\alpha l_0} e^{-j2\beta l_0} \end{bmatrix}, & \text{case } (iii)\,. \end{cases} \tag{73}$$

From (67) and (73) we have the following final results:

$$\langle \mathfrak{L}_T \rangle = \begin{cases} \left\langle \frac{1}{\sqrt{1-c^2}} \right\rangle^N, & \text{cases } (i) \text{ and } (iii) \\ \left(\frac{1}{\sqrt{1-c_0^2}} \right)^N, & \text{case } (ii). \end{cases} \tag{74}$$

The result for case (*ii*) in (74) may be regarded simply as a special case of the results for cases (*i*) and (*iii*). Since in cases (*i*) and (*iii*) the distribution of c is assumed narrow compared to 1, we may in some calculations make the following approximation in (74):

$$\left\langle \frac{1}{\sqrt{1 - c^2}} \right\rangle \approx 1 + \tfrac{1}{2} \langle c^2 \rangle, \tag{75}$$

where $\langle c^2 \rangle$ is the mean square value of the magnitude of the reflection coefficient.

Equation (74) shows that in all three cases the presence of random reflections has increased the expected value of the loss; further, the average loss is independent of β and hence of frequency. Since $\langle \mathfrak{L}_T \rangle \neq 0$, *if* the deviations of $\mathfrak{L}_T$ from its expected value are very small (as they must be in useful amplifiers), then we will have approximately

$$| \langle \mathfrak{L}_T \rangle | \approx \langle | \mathfrak{L}_T | \rangle. \tag{76}$$

This approximate relation permits us to estimate the variance of the magnitude of the loss, as discussed below. We note that

$$| \langle \mathfrak{L}_T \rangle | \leqq \langle | \mathfrak{L}_T | \rangle. \tag{77}$$

Next consider the mean square value of the loss, $\langle | \mathfrak{L}_T |^2 \rangle = \langle \mathfrak{L}_T \mathfrak{L}_T^* \rangle$. First note from (51) that

$$\begin{bmatrix} \mathfrak{L}_T \\ \mathfrak{R} \end{bmatrix} \times \begin{bmatrix} \mathfrak{L}_T^* \\ \mathfrak{R}^* \end{bmatrix} = \begin{bmatrix} \mathfrak{L}_T \mathfrak{L}_T^* \\ \mathfrak{L}_T \mathfrak{R}^* \\ \mathfrak{R} \mathfrak{L}_T^* \\ \mathfrak{R}\mathfrak{R}^* \end{bmatrix} = \begin{bmatrix} | \mathfrak{L}_T |^2 \\ \mathfrak{L}_T \mathfrak{R}^* \\ \mathfrak{R} \mathfrak{L}_T^* \\ | \mathfrak{R} |^2 \end{bmatrix}. \tag{78}$$

From (66), (55), and (78) we have

$$\begin{bmatrix} | \mathfrak{L}_T |^2 \\ \mathfrak{L}_T \mathfrak{R}^* \\ \mathfrak{R} \mathfrak{L}_T^* \\ | \mathfrak{R} |^2 \end{bmatrix} = (Y_1 \times Y_1^*)(Y_2 \times Y_2^*) \cdots (Y_N \times Y_N^*) \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{79}$$

where Y_k is given in (64). Taking the expected value of both sides of (79), again making use of the independence of the different Y_k matrices and the fact that they have the same distribution, we have

$$\begin{bmatrix} \langle | \mathfrak{L}_T |^2 \rangle \\ \langle \mathfrak{L}_T \mathfrak{R}^* \rangle \\ \langle \mathfrak{R} \mathfrak{L}_T^* \rangle \\ \langle | \mathfrak{R} |^2 \rangle \end{bmatrix} = \langle Y \times Y^* \rangle^N \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{80}$$

where $\langle Y \times Y^* \rangle$ is obtained from (64) and (51) as shown in (81).

