Stability of Active Transmission Lines with Arbitrary Imperfections

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Two sufficient conditions for the stability of one-dimensional active transmission lines with arbitrary imperfections (i.e., discrete or continuous reflections) are derived. The first stability condition guarantees stability for any arbitrary distribution of reflection. The second stability condition is restricted to a special case of interest that includes discrete reflectors with nominally equal magnitude and spacing; the stability condition for this restricted class is greatly improved over the general stability condition described above.

These results, aside from their own interest, provide rigorous justification for previous calculations for the gain statistics of such a device with random discrete reflectors.[1] *They may also be used to find an upper bound on the probability of instability of such a device with random reflectors.*

Certain types of optical maser amplifiers and traveling-wave tubes provide examples of practical devices with distributed gain to which these results, or similar ones, might be applied.

I. INTRODUCTION

The preceding paper[1] has considered the theory of active transmission lines with discrete imperfections. First, lines with equally-spaced identical reflectors were studied; in particular, gain-frequency curves were determined as functions of the various parameters, and the stability of the device was studied under these special conditions. It was pointed out that the mathematical expression for gain would yield a perfectly definite result for any values of the parameters, but that this mathematical result would have physical significance only if the device is stable, i.e., does not oscillate.

Next, the case of random imperfections was studied.[1] Here the statistics of the transmission were determined in terms of the statistics of the discrete reflectors, which were assumed to have random position and magnitude. Again, these results have physical significance only if the device is stable (or if the probability of instability is negligible). However, in the random case no precise information about stability was given; the computed statistics of the transmission were felt to be valid if the rms magnitude of the discrete reflectors was sufficiently small, but only intuitive feelings of what was "small enough" were available.

In the present paper we derive a sufficient condition for stability of an active transmission line with arbitrary reflectors; we further show (by one example) that this sufficient condition cannot be greatly improved (if at all) in the general case. This result gives useful information regarding the range of validity of the calculations of the preceding paper[1] for the transmission statistics of active transmission lines with random reflectors. This general bound on stability may be improved if additional information is known about the distribution of reflectors; one such case of interest is treated.

The mathematical model chosen for this problem is discussed in detail in Ref. 1. A line with N discrete reflectors is shown in Fig. 1 (which is identical to Fig. 1 of Ref. 1). The wave traveling to the right at distance z is denoted by $W_0(z)$, the wave traveling to the left by $W_1(z)$; $W_0(L_k+)$ and $W_1(L_k+)$ are the right- and left-traveling waves just to the right of the kth reflector, as indicated in this figure, while $W_0(L_k-)$ and $W_1(L_k-)$ are the right- and left-traveling waves just to the left of the kth reflector.

In the absence of reflections the forward and backward waves vary as

$$\begin{aligned} W_0(z) &\propto e^{-\Gamma z} && \text{— forward wave} \\ W_1(z) &\propto e^{+\Gamma z} && \text{— backward wave} \end{aligned} \tag{1}$$

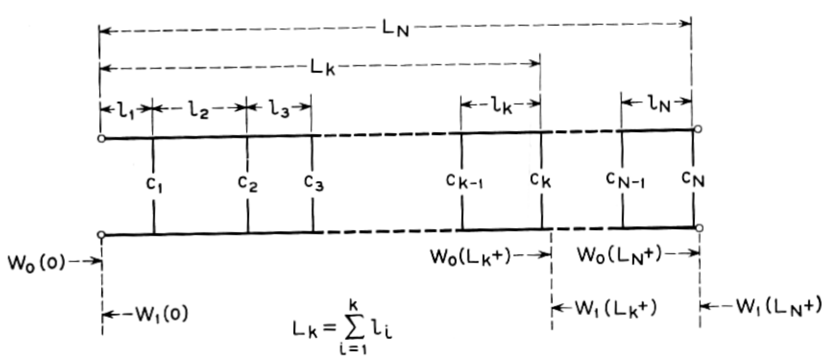

Fig. 1 — Line with N discrete reflectors.

where

$$\Gamma = -\alpha + j\beta, \qquad \alpha > 0. \tag{2}$$

The line has gain, so that $\alpha > 0$. From (12) of Ref. 1, the wave matrix for the cascade connection of the kth line section of length l_k and the kth reflector is

$$X_k = \frac{1}{\sqrt{1 - c_k^2}} \begin{bmatrix} e^{+\Gamma l_k} & -jc_k e^{+\Gamma l_k} \\ +jc_k e^{-\Gamma l_k} & e^{-\Gamma l_k} \end{bmatrix}, \qquad |c_k| \leqq 1, \tag{3}$$

$$\begin{bmatrix} W_0(L_{k-1}+) \\ W_1(L_{k-1}+) \end{bmatrix} = X_k \cdot \begin{bmatrix} W_0(L_k+) \\ W_1(L_k+) \end{bmatrix} \tag{4}$$

where $|c_k|$ is the magnitude of the reflection coefficient for the kth reflector. The over-all transmission matrix for the entire line of Fig. 1, denoted by $\bar{X}$, is given by the matrix product of (13) of Ref. 1:

$$\bar{X} = \prod_{k=1}^{N} X_k, \tag{5}$$

$$\begin{bmatrix} W_0(0) \\ W_1(0) \end{bmatrix} = \bar{X} \cdot \begin{bmatrix} W_0(L_N+) \\ W_1(L_N+) \end{bmatrix}. \tag{6}$$

For convenience, denote the elements of the over-all transmission matrix $\bar{X}$ as in (14) of Ref. 1.

$$\bar{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}. \tag{7}$$

$\bar{X}$ is given by (3) and (5). Assume the device is operated as an amplifier with matched input and output; setting $W_1(L_N+) = 0$, the complex transmission gain $\mathbf{G}_T$ is given by

$$\mathbf{G}_T = \frac{W_0(L_N+)}{W_0(0)} = \frac{1}{x_{11}}. \tag{8}$$

Now x_{11} is a function of Γ and of all of the l_k's and c_k's. We may conceptually investigate stability in the following way. Imagine that c_k is replaced by ϵc_k throughout this analysis; ϵ is a variable parameter that scales the magnitudes of all of the coupling coefficients. Let ϵ be increased from 0, and for each value of ϵ examine x_{11} [which in (8) is the reciprocal of the transmission gain, and so may be regarded as the transmission loss] as a function of frequency ω (or of the phase constant β, which is assumed proportional to frequency, since the line is distortionless)[1] over the entire range $-\infty < \omega < +\infty$. We determine in this way the minimum value of $| x_{11} |$ for each value of ϵ. As ϵ increases, this minimum value of $| x_{11} |$ will eventually just drop to zero, for a critical value of ϵ which we denote by ϵ_c. Thus, as $\epsilon \to \epsilon_c$ the gain $| \mathbf{G}_T | \to \infty$ for a particular value of ω, and the device oscillates. ϵ_c is the dividing line between stability and instability; if $\epsilon_c > 1$, the original device, with the parameters c_k and l_k, is stable.

Such calculations have actually been carried out in Ref. 1 for devices with identical, equally-spaced reflectors. In this case the gain $\mathbf{G}_T$ is a periodic function of frequency ω, so that only a finite portion of the frequency axis (i.e., one period) must be investigated. In general, however, $\mathbf{G}_T$ is not periodic; since we cannot investigate numerically the entire ω-axis, it is not obvious how to investigate stability for the general case.

In the remainder of this paper we determine a sufficient condition that guarantees the stability of a general active line with arbitrary discrete imperfections. In particular, consider such a device, illustrated in Fig. 1, characterized by (3), (5), and (6), with arbitrary α, c_k, and l_k. We show below that any such device satisfying the condition

$$\sum_{i=1}^{N} \tanh^{-1} | c_i | < 2 \sinh^{-1} \frac{e^{-\alpha L_N}}{\sqrt{2}} \tag{9}$$

must be stable. Many practical devices will have large gain, and hence must have small reflections. In such cases $e^{-\alpha L_N} \ll 1$ and $| c_i | \ll 1$; under these conditions a slightly poorer stability condition derived from (9) is useful.

$$\sum_{i=1}^{N} | c_i | \leqq \tanh \left[2 \sinh^{-1} \frac{e^{-\alpha L_N}}{\sqrt{2}} \right]. \tag{10}$$

In the high-gain case the right-hand side of (10) may be made simpler still by further degrading this stability condition. We may show, for example, that

$$\tanh \left[2 \sinh^{-1} \frac{e^{-\alpha L_N}}{\sqrt{2}} \right] \geqq 0.932 \sqrt{2} e^{-\alpha L_N}, \qquad 8.686 \alpha L_N \geqq 10 \text{ db}. \tag{11}$$

Thus a slightly poorer version of (10) is

$$\sum_{i=1}^{N} | c_i | \leqq 0.932 \sqrt{2}\, e^{-\alpha L_N}, \qquad 8.686\, \alpha L_N \geqq 10 \text{ db}. \tag{12}$$

The stability condition of (12) is valid when the one-way gain of the active medium exceeds 10 db. As the lower bound on the one-way gain of the active medium increases beyond 10 db, the numerical factor 0.932 on the right-hand side of (12) increases, approaching 1 as the lower bound on the gain approaches infinity. This is readily seen from (10); as $\alpha L_N \to \infty$, $e^{-\alpha L_N} \to 0$, so that the $\sinh^{-1}$ and tanh functions in (10) may be approximately replaced by their arguments for sufficiently large αL_N. However, direct calculation with (10) is straightforward; the result of (12) (or similar equations) is intended principally to illustrate the general behavior.

Thus (9) or the successively poorer versions of (10) and (12) guarantee that the device will be stable, even for the worst possible choice of the c_k and l_k. Equations (9), (10), and (12) are each sufficient, but not necessary, conditions for stability. These results are derived in Sections II, III, and IV. In addition, a better bound is obtained for a special case in which the reflection coefficient is distributed more or less uniformly with distance z along the active line, in a certain sense to be described more precisely in Section V below; these results include many cases of interest. Finally, some numerical examples illustrating the use of these two different types of bounds are given in Section VI.

II. DIFFERENTIAL EQUATIONS EQUIVALENT TO MATRIX RELATIONS

Consider the following differential equations:

$$\begin{aligned} W_0'(z) &= -\Gamma W_0(z) + jr(z)W_1(z), \\ W_1'(z) &= -jr(z)W_0(z) + \Gamma W_1(z). \end{aligned} \tag{13}$$

These relations have the form of the coupled line equations with a general continuous coupling coefficient. In the present case, $W_0(z)$ and $W_1(z)$ are the right- and left-directed traveling-wave complex amplitudes, and $r(z)$ is the continuous reflection that couples the two waves to each other. Equation 13 is readily obtained as a limiting form of the matrix relations of (3), (5), and (6) by assuming very small, closely spaced discrete reflectors whose magnitude varies slowly with distance. Thus in the matrix relations of Section I above set

$$l_k = \Delta z. \tag{14}$$

Assume that c_k varies slowly with k. Then we set

$$c_k = r(k\Delta z)\cdot \Delta z, \tag{15}$$

where $r(z)$ is a continuous function. We now let $\Delta z \to 0$ so that the number of discrete reflectors $\to \infty$; during this process the continuous function $r(z)$ is fixed and the c_k determined by (15), so that the magnitudes of the individual reflectors $\to 0$ as $\Delta z \to 0$. Then the matrix relations of (3), (5), and (6) will yield the continuous differential equations of (13). The analysis is straightforward and quite similar to that of Ref. 2 for a similar problem, and so will not be given here. The above discussion of (13) as an appropriate limiting continuous form of the matrix relations of Section I is given only to provide some physical motivation for considering (13), and plays no part in the mathematical analysis to follow.

The case of isolated, discrete reflectors, characterized by (3), (5), and (6), may conversely be regarded as a special case of continuous reflection in (13), in which the continuous reflection $r(z)$ becomes a sum of suitable δ-functions, one located at each discrete reflector. Thus we show that if $r(z)$ in (13) is given by

$$r(z) = \sum_{i=1}^{N} \tanh^{-1} c_i \cdot \delta(z - L_i), \tag{16}$$

where in Fig. 1 L_i is the total distance from the input of the line to the ith reflector, then the solutions to (13) at the output of the line, i.e., $W_0(L_N+)$ and $W_1(L_N+)$, are given in terms of the input conditions $W_0(0)$ and $W_1(0)$ by (3), (5), and (6).

Consider the typical kth section of line, of length l_k, followed by the kth discrete reflector, as illustrated in Fig. 1. In the line section between the $(k-1)$th and the kth reflectors $r(z) = 0$, from (16). Therefore in this region the solution to (13) has the form of (1); the forward and backward waves are uncoupled, and have the same propagation constant. We may thus write the solution between the $(k-1)$th and kth reflectors in the matrix form

$$\begin{bmatrix} W_0(L_{k-1}+) \\ W_1(L_{k-1}+) \end{bmatrix} = \begin{bmatrix} e^{+\Gamma l_k} & 0 \\ 0 & e^{-\Gamma l_k} \end{bmatrix} \cdot \begin{bmatrix} W_0(L_k-) \\ W_1(L_k-) \end{bmatrix}, \tag{17}$$

where $W(L_k-)$ indicates a wave amplitude evaluated just to the left of the kth reflector, $W(L_k+)$ just to the right.

We next evaluate the transmission matrix for the kth reflector, i.e., the kth δ-function of (16). This calculation may be performed by setting

$$r(z) = \begin{cases} \dfrac{\tanh^{-1} c_k}{\Delta}, & L_k < z < L_k + \Delta. \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

We then determine the matrix $T(\Delta)$,

$$\begin{bmatrix} W_0(L_k + \Delta) \\ W_1(L_k + \Delta) \end{bmatrix} = T(\Delta) \cdot \begin{bmatrix} W_0(L_k) \\ W_1(L_k) \end{bmatrix}. \tag{19}$$

Then as $\Delta \to 0$, $r(z) \to \tanh^{-1} c_k \cdot \delta(z - L_k)$, and $\lim_{\Delta \to 0} T(\Delta) = T(0)$ yields a matrix relating the wave amplitudes W_0 and W_1 on the two sides of the kth δ-function of $r(z)$ [see (16)]. This analysis is again similar in motivation, although different in detail, to that of Ref. 2 for a similar problem. Since $r(z)$ in (18) is constant throughout the region of interest, (13) becomes a linear differential equation with constant coefficients, and is readily solved by the usual techniques. The solution for general Δ may be written in matrix form, yielding $T(\Delta)$ of (19), as follows:

$$T(\Delta) = \frac{1}{K_+ - K_-}\begin{bmatrix} -K_- e^{\Gamma\Delta\sqrt{}} + K_+ e^{-\Gamma\Delta\sqrt{}} & e^{\Gamma\Delta\sqrt{}} - e^{-\Gamma\Delta\sqrt{}} \\ -e^{\Gamma\Delta\sqrt{}} + e^{-\Gamma\Delta\sqrt{}} & K_+ e^{\Gamma\Delta\sqrt{}} - K_- e^{-\Gamma\Delta\sqrt{}} \end{bmatrix} \tag{20}$$

$$K_\pm = -j \frac{1 \pm \sqrt{}}{\dfrac{\tanh^{-1} c_k}{\Gamma\Delta}}; \qquad K_+K_- = 1 \tag{21}$$

$$\frac{1}{K_+ - K_-} = \frac{j}{2}\,\frac{\dfrac{\tanh^{-1} c_k}{\Gamma\Delta}}{\sqrt{}} \tag{22}$$

$$\sqrt{} = \sqrt{1 + \left(\frac{\tanh^{-1} c_k}{\Gamma\Delta}\right)^2} \tag{23}$$

Taking the limit as $\Delta \to 0$, (20)–(23) yield

$$\begin{bmatrix} W_0(L_k+) \\ W_1(L_k+) \end{bmatrix} = T(0)\cdot\begin{bmatrix} W_0(L_k-) \\ W_1(L_k-) \end{bmatrix} \tag{24}$$

where

$$T(0) \equiv \lim_{\Delta \to 0} T(\Delta) = \frac{1}{\sqrt{1 - c_k^2}}\begin{bmatrix} 1 & jc_k \\ -jc_k & 1 \end{bmatrix}. \tag{25}$$

Inverting (24),

$$\begin{bmatrix} W_0(L_k-) \\ W_1(L_k-) \end{bmatrix} = T^{-1}(0)\cdot\begin{bmatrix} W_0(L_k+) \\ W_1(L_k+) \end{bmatrix} \tag{26}$$

where, from (25)

$$T^{-1}(0) = \frac{1}{\sqrt{1 - c_k^2}}\begin{bmatrix} 1 & -jc_k \\ +jc_k & 1 \end{bmatrix}. \tag{27}$$

From (17), (26), and (27) we now have

$$\begin{bmatrix} W_0(L_{k-1}+) \\ W_1(L_{k-1}+) \end{bmatrix} = X_k \cdot \begin{bmatrix} W_0(L_k+) \\ W_1(L_k+) \end{bmatrix} \tag{28}$$

where X_k is as given in (3). Equation (28) is identical to (4). Finally, the solution to (13), with $r(z)$ given by (16), is given by (3), (5), and (6).