$$\langle Y \times Y^* \rangle = \begin{bmatrix} \left\langle \frac{1}{1-c^2} \right\rangle & +j\left\langle \frac{c}{1-c^2} \right\rangle & -j\left\langle \frac{c}{1-c^2} \right\rangle & \left\langle \frac{c^2}{1-c^2} \right\rangle \\ -j\left\langle \frac{c}{1-c^2} \right\rangle \langle e^{-2\Gamma^* l} \rangle & \left\langle \frac{1}{1-c^2} \right\rangle \langle e^{-2\Gamma^* l} \rangle & -\left\langle \frac{c^2}{1-c^2} \right\rangle \langle e^{-2\Gamma^* l} \rangle & -j\left\langle \frac{c}{1-c^2} \right\rangle \langle e^{-2\Gamma^* l} \rangle \\ +j\left\langle \frac{c}{1-c^2} \right\rangle \langle e^{-2\Gamma l} \rangle & -\left\langle \frac{c^2}{1-c^2} \right\rangle \langle e^{-2\Gamma l} \rangle & \left\langle \frac{1}{1-c^2} \right\rangle \langle e^{-2\Gamma l} \rangle & +j\left\langle \frac{c}{1-c^2} \right\rangle \langle e^{-2\Gamma l} \rangle \\ \left\langle \frac{c^2}{1-c^2} \right\rangle \langle e^{+4\alpha l} \rangle & +j\left\langle \frac{c}{1-c^2} \right\rangle \langle e^{+4\alpha l} \rangle & -j\left\langle \frac{c}{1-c^2} \right\rangle \langle e^{+4\alpha l} \rangle & \left\langle \frac{1}{1-c^2} \right\rangle \langle e^{+4\alpha l} \rangle \end{bmatrix} \quad (81)$$

We again omit the subscript k in the above, since the statistics of the different c_k's and of the different l_k's are assumed identical.

We now apply the same assumptions used above to (81). As in (69), neglecting the small variations in gain per section leads to

$$\begin{aligned} \langle e^{-2\Gamma l} \rangle &\approx e^{2\alpha l_0} \langle e^{-j2\beta l} \rangle , \\ \langle e^{-2\Gamma^* l} \rangle &\approx e^{2\alpha l_0} \langle e^{+j2\beta l} \rangle , \\ \langle e^{4\alpha l} \rangle &\approx e^{4\alpha l_0} , \end{aligned} \tag{82}$$

where l_0 as before is the average length of the sections [see (49)]. Further, we make use of (71) for cases (*i*) and (*iii*), and (72) for cases (*i*) and (*ii*). The resulting forms for $\langle Y \times Y^* \rangle$ differ in the three cases, but after some simplification the final quantity of interest, $\langle | \mathcal{L}_T |^2 \rangle = \langle \mathcal{L}_T \mathcal{L}_T^* \rangle$, is given by the following single relation in all three cases:

$$\begin{bmatrix} \langle | \mathcal{L}_T |^2 \rangle \\ \\ \langle | \mathcal{R} |^2 \rangle \end{bmatrix} = \begin{bmatrix} \left\langle \dfrac{1}{1 - c^2} \right\rangle & \left\langle \dfrac{c^2}{1 - c^2} \right\rangle \\ e^{4\alpha l_0} \left\langle \dfrac{c^2}{1 - c^2} \right\rangle & e^{4\alpha l_0} \left\langle \dfrac{1}{1 - c^2} \right\rangle \end{bmatrix}^N \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} . \tag{83}$$

In case (*ii*), we have in (83)

$$\left\langle \frac{1}{1 - c^2} \right\rangle = \frac{1}{1 - c_0^2}, \qquad \left\langle \frac{c^2}{1 - c^2} \right\rangle = \frac{c_0^2}{1 - c_0^2}. \tag{84}$$

Equation (83) gives the desired result $\langle | \mathcal{L}_T |^2 \rangle$ in terms of the nth power of a real matrix. The matrix power may of course be written out explicitly in the usual way, but for the sake of simplicity this will not be done here. Some numerical examples are worked out in the next section. The variance of the loss, denoted $\sigma_{\mathcal{L}_T}^2$, is given by

$$\begin{aligned} \sigma_{\mathcal{L}_T}^2 &\equiv \langle | \mathcal{L}_T - \langle \mathcal{L}_T \rangle |^2 \rangle \\ &= \langle | \mathcal{L}_T |^2 \rangle - | \langle \mathcal{L}_T \rangle |^2 . \end{aligned} \tag{85}$$

The variance of the *magnitude* of the loss is given by

$$\begin{aligned} \sigma_{|\mathcal{L}_T|}^2 &\equiv \langle [| \mathcal{L}_T | - \langle | \mathcal{L}_T | \rangle]^2 \rangle = \langle | \mathcal{L}_T |^2 \rangle - \langle | \mathcal{L}_T | \rangle^2 \\ &\approx \langle | \mathcal{L}_T |^2 \rangle - | \langle \mathcal{L}_T \rangle |^2 \equiv \sigma_{\mathcal{L}_T}^2 , \end{aligned} \tag{86a}$$

where the approximation of (86a) follows from (76). From (77) we have

$$\sigma_{| \mathcal{L}_T |}^2 \leqq \sigma_{\mathcal{L}_T}^2 . \tag{86b}$$

In these results $\langle | \mathcal{L}_T |^2 \rangle$ is given by (83), $\langle \mathcal{L}_T \rangle$ by (74); the approximation of (86a) should be good when $\sigma_{|\mathfrak{L}_T|}/\langle\mathfrak{L}_T\rangle \ll 1$. We see that for all three cases $\langle | \mathfrak{L}_T |^2\rangle$ and $\sigma_{\mathfrak{L}_T}^{\ 2}$ are independent of β and hence of frequency.

Finally we study the covariance of the loss $\mathfrak{L}_T$, denoted $R_{\mathfrak{L}_T}(\tau)$, defined by

$$R_{\mathfrak{L}_T}(\tau) = \langle \mathfrak{L}_T(\beta + \tau)\mathfrak{L}_T^*(\beta)\rangle = R_{\mathfrak{L}_T}^*(-\tau). \qquad (87)$$

It will appear below that the expected value in (87) is indeed dependent only on τ, and not on β, within the approximations of the present treatment. If we regard the loss $\mathfrak{L}_T(\beta)$ as a random process, then the Fourier transform of $R_{\mathfrak{L}_T}(\tau)$ yields the power spectrum of the random processes $\mathfrak{L}_T(\beta)$. $R_{\mathfrak{L}_T}(\tau)$ thus gives information about both the dc and ac components of $\mathfrak{L}_T(\beta)$; of particular interest are the mean square magnitude and the rate of fluctuation of the ac component of the loss. The total "power" (dc plus ac) P_T of the random process $\mathfrak{L}_T(\beta)$ is

$$P_T = R_{\mathfrak{L}_T}(0) = \langle | \mathfrak{L}_T(\beta) |^2\rangle. \qquad (88)$$

The dc "power" P_{dc} of $\mathfrak{L}_T(\beta)$ is

$$P_{dc} = R_{\mathfrak{L}_T}(\infty) = R_{\mathfrak{L}_T}(-\infty), \qquad (89)$$

where the limits as $\tau \rightarrow \pm\infty$ exist. Both ac and dc "powers" are necessarily pure real, and are of course independent of β, since $R_{\mathfrak{L}_T}(\tau)$ is independent of β in general. Let us define the dc component of a given $\mathfrak{L}_T(\beta)$ curve as

$$\mathfrak{L}_{T_{dc}} = \overline{\mathfrak{L}_T(\beta)} = \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} \mathfrak{L}_T(\beta)\, d\beta, \qquad (90)$$

where the bar indicates an average over β. Then it is easy to show that the dc power of (89) is also equal to

$$P_{dc} = R_{\mathfrak{L}_T}(\infty) = R_{\mathfrak{L}_T}(-\infty) = \langle | \mathfrak{L}_{T_{dc}} |^2\rangle, \qquad (91)$$

where $\mathfrak{L}_{T_{dc}}$ is given by (90). Let us now define the ac component of a given $\mathfrak{L}_T(\beta)$ curve by

$$\mathfrak{L}_{T_{ac}}(\beta) = \mathfrak{L}_T(\beta) - \mathfrak{L}_{T_{dc}}. \qquad (92)$$