The equivalence of (13) and (16) with (3), (5) and (6) is useful because the original matrix problem may thus be regarded as a special case of a pair of differential equations. Stability appears to be more readily studied for the more general continuous case described by the differential equations; these results may then be applied to the special discrete case of interest here.

III. SOLUTION BY SUCCESSIVE APPROXIMATIONS (PICARD'S METHOD)

We summarize the solution of (13) by successive approximation, following the same general approach as in Ref. 3 for a similar problem. First, it is convenient to make the following transformations:

$$\begin{aligned} W_0(z) &= e^{-\Gamma z} \cdot G_0(z) \\ W_1(z) &= e^{+\Gamma z} \cdot G_1(z). \end{aligned} \tag{29}$$

Substituting (29) into (13), we have

$$\begin{aligned} G'_0(z) &= jr(z)\, e^{+2\Gamma z} G_1(z) \\ G_1'(z) &= -jr(z)\, e^{-2\Gamma z} G_0(z). \end{aligned} \tag{30}$$

Assume that the device is operated as an amplifier with matched input and output. It proves convenient in the following analysis to take the input at the right-hand end of the amplifier, i.e., at $z = L_N$, where L_N is the total length, and the output at the left-hand end, i.e., $z = 0$; this is just opposite to the choice made in Ref. 1 and in Section I above [particularly in (8)]. The useful output is then the left-directed traveling wave at $z = 0$, i.e., $W_1(0)$, corresponding to an input taken to be the left-directed traveling wave at $z = L_N$, $W_1(L_N)$. Since the device is matched at both ends, $W_0(0) = 0$; $W_0(L_N) \neq 0$, since this quantity corresponds to the reflected wave at the input end (i.e., at $z = L_N$) of the amplifier.

Now assume for convenience a unit-amplitude output wave:

$$W_1(0) = 1. \tag{31}$$

As noted above, since the output is matched,

$$W_0(0) = 0. \tag{32}$$

We seek $W_1(L_N)$, the input corresponding to the output of (31); since unit output has been assumed in (31), the complex transmission gain $\mathbf{G}_T$ will be

$$\mathbf{G}_T = \frac{1}{W_1(L_N)}, \tag{33}$$

where $W_1(L_N)$ is the solution to (13) subject to the initial conditions of (31) and (32).

The transmission gain is readily stated in terms of the solutions to (30), which were obtained from (13) via the transformation of (29). Thus, consider (30) subject to the initial conditions

$$\begin{aligned} G_0(0) &= 0, \\ G_1(0) &= 1, \end{aligned} \tag{34}$$

obtained from (31) and (32) via (29). The complex transmission gain $\mathbf{G}_T$ of the amplifier is then given by

$$\mathbf{G}_T = e^{-\Gamma L_N} \cdot \frac{1}{G_1(L_N)}, \tag{35}$$

where $G_1(L_N)$ is the solution to (30) subject to the initial conditions of (34).

We now seek the solution to (30), with the initial conditions of (34), via Picard's method of successive approximations.[4,5] Assume the $(n - 1)$th approximation to the solution is available; let us denote this approximation by $G_{0(n-1)}(z)$ and $G_{1(n-1)}(z)$. Then the $(n - 1)$th approximation is substituted into the right-hand side of (30) and the right-hand side integrated to yield the nth approximation.

$$\begin{aligned} G_{0(n)}(z) &= j \int_0^z r(s)\, e^{+2\Gamma s} G_{1(n-1)}(s)\, ds. \\ G_{1(n)}(z) &= 1 - j \int_0^z r(s)\, e^{-2\Gamma s} G_{0(n-1)}(s)\, ds. \end{aligned} \tag{36}$$

We take the initial (0th) approximation as simply the initial conditions of (34):

$$\begin{aligned} G_{0(0)}(z) &= 0, \\ G_{1(0)}(z) &= 1. \end{aligned} \tag{37}$$

Writing

$$\begin{aligned} G_{0(n)}(z) - G_{0(n-1)}(z) &= g_{0(n)}(z), \\ G_{1(n)}(z) - G_{1(n-1)}(z) &= g_{1(n)}(z), \end{aligned} \tag{38}$$

we have

$$\begin{aligned} G_{0(n)}(z) &= \sum_{k=1}^{n} g_{0(k)}(z), \\ G_{1(n)}(z) &= 1 + \sum_{k=1}^{n} g_{1(k)}(z). \end{aligned} \tag{39}$$

From (36) and (38), the g's of (39) are given as follows:

$$g_{0(n)}(z) = j \int_0^z r(s)\, e^{+2\Gamma s} g_{1(n-1)}(s)\, ds, \qquad n \geqq 1. \tag{40}$$

$$g_{1(n)}(z) = -j \int_0^z r(s)\, e^{-2\Gamma s} g_{0(n-1)}(s)\, ds, \qquad n \geqq 1. \tag{41}$$

$$g_{0(0)}(z) = 0, \qquad g_{1(0)}(z) = 1. \tag{42}$$

From (40)–(42)

$$\begin{aligned} g_{0(n)}(z) &= 0, \qquad n \text{ even.} \\ g_{1(n)}(z) &= 0, \qquad n \text{ odd.} \end{aligned} \tag{43}$$

Thus only odd terms appear in the top summation of (39), and only even terms appear in the bottom summation of (39).

We next obtain bounds on the magnitudes of the terms in the series of (39), thus showing that these series converge as $n \to \infty$ for all finite z, so that the solutions to (30) subject to the initial conditions of (34) are

$$\begin{aligned} G_0(z) &= \sum_{n=0}^{\infty} g_{0(n)}(z), \\ G_1(z) &= \sum_{n=0}^{\infty} g_{1(n)}(z), \end{aligned} \tag{44}$$

with $g_{0(n)}(z)$ and $g_{1(n)}(z)$ as given by (40)–(42). The analysis is suggested by that of Ref. 3. We show that:

$$| g_{0(n)}(z) | \begin{cases} = 0, & n \text{ even.} \\ \leqq \dfrac{\left[\int_0^z | r(s) |\, ds \right]^n}{n!}, & n \text{ odd.} \end{cases} \tag{45}$$

$$| g_{1(n)}(z) | \begin{cases} \leqq e^{2\alpha z} \dfrac{\left[\int_0^z | r(s) | \, ds\right]^n}{n!}, & n \text{ even.} \\ = 0, & n \text{ odd.} \end{cases} \tag{46}$$

where from (2)

$$\Gamma = -\alpha + j\beta, \qquad \alpha = -\text{Re } \Gamma > 0. \tag{47}$$

Suppose that (46) is true for some even n. Then from (40)

$$\begin{aligned} | g_{0(n+1)}(z) | &\leqq \int_0^z | r(t) | \, e^{-2\alpha t} \, e^{+2\alpha t} \frac{\left[\int_0^t | r(s) | \, ds\right]^n}{n!} \, dt \\ &= \frac{1}{n!} \int_0^{t=z} \left[\int_0^t | r(s) | \, ds\right]^n d\left[\int_0^t | r(s) | \, ds\right] \\ &= \frac{\left[\int_0^z | r(s) | \, ds\right]^{n+1}}{(n+1)!}, \end{aligned} \tag{48}$$

in agreement with (45). Substituting this result into (41),

$$\begin{aligned} | g_{1(n+2)}(z) | &\leqq \int_0^z | r(t) | \, e^{+2\alpha t} \frac{\left[\int_0^t | r(s) | \, ds\right]^{n+1}}{(n+1)!} \, dt \\ &\leqq \frac{e^{+2\alpha z}}{(n+1)!} \int_0^{t=z} \left[\int_0^t | r(s) | \, ds\right]^{n+1} \\ &\qquad \cdot d\left[\int_0^t | r(s) | \, ds\right] \\ &= e^{+2\alpha z} \frac{\left[\int_0^z | r(s) | \, ds\right]^{n+2}}{(n+2)!}, \end{aligned} \tag{49}$$

in agreement with (46). Noting (42) and (43), the results of (45) and (46) hold for all n by induction.

The bounds of (45) and (46) guarantee the convergence of the series solutions of (44) under quite general conditions. It is readily seen that

$$\begin{aligned} | G_0(z) | &\leqq \sinh\left[\int_0^z | r(s) | \, ds\right], \\ | G_1(z) | &\leqq e^{+2\alpha z} \cosh\left[\int_0^z | r(s) | \, ds\right]. \end{aligned} \tag{50}$$

The series solutions of (44) converge for all finite z, so long as the continuous reflection coefficient is absolutely integrable,

$$\int_0^z | r(s) | \, ds < \infty. \tag{51}$$

In particular, note that $r(z)$ may contain δ-functions, as in (16), so that the above bounds may be applied directly to the discrete case of Section I.