Then the covariance $R_{\mathfrak{L}_{T_{ac}}}(\tau)$ of the ac component $\mathfrak{L}_{T_{ac}}(\beta)$ and the ac "power" P_{ac} of the normalized loss $\mathfrak{L}_T(\beta)$ are given as follows:

$$R_{\mathfrak{L}_{T_{ac}}}(\tau) = \langle \mathfrak{L}_{T_{ac}}(\beta + \tau)\mathfrak{L}_T^{ac*}(\beta)\rangle = R_{\mathfrak{L}_T}(\tau) - R_{\mathfrak{L}_T}(\infty), \qquad (93a)$$

$$P_{ac} = \langle \overline{| \mathfrak{L}_{T_{ac}}(\beta) |^2}\rangle = R_{\mathfrak{L}_T}(0) - R_{\mathfrak{L}_T}(\infty) = R_{\mathfrak{L}_{T_{ac}}}(0). \qquad (93b)$$

For convenience we define the covariance of $\mathcal{R}(\beta)$ as an auxiliary quantity, although this quantity is not of present interest to us:

$$R_{\mathcal{R}}(\tau) = \langle \mathcal{R}(\beta + \tau)\mathcal{R}^*(\beta)\rangle. \tag{94}$$

We have

$$\left\langle \begin{bmatrix} \mathcal{L}_T(\beta + \tau) \\ \mathcal{R}(\beta + \tau) \end{bmatrix} \times \begin{bmatrix} \mathcal{L}_T^*(\beta) \\ \mathcal{R}^*(\beta) \end{bmatrix} \right\rangle = \begin{bmatrix} R_{\mathcal{L}_T}(\tau) \\ \langle \mathcal{L}_T(\beta + \tau)\mathcal{R}^*(\beta)\rangle \\ \langle \mathcal{R}(\beta + \tau)\mathcal{L}_T^*(\beta)\rangle \\ R_{\mathcal{R}}(\tau) \end{bmatrix}. \tag{95}$$

From (66), (55), and (95)

$$\begin{bmatrix} R_{\mathcal{L}_T}(\tau) \\ \langle \mathcal{L}_T(\beta + \tau)\mathcal{R}^*(\beta)\rangle \\ \langle \mathcal{R}(\beta + \tau)\mathcal{L}_T^*(\beta)\rangle \\ R_{\mathcal{R}}(\tau) \end{bmatrix} = \langle Y(\beta + \tau) \times Y^*(\beta)\rangle^N \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{96}$$

where we again make use of the independence of the different Y_k and the fact that they have the same distribution. Using the various assumptions given above in (69), (71), (72), and (82), and making appropriate simplifications in the different cases, we obtain the following final common result for cases (*i*), (*ii*), and (*iii*):

$$\begin{bmatrix} R_{\mathcal{L}_T}(\tau) \\ \\ R_{\mathcal{R}}(\tau) \end{bmatrix} = \begin{bmatrix} \left\langle \dfrac{1}{1 - c^2} \right\rangle & \left\langle \dfrac{c^2}{1 - c^2} \right\rangle \\ e^{4\alpha l_0} \left\langle \dfrac{c^2}{1 - c^2} \right\rangle \langle e^{-j2\tau l}\rangle & e^{4\alpha l_0} \left\langle \dfrac{1}{1 - c^2} \right\rangle \langle e^{-j2\tau l}\rangle \end{bmatrix}^N \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix}. \tag{97}$$