The solutions to (30) given by (44) and (40)–(43) thus converge for all finite z in the case of interest. However these formal mathematical solutions have physical significance only when the device to which they apply is stable, i.e., does not oscillate. In the following section we use the bounds of (45) and (46) to obtain a sufficient condition guaranteeing stability in the general case.

IV. BOUNDS ON STABILITY — GENERAL CASE

Consider a general amplifier described by (13) or equivalently by (30). Assume the total length is given by L_N. We may investigate stability as indicated following (8). Replace the continuous reflection coefficient $r(z)$ by $\epsilon \cdot r(z)$, where ϵ is a numerical parameter. Let ϵ be increased from 0, and for each value of ϵ determine the maximum value of the transmission gain $| \mathbf{G}_T |$ as a function of frequency ω. From (35) the maximum value of $| \mathbf{G}_T |$ corresponds to the minimum value of $| G_1(L_N) |$. As ϵ approaches a critical value, denoted above by ϵ_c, $| \mathbf{G}_T |_{\max} \to \infty$ and $| G_1(L_N) |_{\min} \to 0$; if $\epsilon_c > 1$ the original device is stable.

From (40)–(44),

$$G_1(L_N) = 1 + \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} g_{1(n)}(L_N). \tag{52}$$

Noting that $r(z)$ has been temporarily replaced by $\epsilon \cdot r(z)$, for sufficiently small ϵ a lower bound on the magnitude of $G_1(L_N)$ is given by

$$| G_1(L_N) | \geqq 1 - \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} | g_{1(n)}(L_N) | . \tag{53}$$

Both sides of (52) and (53) are functions of frequency ω, through their dependence on the propagation constant β. Using the result of (46) in (53),

$$| G_1(L_N) | \geqq 1 - \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} e^{2\alpha L_N} \frac{\left[\int_0^{L_N} | \epsilon \cdot r(s) | \, ds \right]^n}{n!} . \tag{54}$$

Since the expression on the right-hand side of (54) is independent of the propagation constant β and hence of the frequency ω, this expression is also a lower bound on $| G_1(L_N) |_{\min}$, the minimum value of $| G_1(L_N) |$ as a function of ω.

$$| G_1(L_N) |_{\min} \geqq 1 - \sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} e^{2\alpha L_N} \frac{\left[\int_0^{L_N} | \epsilon \cdot r(s) | \, ds \right]^n}{n!} . \tag{55}$$

As ϵ increases from 0, the lower bound on $| G_1(L_N) |_{\min}$ given by (55) steadily decreases, and for some particular value of $\epsilon \leqq \epsilon_c$ approaches 0. Therefore if

$$\sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{\left[\int_0^{L_N} | \epsilon \cdot r(s) | \, ds \right]^n}{n!} < e^{-2\alpha L_N} \tag{56}$$

stability is guaranteed. If (56) is satisfied for $\epsilon = 1$, then stability is guaranteed for the original amplifier, with reflection coefficient $r(z)$.

Consequently, a sufficient stability condition for an active transmission line with a general continuous reflection coefficient $r(z)$, described by either (13) or (30), assuming the device to be matched at both ends, is given by

$$\sum_{\substack{n=2 \\ n \text{ even}}}^{\infty} \frac{\left[\int_0^{L_N} | r(s) | \, ds \right]^n}{n!} < e^{-2\alpha L_N}. \tag{57}$$

This may be written

$$\cosh\left[\int_0^{L_N} | r(s) | \, ds \right] - 1 < e^{-2\alpha L_N} \tag{58}$$

or further

$$\sinh^2 \left[\frac{\int_0^{L_N} | r(s) | \, ds}{2} \right] < \frac{1}{2} e^{-2\alpha L_N} . \tag{59}$$

Finally, taking the square root of both sides of (59) we obtain

$$\sinh \left[\frac{\int_0^{L_N} | r(s) | \, ds}{2} \right] < \frac{e^{-\alpha L_N}}{\sqrt{2}} \tag{60}$$

or equivalently

$$\int_0^{L_N} | r(s) | \, ds < 2 \sinh^{-1} \frac{e^{-\alpha L_N}}{\sqrt{2}} \tag{61}$$

as sufficient conditions for stability for a general active transmission line with an arbitrary continuous reflection coefficient $r(z)$.

We may now apply the result of (61) to the discrete case of Section I above by making use of the results of Section II. As noted in Section II, if the continuous coupling coefficient $r(z)$ is a series of δ-functions of the form given in (16), then the solution to (13) is identical to that for the discrete case, given in (3), (5), and (6). Since the stability condition of (61) holds true in general, it may be applied to the discrete case by substituting (16) into (61), yielding

$$\sum_{i=1}^{N} \tanh^{-1} | c_i | < 2 \sinh^{-1} \frac{e^{-\alpha L_N}}{\sqrt{2}}. \tag{62}$$

Equation (62) is a sufficient condition for stability for a general active transmission line with arbitrary discrete reflectors, having reflection coefficients c_i located at arbitrary positions along the line. Equation (62) is the result stated in Section I as (9). This inequality is a sufficient condition for stability; if the inequality is satisfied, the device must be stable. This condition is *not* necessary for stability; many devices that violate (62) or (9) are stable.

The weaker bounds of (10) and (12) are readily obtained from the basic result of (62) or (9) by straightforward use of inequalities. From (62) or (9) we must have

$$\tanh^{-1} | c_i | < 2 \sinh^{-1} \frac{e^{-\alpha L_N}}{\sqrt{2}} \qquad i = 1, 2, \cdots N. \tag{63}$$

Since the function $y = \tanh^{-1} x$ is concave upward for $x > 0$,

$$\tanh^{-1} x < \frac{\tanh^{-1} x_m}{x_m} \cdot x, \qquad 0 < x < x_m < 1. \tag{64}$$

Therefore, from (63),

$$\tanh^{-1} | c_i | < \frac{2 \sinh^{-1} \dfrac{e^{-\alpha L_N}}{\sqrt{2}}}{\tanh \left[2 \sinh^{-1} \dfrac{e^{-\alpha L_N}}{\sqrt{2}} \right]} \cdot | c_i |. \tag{65}$$

Therefore if the relation

$$\sum_{i=1}^{N} | c_i | \leqq \tanh \left[2 \sinh^{-1} \frac{e^{-\alpha L_N}}{\sqrt{2}} \right] \tag{66}$$

is satisfied, then the condition of (62) must also be satisfied, so that (66) is a slightly poorer sufficient condition for stability; this result was given in (10). Finally, since the function $y = \tanh [2 \sinh^{-1} x]$ is concave downward for $x > 0$,

$$\tanh [2 \sinh^{-1} x] \geqq \frac{\tanh [2 \sinh^{-1} x_m]}{x_m} \cdot x, \qquad 0 \leqq x \leqq x_m . \tag{67}$$

As a particular instance let us choose $x_m = (1/\sqrt{20}) = 0.2236$; then (67) becomes

$$\tanh [2 \sinh^{-1} x] \geqq 1.863\, x, \qquad 0 \leqq x \leqq \frac{1}{\sqrt{20}} = 0.2236. \tag{68}$$

By using (68) to decrease the right-hand side of (66), we obtain the slightly poorer sufficient condition for stability

$$\begin{aligned} \sum_{i=1}^{N} | c_i | &\leqq 1.863 \frac{e^{-\alpha L_N}}{\sqrt{2}} \\ &= 0.932 \sqrt{2}\, e^{-\alpha L_N}, \qquad 20 \log_{10} e^{\alpha L_N} \geqq 10\, db \end{aligned} \tag{69}$$

given in (12).

V. BOUNDS ON STABILITY — SPECIAL CASE, INCLUDING REFLECTORS OF NOMINALLY EQUAL MAGNITUDE AND SPACING

The bounds on stability derived in Section IV in the general case guarantee stability for the worst possible arrangement of reflectors. Thus in many cases the sum of the magnitudes of the reflectors may far exceed the bound given by (9), (10), or (12) without causing instability.

These general bounds guarantee stability even if we have no information whatever about the distribution of reflectors. If we do have such additional information, it should be possible to make use of it to find improved bounds. As a trivial example, in the treatment of equally spaced, identical reflectors in the previous paper[1] exact stability conditions were obtained; we will see in Section VI that for this case the sum of the magnitudes of the reflectors at the boundary of instability may far exceed that given by (9), (10), or (12).