In addition to the usual approximations, we have used

$$\langle e^{-j2(\beta+\tau)l}\rangle \approx 0 \tag{98}$$

in cases (*i*) and (*ii*) in obtaining the result of (97). This approximation implies that $|\tau| \ll \beta$; i.e., we examine the covariance and hence the loss over only a relatively narrow (electrical) band. In the analysis we often use the quantity $R_{\mathcal{L}_T}(\infty)$, which gives the dc "power" [see (91)]; this is justified because the covariance computed from (97) will approach its asymptotic value $R_{\mathcal{L}_T}(\infty)$ for values of τ satisfying the requirement $|\tau| \ll \beta$. We assume the distribution of l is the Gaussian distribution of (48), and note that $\langle e^{-j2\tau l}\rangle$ is simply related to the corresponding characteristic function.[4] Thus

$$\langle e^{-j2\tau l}\rangle = e^{-j2\tau l_0} e^{-2(\tau\sigma_l)^2}.\dagger \tag{99}$$

† Note that this result justifies the approximations of (72) and (98) [subject to the condition of (50)]. A similar result for $\langle e^{\Gamma l}\rangle$, where Γ is complex, may be readily derived, and justifies the approximation of (69) and (82).

In case (*iii*) we have $\sigma_l = 0$ in (99). Thus we have as our final result:

$$\begin{bmatrix} R_{\mathfrak{L}_T}(\tau) \\ R_{\mathfrak{R}}(\tau) \end{bmatrix} = \begin{bmatrix} \left\langle \dfrac{1}{1-c^2} \right\rangle & \left\langle \dfrac{c^2}{1-c^2} \right\rangle \\ e^{4\alpha l_0} \left\langle \dfrac{c^2}{1-c^2} \right\rangle e^{-j2\tau l_0} e^{-2(\tau\sigma_l)^2} & e^{4\alpha l_0} \left\langle \dfrac{1}{1-c^2} \right\rangle e^{-j2\tau l_0} e^{-2(\tau\sigma_l)^2} \end{bmatrix}^N \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{100}$$

$$\left\langle \frac{1}{1-c^2} \right\rangle = \frac{1}{1-c_0^2}, \left\langle \frac{c^2}{1-c^2} \right\rangle = \frac{c_0^2}{1-c_0^2}; \qquad \text{case } (ii)$$
$$\sigma_l = 0; \qquad \text{case } (iii). \tag{101}$$

Certain general properties of $R_{\mathfrak{L}_T}(\tau)$ are readily deduced from (100). First, $R_{\mathfrak{L}_T}(\tau)$ is independent of β and dependent only on τ, as assumed above in (87). Second, for $\tau = 0$, (100) becomes identical to (83), as it must. Finally, for $\tau \to \infty$, we have in cases (*i*) and (*ii*) from (100) and (101)

$$R_{\mathfrak{L}_T}(\infty) = \begin{cases} \left\langle \dfrac{1}{1-c^2} \right\rangle^N, & \text{case } (i) \ \text{[also case } (iii) \text{ — see below]} \\ \left(\dfrac{1}{1-c_0^2} \right)^N, & \text{case } (ii). \end{cases} \tag{102}$$

$R_{\mathfrak{L}_T}(\infty)$ is real, as stated above. We recall from (91) that $R_{\mathfrak{L}_T}(\infty)$ is the dc "power" of $\mathfrak{L}_T(\beta)$. The ac "power" is given by (93).

Now, in case (*iii*) the covariance $R_{\mathfrak{L}_T}(\tau)$ is periodic, which implies that the random process $\mathfrak{L}_T(\beta)$ is periodic;[4] however, this is obvious from the original formulation of the problem. $R_{\mathfrak{L}_T}(\infty)$ no longer exists in the strict sense; the dc "power" is now the average value (over τ) of $R_{\mathfrak{L}_T}(\tau)$. It turns out that we may approach case (*iii*) by considering case (*i*) and allowing σ_l to approach 0 in (100). [This violates the condition imposed by (50) and used in the approximations of (72) and (98) and so the limiting process $\sigma_l \to 0$ is forbidden in some of the above results; careful examination shows that it *is* valid to allow $\sigma_l \to 0$ in (100).] Then $R_{\mathfrak{L}_T}(\tau)$ does approach the limit of (102) as $\tau \to \infty$; and so we take the first result of (102) as the dc "power" in case (*iii*), as well as in case (*i*).