In the present section we consider a somewhat restricted special case in which the reflection coefficient is almost uniformly distributed in a certain sense. We assume that

$$R \cdot (z - f) \leqq \int_0^z | r(s) | \, ds \leqq R \cdot (z + g), \qquad R > 0, \quad f \geqq 0, \quad g \geqq 0, \tag{70}$$

where R, f, and g are constants. Equation (70) states that the indefinite integral of the absolute magnitude of the reflection coefficient is constrained to lie between two straight lines of the same slope R, separated by the horizontal distance h given by

$$h \equiv f + g, \qquad h \geqq 0. \tag{71}$$

It turns out that the final bounds of this section are better the smaller the separation h. This is to be expected, since the smaller the separation of the two straight lines given by the right- and left-hand sides of (70), the more constrained is the reflection coefficient $r(z)$.

The presence of sufficient length of perfect (i.e., reflectionless) active line at either end will needlessly increase f and hence h in (70) and (71), and hence needlessly degrade the final stability condition given below. Such a length of perfect line cannot affect the stability, but merely alters the gain of the device (assuming it is stable). Therefore for purposes of the present stability analysis sufficient lengths of perfect active line should be removed from each end so that h is minimized, and hence the best possible bound is obtained. Removal of any additional lengths of perfect active line from either end will do neither good nor harm to the final stability condition.

A few examples serve to illustrate the general nature of the restriction of (70). First suppose that $r(z)$ is equal to a (positive) constant,

$$r(z) = r_0 . \tag{72}$$

Then (70) is true with

$$R = r_0 \qquad f = 0, \quad g = 0 \qquad h \equiv f + g = 0. \tag{73}$$

The separation h [of (71)] between the straight lines of the two sides of the inequality of (70) is zero in this case. Equations (13) or (30) are readily solved exactly for the reflection coefficient of (72) by slight modification of the results of (18)–(23), in particular by first replacing

$\tanh^{-1} c_k \to r_0\Delta$ and subsequently replacing any remaining Δ's by $\Delta \to L$, where L is the total length, in these equations. From this exact solution precise stability conditions may be obtained for the case of constant (continuous) reflection coefficient; we expect the bounds of the present section to agree with this exact result when we set $f = g = 0$.

Similarly, the parameters of (73) apply to the bounds of (70) when the (continuous) reflection coefficient is a square wave of constant absolute value r_0, with arbitrary transitions between the $+r_0$ and the $-r_0$ sections.

The above two examples utilize a continuous reflection coefficient. However, our particular present interest lies in some of the discrete cases of the preceding paper.[1] First, consider the case of identical, equally-spaced reflectors of Section II, Ref. 1; the relations of (70) are illustrated for this case in Fig. 2. A less-restricted case is provided by the case of reflectors of identical magnitude but random spacing, where the fluctuation in spacing is very small compared to the average spacing, treated in Section III of Ref. 1. The relations of (70) for this case are shown in Fig. 3; the randomness in spacing has resulted in a slightly wider separa-

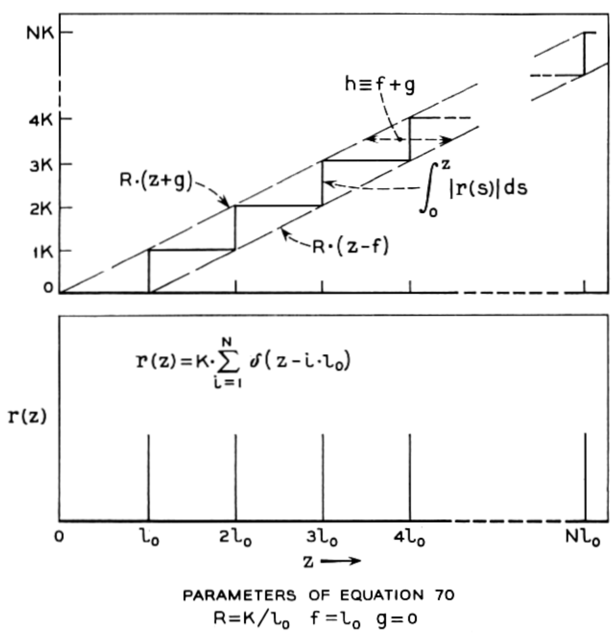

Fig. 2 — Identical, equally spaced reflectors.

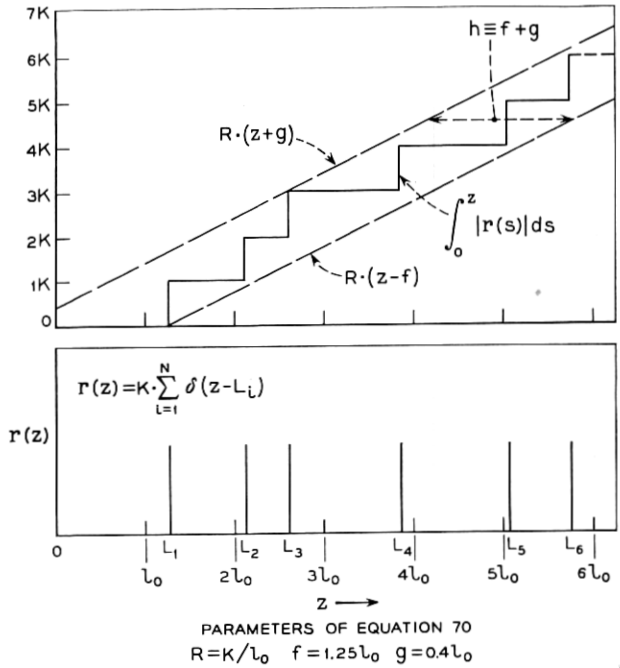

Fig. 3 — Identical, randomly spaced reflectors.

tion than in Fig. 2 between the dashed lines that enclose the staircase curve of

$$\int_0^z | r(s) | \, ds.$$

Since in this case the magnitudes of the reflectors are strictly constant, the "risers" of the staircase have the same size, while the "treads" vary in length. It is clear that if the magnitudes as well as the spacings of the reflectors vary slightly, both the "risers" and the "treads" of the staircase will vary slightly, but otherwise the behavior will be much the same as in Fig. 3, so that the restriction of (70) may be satisfied with small separation between the straight-line bounds.

While the discrete cases of the preceding paragraph, which have reflectors of nominally equal magnitude and spacing, are of principal interest here and supply the motivation for the analysis of the present section, discrete reflectors having quite different distributions from the above may also fall within the restriction of (70) with small separation of the bounding lines; one such case is illustrated in Fig. 4. (Note that reflectors of both signs are indicated in the lower drawing of this figure, by δ-functions with both positive and negative magnitudes.)

The above cases, which satisfy the restriction of (70), may be regarded as having the absolute magnitude of the reflection coefficient more or less constant in a certain sense, in that

$$\int_0^z | r(s) | \, ds$$

is approximately proportional to z [see (70)]. Thus we seek bounds on stability in the case of (70) that are similar to those obtained for constant reflection coefficient [see (72)].

We again use the solution by successive approximation given in Section III above. The discussion of (29)–(43) remains appropriate for our

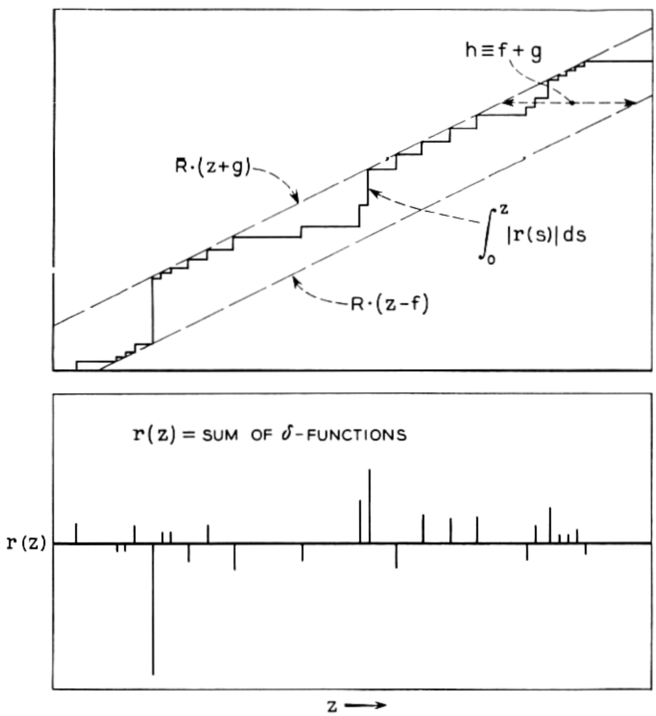

Fig. 4 — More general case satisfying the restrictions of (70).

present purposes. However, greatly improved bounds over those obtained in (44)–(51) may be obtained because of the additional restriction of (70) imposed in the present section; in contrast, the bounds of (44)–(47) of Section III hold true in general, and specifically when the restriction of (70) is not satisfied.