In general

$$\langle \mathfrak{L}_T(\beta) \rangle^2 \neq P_{\text{dc}} \equiv \langle \mid \overline{\mathfrak{L}_T(\beta)} \mid^2 \rangle, \tag{103}$$

$$\sigma_{\mathfrak{L}_T}{}^2 \neq P_{\text{ac}} \equiv \langle \mid \overline{\mathfrak{L}_{T_{\text{ac}}}(\beta) \mid^2} \rangle . \tag{104}$$

However, in case (*ii*) only — i.e., reflectors of identical magnitude and random spacing — (103) and (104) are true with the $\neq$ replaced by $=$, as seen from (74) and (102).

The matrix power of (100) is easily written explicitly in the usual way, but the results would be rather complicated. Numerical examples are worked out in the next section.

IV. NUMERICAL EXAMPLE — RANDOM REFLECTORS

Consider an optical amplifier with random reflectors of the type given in case (*ii*) of Section III: i.e., the reflectors have identical magnitude but random spacing. Assume:

$$20 \log_{10} e^{\alpha l_0} = 1 \text{ db, nominal gain per section}$$

$$N = 30\text{, number of sections}$$

$$20 \log_{10} e^{N\alpha l_0} = 30 \text{ db, nominal total gain.}$$

Fig. 9 shows the average normalized loss and the rms fluctuation of the normalized loss about its average value, plotted versus c_0, the magnitude of the reflectors. As seen from example (*i*), Section II, instability is possible if $|c_0| > 0.00860$. Therefore the curves of Fig. 9 are solid for $c_0 < 0.00860$, dotted for $c_0 > 0.00860$. However, this is intended only as a symbolic reminder of the question of stability. We do not know whether or not instability can occur for $|c_0| < 0.00860$. Even though we know that instability can occur for $|c_0| > 0.00860$, the probability of instability might remain so small for some greater range of c_0 that these curves would provide a useful approximation. In Ref. 1, Section VI, equations (122)–(131) we show that stability is guaranteed for $|c_0| < 0.00590$, assuming that the maximum fractional variation in spacing of the reflectors [ν in (124) of Ref. 1] is small compared to 1. This is indicated in Fig. 9.

All of the above results have been independent of the precise distribution of the l_k, the spacing between reflectors, except that the conditions of (47) and the following sentence must be satisfied. However, the covariance of the loss depends explicitly on the probability distribution of the l_k. For our present example we therefore assume that the differ-

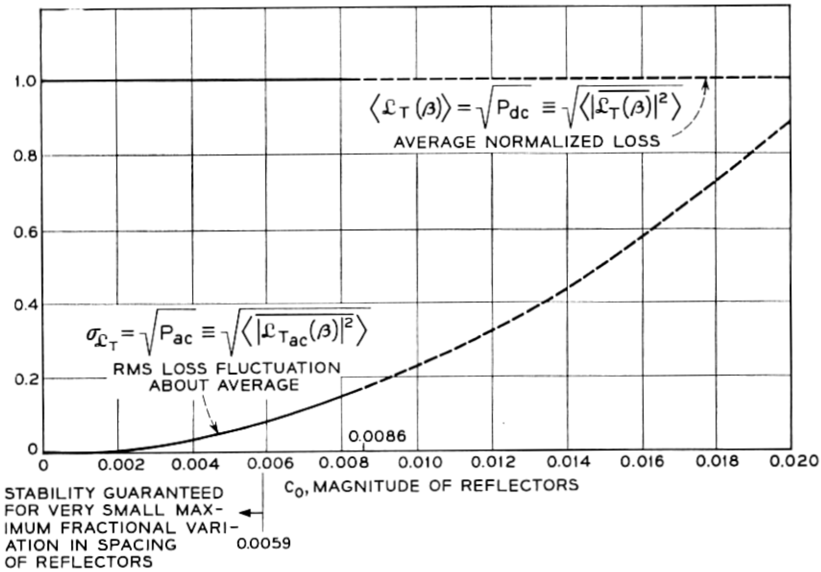

Fig. 9 — Average normalized loss and rms fluctuation about the average for one-dimensional active medium with randomly spaced reflectors of identical magnitude. $N = 30$, number of sections; $20 \log_{10} e^{\alpha l_0} = 1$ db, nominal gain per section; nominal total gain $= 30$ db.