Consider the series solutions of (44). From (42)

$$g_{1(0)}(z) = 1, \qquad g_{0(0)}(z) = 0. \tag{74}$$

Note also (43). We show that:

$$\begin{aligned} | g_{1(n)}(z) | &< R^2 \left(\frac{1}{2\alpha} + h \right)^2 e^{2\alpha z} \\ &\quad \cdot \frac{\left\{ R^2 \left(\frac{1}{2\alpha} + h \right) \left[z + \left(\frac{n}{2} - 1 \right) h \right] \right\}^{(n/2)-1}}{\left(\frac{n}{2} - 1 \right) !} \\ &\qquad\qquad n \text{ even}, n \geqq 2. \\ | g_{1(n)}(z) | &= 0, \qquad\qquad n \text{ odd}. \end{aligned} \tag{75}$$

$$\begin{aligned} | g_{0(n)}(z) | &= 0, \qquad\qquad n \text{ even}. \\ | g_{0(n)}(z) | &< R \left(\frac{1}{2\alpha} + h \right) \\ &\quad \cdot \frac{\left\{ R^2 \left(\frac{1}{2\alpha} + h \right) \left[z + \left(\frac{n-1}{2} \right) h \right] \right\}^{(n-1)/2}}{\left(\frac{n-1}{2} \right) !}, \\ &\qquad\qquad n \text{ odd}. \end{aligned} \tag{76}$$

In (75) and (76), R and h are the parameters of (70) and (71).

First, from (40), (42) or (74), and (47),

$$\begin{aligned} | g_{0(1)}(z) | &\leqq \int_0^z e^{-2\alpha s} | r(s) | \, ds = \int_0^z e^{-2\alpha s} \, d \left[\int_0^s | r(t) | \, dt \right] \\ &= e^{-2\alpha z} \int_0^z | r(t) | \, dt + 2\alpha \int_0^z e^{-2\alpha s} \left[\int_0^s | r(t) | \, dt \right] ds, \end{aligned} \tag{77}$$

where we have made use of integration by parts. Using (70) in (77),

$$
\begin{aligned}
| g_{0(1)}(z) | &\leqq e^{-2\alpha z}\cdot R(z + g) + 2\alpha R \int_0^z e^{-2\alpha s}\cdot(s + g)\, ds \\
&= e^{-2\alpha z}\cdot R(z + g) \\
&\qquad + R\left[\frac{1 - e^{-2\alpha z}}{2\alpha} - ze^{-2\alpha z} + g(1 - e^{-2\alpha z})\right] \\
&= \frac{R}{2\alpha}(1 - e^{-2\alpha z}) + Rg < \frac{R}{2\alpha} + R(f + g),
\end{aligned}
\tag{78}
$$

where in the final step we have used the fact that $f \geqq 0$. Finally, substituting the definition of h from (71) into (78),

$$
| g_{0(1)}(z) | < R\left(\frac{1}{2\alpha} + h\right). \tag{79}
$$

Equation (79) agrees with (76) for $n = 1$.

Next, from (41), (47), and (79),

$$
\begin{aligned}
| g_{1(2)}(z) | &< R\left(\frac{1}{2\alpha} + h\right) \int_0^z e^{+2\alpha s}\, | r(s) |\, ds \\
&= R\left(\frac{1}{2\alpha} + h\right) \int_0^z e^{+2\alpha s}\, d\left[\int_0^s | r(t) |\, dt\right] \\
&= R\left(\frac{1}{2\alpha} + h\right) e^{+2\alpha z} \int_0^z | r(t) |\, dt \\
&\qquad - R\left(\frac{1}{2\alpha} + h\right) 2\alpha \int_0^z e^{+2\alpha s}\left[\int_0^s | r(t) |\, dt\right] ds.
\end{aligned}
\tag{80}
$$

Using (70), (80) becomes

$$
\begin{aligned}
| g_{1(2)}(z) | &< R^2\left(\frac{1}{2\alpha} + h\right) e^{2\alpha z}\cdot(z + g) \\
&\qquad - R^2\left(\frac{1}{2\alpha} + h\right) 2\alpha \int_0^z e^{2\alpha s}(s - f)\, ds \\
&= R^2\left(\frac{1}{2\alpha} + h\right) e^{2\alpha z}\cdot(z + g) \\
&\qquad - R^2\left(\frac{1}{2\alpha} + h\right)\left[\frac{1 - e^{2\alpha z}}{2\alpha} + ze^{2\alpha z} + f(1 - e^{2\alpha z})\right] \\
&< R^2\left(\frac{1}{2\alpha} + h\right) e^{2\alpha z}\left[\frac{1}{2\alpha} + f + g\right].
\end{aligned}
\tag{81}
$$

Finally from (71), (81) becomes

$$| g_{1(2)}(z) | < R^2 \left(\frac{1}{2\alpha} + h \right)^2 e^{2\alpha z}, \tag{82}$$

which agrees with (75) for $n = 2$.

We now establish the bounds of (75) and (76) by induction. Suppose that (75) is true for some even $n \geqq 2$. Then from (40) and (47),

$$| g_{0(n+1)}(z) | < \frac{R^n \left(\frac{1}{2\alpha} + h \right)^{(n/2)+1}}{\left(\frac{n}{2} - 1 \right)!} I \tag{83}$$

where

$$I \equiv \int_0^z \left[s + \left(\frac{n}{2} - 1 \right) h \right]^{(n/2)-1} d \left[\int_0^s | r(t) | \, dt \right]. \tag{84}$$

Integrating (84) by parts,

$$\begin{aligned} I = {} & \left[z + \left(\frac{n}{2} - 1 \right) h \right]^{(n/2)-1} \left[\int_0^z | r(t) | \, dt \right] \\ & - \left(\frac{n}{2} - 1 \right) \int_0^z \left[s + \left(\frac{n}{2} - 1 \right) h \right]^{(n/2)-2} \\ & \cdot \left[\int_0^s | r(t) | \, dt \right] ds. \end{aligned} \tag{85}$$

Using (70) and (71), we have from (85)

$$\begin{aligned} I \leqq {} & \left[z + \left(\frac{n}{2} - 1 \right) h \right]^{(n/2)-1} R(z + g) \\ & - R \left(\frac{n}{2} - 1 \right) \int_0^z \left[s + \left(\frac{n}{2} - 1 \right) h \right]^{(n/2)-2} (s - f) \, ds \\ = {} & \left[z + \left(\frac{n}{2} - 1 \right) h \right]^{(n/2)-1} R(z + g) \\ & - R \int_0^z (s - f) \, d \left[s + \left(\frac{n}{2} - 1 \right) h \right]^{(n/2)-1} \end{aligned}$$

$$
\begin{aligned}
&= \left[z + \left(\frac{n}{2} - 1\right)h\right]^{(n/2)-1} R(z+g) \\
&\quad - R(z-f)\left[z + \left(\frac{n}{2} - 1\right)h\right]^{(n/2)-1} - Rf\left[\left(\frac{n}{2} - 1\right)h\right]^{(n/2)-1} \\
&\quad + R\int_0^z \left[s + \left(\frac{n}{2} - 1\right)h\right]^{(n/2)-1} ds \qquad (86) \\
&= Rh\left[z + \left(\frac{n}{2} - 1\right)h\right]^{(n/2)-1} - Rf\left[\left(\frac{n}{2} - 1\right)h\right]^{(n/2)-1} \\
&\quad + \frac{R}{\left(\frac{n}{2}\right)}\left[z + \left(\frac{n}{2} - 1\right)h\right]^{n/2} - \frac{R}{\left(\frac{n}{2}\right)}\left[\left(\frac{n}{2} - 1\right)h\right]^{n/2} \\
&\leqq \frac{R}{\left(\frac{n}{2}\right)}\left\{\left[z + \left(\frac{n}{2} - 1\right)h\right]^{n/2} + \frac{n}{2}h\left[z + \left(\frac{n}{2} - 1\right)h\right]^{(n/2)-1}\right\},
\end{aligned}
$$

where the last step follows from the preceding one because $n \geqq 2$ [from (75)], $f \geqq 0$ [from (70)], and $h \geqq 0$ [from (71)]. Using the inequality

$$x^k + \epsilon x^{k-1} < [x + (\epsilon/k)]^k, \qquad x \geqq 0 \quad \text{and} \quad \epsilon > 0, \tag{87}$$

(86) yields

$$I < \frac{R}{\left(\frac{n}{2}\right)}\left[z + \left(\frac{n}{2} - 1\right)h + h\right]^{n/2} = \frac{R}{\left(\frac{n}{2}\right)}[z + nh]^{n/2}. \tag{88}$$

Substituting (88) into (83),

$$|g_{0(n+1)}(z)| < R\left(\frac{1}{2\alpha} + h\right)\frac{\left[R^2\left(\frac{1}{2\alpha} + h\right)(z + nh)\right]^{n/2}}{\left(\frac{n}{2}\right)!}. \tag{89}$$

Recalling that n is some even integer $\geqq 2$ in (89), (89) agrees with (76).