ent l_k are independent, with the Gaussian probability density given in (48)–(50). We further assume the following numerical values:

$$(\sigma_l/l_0) = 0.01, \qquad c_0 = 0.005. \tag{105}$$

Thus, the spacing between successive reflectors is accurate to about 1 per cent, and the magnitude of the reflectors would guarantee stability in the equally spaced case of Section II. Of course a practical device would probably be built much more accurately, but the values in (105) are suitable for illustrating the general behavior. Fig. 10 shows the (complex) covariance $R_{\mathcal{L}_{T_{ac}}}(\tau)$ of the ac component $\mathcal{L}_{T_{ac}}(\beta)$ of the normalized loss for this case as a function of the normalized variable $(l_0/\pi)\tau$, for $0 < (l_0/\pi)\tau < 4$. Fig. 10(a) shows the magnitude $| R_{\mathcal{L}_{T_{ac}}}(\tau) |$ and Fig. 10(b) the phase $\angle R_{\mathcal{L}_{T_{ac}}}(\tau) + 58\, l_0\tau$; note that the linear component of phase has been removed in the plot of Fig. 10(b). The covariance is seen to be approximately a damped periodic function of τ; Fig. 11 shows a plot of the magnitude of the covariance at the points $\tau = n(\pi/l_0)$, which correspond closely to the maxima of $| R_{\mathcal{L}_{T_{ac}}}(\tau) |$.

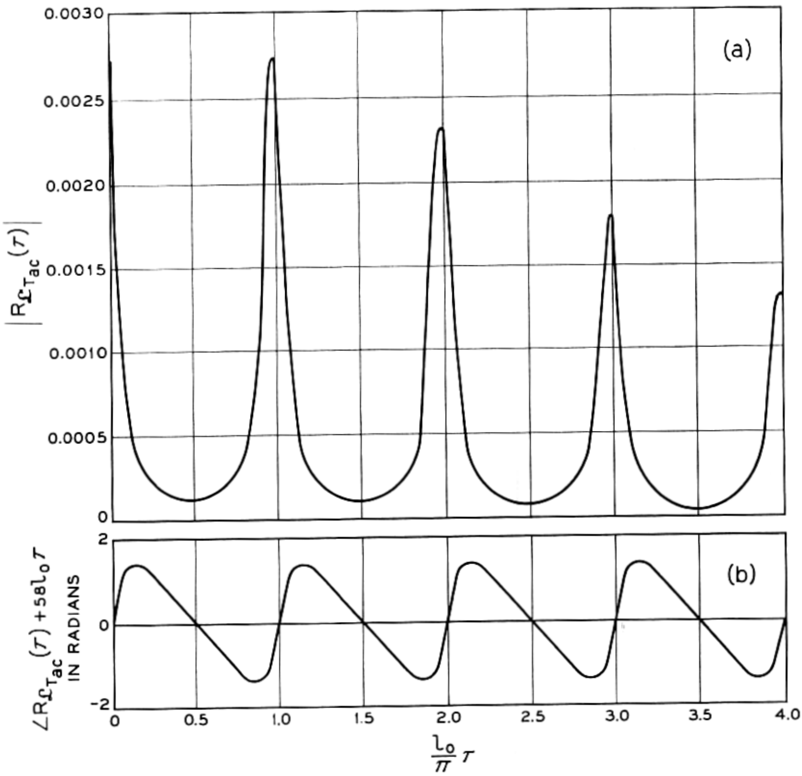

Fig. 10 — Covariance of ac component of normalized loss for one-dimensional active medium with randomly spaced reflectors. $\sigma_l/l_0 = 0.01$; $c_0 = 0.005$, magnitude of reflectors; $N = 30$, number of sections; $20 \log_{10} e^{\alpha l_0} = 1$ db, nominal gain per section; nominal total gain = 30 db.

We would expect some resemblance between the covariance of Figs. 10 and 11, for reflectors with identical magnitude but random spacing, and the (nonrandom) case of Section II for reflectors with identical magnitude and spacing. For the nonrandom case we have seen that the loss is periodic; consequently the covariance will also be periodic, and will look something like that of Figs. 10 and 11 for the random case except that it will not be damped. Note that the large linear component $-58\ l_0\tau$ that has been removed from the phase curve of Fig. 10(b) implies that the power spectrum of the random process $\mathfrak{L}_{T_{ac}}(\beta)$ is concentrated around the angular "frequency" $-58\ l_0$; this angular "frequency" corresponds to the rate of variation of the loss for two reflectors whose separation is equal to the nominal spacing of the two end reflectors in the random case.

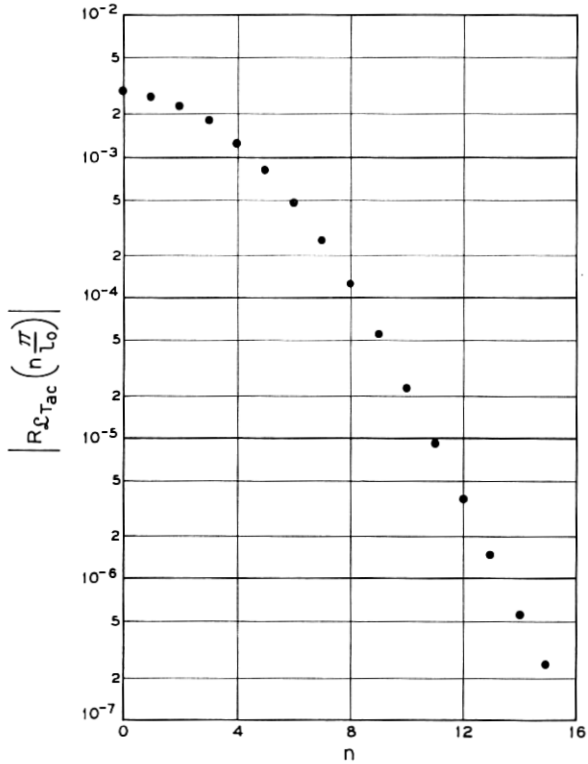

Fig. 11 — Approximate maxima of covariance of ac component of normalized loss for one-dimensional active medium with randomly spaced reflectors (see Fig. 10). $\sigma_l/l_0 = 0.01$; $c_0 = 0.005$, magnitude of reflectors; $N = 30$, number of sections; $20 \log_{10} e^{\alpha l_0} = 1$ db, nominal gain per section; nominal total gain = 30 db.

V. DISCUSSION

The question of stability has been discussed for the periodic case at the end of Section II. There it is pointed out that these calculations are valid only if the device is stable, i.e., does not oscillate. The same is true in the random case. In the periodic case we can determine by calculation the limits of stability, and this has been done in the examples of Section II. Stability in the random case is studied in Ref. 1.

Various higher-order transmission statistics may be calculated by methods similar to those used above, but the complexity of the calculations increases with the order of the statistics. In addition, statistics of the real and imaginary parts of the normalized loss $\mathcal{L}_T$ may be readily determined by similar methods.

VI. ACKNOWLEDGMENT

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REFERENCES

1. Rowe, H. E., Stability of Active Transmission Lines with Arbitrary Imperfections, B.S.T.J., this issue, p. 293.
2. Montgomery, C. G., Dicke, R. H., and Purcell, E. M., *Principles of Microwave Circuits*, McGraw-Hill, New York, 1948.
3. Bellman, R., *Introduction to Matrix Analysis*, McGraw-Hill, New York, 1960.
4. Davenport, W. B., and Root, W. L., *Random Signals and Noise*, McGraw-Hill, New York, 1958.