Next, from (41) and (47), using the result of (89)

$$|g_{1(n+2)}(z)| < \frac{R^{n+1}\left(\frac{1}{2\alpha} + h\right)^{(n/2)+1}}{\left(\frac{n}{2}\right)!} J \tag{90}$$

where

$$J \equiv \int_0^z e^{+2\alpha s}(s + nh)^{n/2}\, d\left[\int_0^s |\, r(t) \,|\, dt\right]. \tag{91}$$

Integrating (91) by parts,

$$\begin{aligned} J = {} & e^{2\alpha z}(z + nh)^{n/2}\left[\int_0^z |\, r(t)\,|\, dt\right] \\ & - 2\alpha \int_0^z e^{+2\alpha s}(s + nh)^{n/2}\left[\int_0^s |\, r(t)\,|\, dt\right] ds \\ & - \frac{n}{2}\int_0^z e^{+2\alpha s}(s + nh)^{(n/2)-1}\left[\int_0^s |\, r(t)\,|\, dt\right] ds. \end{aligned} \tag{92}$$

Using (70) and (71), we have from (92)

$$\begin{aligned} J \leqq {} & e^{2\alpha z}(z + nh)^{n/2}R(z + g) \\ & - R2\alpha \int_0^z e^{+2\alpha s}(s + nh)^{n/2}\,(s - f)\, ds \\ & - R\frac{n}{2}\int_0^z e^{+2\alpha s}(s + nh)^{(n/2)-1}(s - f)\, ds \\ = {} & e^{2\alpha z}(z + nh)^{n/2}R(z + g) \\ & - R\int_0^z (s - f)\, d\,[e^{+2\alpha s}(s + nh)^{n/2}] \\ = {} & e^{2\alpha z}(z + nh)^{n/2}R(z + g) - R(z - f)e^{2\alpha z}(z + nh)^{n/2} \\ & - Rf(nh)^{n/2} + R\int_0^z e^{+2\alpha s}(s + nh)^{n/2}\, ds \\ = {} & Rhe^{2\alpha z}(z + nh)^{n/2} - Rf(nh)^{n/2} \\ & + \frac{R}{2\alpha}\int_0^z (s + nh)^{n/2}\, d\,(e^{2\alpha s}) \\ = {} & Rhe^{2\alpha z}(z + nh)^{n/2} - Rf(nh)^{n/2} + \frac{R}{2\alpha}e^{2\alpha z}(z + nh)^{n/2} \\ & - \frac{R}{2\alpha}(nh)^{n/2} - \frac{R}{2\alpha}\frac{n}{2}\int_0^z e^{2\alpha s}(s + nh)^{(n/2)-1}\, ds. \end{aligned} \tag{93}$$

From (71), $h \geqq 0$, so that (93) yields

$$J < R\left(\frac{1}{2\alpha} + h\right)e^{2\alpha z}(z + nh)^{n/2}. \tag{94}$$

Substituting (94) into (90),

$$| g_{1(n+2)}(z) | < R^2 \left(\frac{1}{2\alpha} + h\right)^2 e^{2\alpha z} \frac{\left[R^2 \left(\frac{1}{2\alpha} + h\right) (z + nh)\right]^{n/2}}{\left(\frac{n}{2}\right)!}. \quad (95)$$

Recalling that n is some even integer $\geqq 2$ in (95), (95) agrees with (75). Noting (79) and (82), the results of (75) and (76) hold for all n by induction.

We now use the results of (75) together with (74) to obtain bounds on stability for those cases where the reflection coefficient $r(z)$ is restricted as in (70). This analysis is almost identical to that of Section IV, (52)–(57), for the general case, modified by replacing the relation of (46) by that of (75). Thus, making the substitution

$$\frac{\left[\int_0^z | r(s) | \, ds\right]^n}{n!} \to R^2 \left(\frac{1}{2\alpha} + h\right)^2 \cdot \frac{\left\{R^2 \left(\frac{1}{2\alpha} + h\right) \left[z + \left(\frac{n}{2} - 1\right) h\right]\right\}^{(n/2)-1}}{\left(\frac{n}{2} - 1\right)!} \quad (96)$$

throughout (54)–(57), we obtain, corresponding to (57), the following sufficient condition for stability in the present case, after a minor modification of the summation index:

$$R^2 \left(\frac{1}{2\alpha} + h\right)^2 \sum_{m=0}^{\infty} \frac{\left[R^2 \left(\frac{1}{2\alpha} + h\right) (L_N + mh)\right]^m}{m!} < e^{-2\alpha L_N}. \quad (97)$$

L_N is the total length of the device. The summation of (97) is found in closed form by the analysis given in the Appendix. Using the final result of the Appendix (137), the final results of this section may be summarized as follows:

If the reflection coefficient $r(z)$ (continuous, discrete, or a combination of both) satisfies the condition

$$R \cdot (z - f) \leqq \int_0^z | r(s) | \, ds \leqq R \cdot (z + g); \qquad R > 0, f \geqq 0, g \geqq 0 \quad (98a)$$

$$h \equiv f + g; \qquad h \geqq 0.$$

then a sufficient condition for stability of the active line (with reflection) is

$$\frac{\delta\left(1+\frac{1}{2\alpha h}\right)}{1-\delta r_1} < \exp\left[-2\alpha L_N\left(1+\frac{\delta r_1}{2\alpha h}\right)\right] \tag{98b}$$

where

$$\delta \equiv R^2\left(\frac{1}{2\alpha}+h\right)h < \frac{1}{e} \tag{98c}$$

and r_1 is given by

$$r_1 = e^{\delta r_1}, \qquad r_1 < e. \tag{98d}$$

The results of (98) are illustrated in Fig. 5, which shows the maximum value of R for which stability is guaranteed by (98) versus the nominal total gain $20 \log_{10} e^{\alpha L_N}$, with $20 \log_{10} e^{\alpha h}$ as a parameter.

A greatly simplified but slightly poorer version of the stability condition of (98) may be obtained in the high-gain case. As one example, suppose the one-way gain of the active line exceeds 10 db,

$$e^{2\alpha L_N} \geqq 10, \qquad 8.686\ \alpha L_N \geqq 10 \text{ db}, \qquad \alpha L_N \geqq 1.151. \tag{99}$$

If δ satisfies the sufficient stability condition of (98b), it must also satisfy the weaker inequality

$$\delta < \frac{2\alpha h}{1+2\alpha h} e^{-2\alpha L_N}. \tag{100}$$

Substituting (99) into (100),

$$\delta < 0.1. \tag{101}$$

From (98d), r_1 is a monotonic increasing function of δ. Therefore

$$r_1 < 1.118. \tag{102}$$

Further, since from (98d)

$$\delta r_1 = \ln r_1, \tag{103}$$

δr_1 is a monotonic increasing function of r_1, so that

$$\delta r_1 < 0.1118. \tag{104}$$

Now writing out the right-hand side of (98b),

$$\exp\left[-2\alpha L_N\left(1+\frac{\delta r_1}{2\alpha h}\right)\right] = \exp(-2\alpha L_N)\exp\left(-\frac{L_N}{h}\delta r_1\right), \tag{105}$$

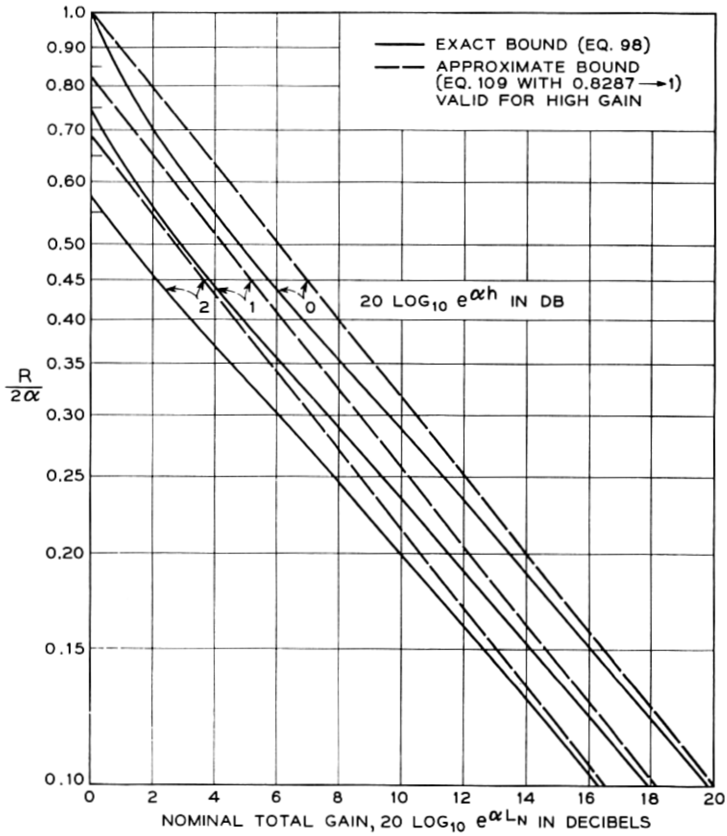

Fig. 5 — Exact and approximate bounds on R for which stability is guaranteed.

we investigate the exponent of the second factor on the right-hand side of (105). From (100),

$$\frac{L_N}{h}\,\delta r_1 < \frac{2\alpha L_N}{1 + 2\alpha h}\,e^{-2\alpha L_N}\cdot r_1 < 2\alpha L_N\,e^{-2\alpha L_N}\cdot r_1\,. \tag{106}$$

The right-hand side of (106) is a monotonic decreasing function of $2\alpha L_N$ for $2\alpha L_N > 1$. Therefore, substituting from (99) and (102), (106) yields

$$\frac{L_N}{h}\,\delta r_1 < 0.2574. \tag{107}$$

$$\exp\left[-\frac{L_N}{h}\,\delta r_1\right] > 0.7731. \tag{108}$$

Finally, using (104) and (108) in (98b), we obtain the following sufficient condition for stability, subject to (98a);

$$R < 0.8287 \frac{2\alpha}{1 + 2\alpha h} e^{-\alpha L_N}; \qquad 8.686\alpha L_N \geqq 10\ db. \tag{109}$$

The stability condition of (109) is slightly poorer than the stability condition of (98b), (98c), and (98d), from which it was derived. As the lower bound on the gain of the active line increases beyond 10 db and approaches ∞, the numerical factor 0.8287 in (109) increases and approaches 1. Equation (109) or a similar result is useful in illustrating the general behavior; however calculations using the basic result of (98) are straightforward. The result of (109), with the numerical factor $0.8287 \rightarrow 1$, is also shown as the dashed curves of Fig. 5, illustrating the way in which this approximate stability condition approaches the exact result of (98) in the high-gain case.

VI. EXAMPLES AND DISCUSSION

Consider first an active line with two discrete reflectors of equal magnitude c at the ends of the line, $z = 0$ and $z = L_2$. c is of course real; for convenience we assume $c > 0$. In this simple case the exact stability condition is readily found, and may be compared with the two bounds derived above. From (8) of Section I, the transmission gain of this device in the stable region is

$$\mathbf{G}_T = \frac{1}{x_{11}}, \tag{110}$$

where from (1)–(7)

$$x_{11} = e^{\Gamma L_2}(1 + c^2 e^{-2\Gamma L_2}). \tag{111}$$

The condition for stability is readily found as described following (8) [this procedure is similar to that used in Section IV, (52)–(57), and Section V, (96)–(97), in obtaining bounds on stability]. Replacing c by ϵc, where ϵ is a numerical parameter greater than 0, and using (2),

$$x_{11} = e^{\Gamma L_2}[1 + (\epsilon c)^2 e^{+2\alpha L_2} e^{-j2\beta L_2}]. \tag{112}$$

For small enough ϵ the minimum value of x_{11}, and hence the maximum value of gain $\mathbf{G}_T$ of (110), occurs at

$$2\beta L_2 = \pm\pi, \pm 3\pi, \cdots . \tag{113}$$

Hence

$$|x_{11}|_{\min} = e^{-2\alpha L_2}[1 - (\epsilon c)^2 e^{+2\alpha L_2}]. \tag{114}$$

As ϵ increases from zero, instability will take place at a value of ϵ for which

$$\begin{aligned} |x_{11}|_{\min} &= 0, \\ (\epsilon c)^2 e^{2\alpha L_2} &= 1. \end{aligned} \tag{115}$$

Hence the original device (with $\epsilon = 1$) will be stable if

$$c < e^{-\alpha L_2}. \tag{116}$$

Equation (116) is an exact condition for stability for the active line described above, with two equal reflectors at the ends. We now compare this exact result with the bounds described above.

Consider first the bound of (9) or (62). This result is a sufficient condition for stability for any arbitrary distribution of discrete reflectors, and so must apply to the special case above. Setting $N = 2$, $c_1 = c_2 = c$, this general bound guarantees stability if

$$\tanh^{-1} c < \sinh^{-1} \frac{e^{-\alpha L_2}}{\sqrt{2}}. \tag{117}$$

Equation (117) yields

$$c < \frac{1}{\sqrt{2}} \frac{e^{-\alpha L_2}}{\sqrt{1 + \frac{1}{2} e^{-2\alpha L_2}}} \tag{118}$$

as a sufficient condition for stability for an active device with two equal reflectors of magnitude c at the ends. Comparing the bound of (118) with the exact stability condition of (116), we see that the general bound of (9) or (62) is conservative in the present special case; i.e., the device with two equal reflectors at the ends remains stable for the reflector magnitude c larger than that guaranteed by the general bound of (9) or (62) by a numerical factor that varies from $\sqrt{3}$ to $\sqrt{2}$ as the gain αL_2 varies from 0 to ∞. Therefore the general bound on stability given in (9) or (62) cannot be improved by a factor greater than $\sqrt{2}$ [i.e., this factor to multiply the right-hand side of (9) or (62)]; of course it may be that no improvement at all is possible, and that some distribution of reflectors can be found for which (9) is satisfied as an equality at the boundary of instability.

Next, consider the bound of Section V, (98), applied to the above special case, i.e., two discrete reflectors of identical magnitude c at the ends of the active line. In (98) we set $h = L_2$, $R = (\tanh^{-1} c)/L_2$, to yield the following (precise) bound on stability:

$$\frac{\delta r_1}{1 - \delta r_1} < \frac{2\alpha L_2}{1 + 2\alpha L_2} e^{-2\alpha L_2} \tag{119a}$$

where

$$\delta \equiv (\tanh^{-1} c)^2 \cdot \frac{1 + 2\alpha L_2}{2\alpha L_2} \tag{119b}$$

and r_1 is given by

$$r_1 = e^{\delta r_1}, \qquad r_1 < e. \tag{119c}$$

The bound on c for stability is readily determined numerically from (119) as a function of αL_2. However, when the one-way gain of the active line is large, $\alpha L_2 \gg 1$, the bound of (98) takes on the form of (109), with the numerical factor $0.8287 \rightarrow 1$ since $\alpha L_2 \gg 1$ (i.e., the gain is taken to be very large, not simply greater than 10 db). Thus the approximate bound on stability in the present case becomes

$$\tanh^{-1} c \lesssim \frac{2\alpha L_2}{1 + 2\alpha L_2} e^{-\alpha L_2}; \qquad \alpha L_2 \gg 1. \tag{120}$$

The symbol $\lesssim$ indicates that the relation of (120) is not a precise bound, but merely gives a good numerical approximation to the precise bound if αL_2 is large enough. Comparison of the (imprecise) bound of (120) with the exact stability condition of (116) shows that in the high-gain case, $\alpha L_2 \gg 1$, where $c \ll 1$, the specialized bound of Section V, (98), yields bounds on the magnitude of the reflection c in the present special case (two equal reflectors at the ends of the active line) that approach those of the exact condition for stability. Consequently the bounds of (98) cannot be further improved (in their present form).

The case of N identical, equally spaced reflectors was studied in Section II of Ref. 1, where simple expressions for stability were found in the high-gain case. If the total gain is large and the gain per section small, comparison of (109) (with the factor $0.8287 \rightarrow 1$) and (98a) with (43) of Ref. 1 shows again that the bound on stability of (98) cannot be further improved. It is of interest to see how close the bounds of (98) come to the exact value corresponding to instability in a few cases of interest. For this purpose we consider examples (*i*), (*ii*), and (*iii*) of

Section II, Ref. 1. In (98) we set

$$h = l, \qquad R = \frac{\tanh^{-1} c}{l}, \tag{121}$$

and compute upper bounds on $| c |$ that guarantee stability. It is also of interest to compare the general bound of (9) or (62) for this case. Table I summarizes these results. The bounds of (98) are quite good when the total gain is high, $\alpha L_N \gg 1$, and when the gain corresponding to the distance l is small, $\alpha l \ll 1$; for these conditions the stability condition of (98) gives much better results than the more general stability condition of (9), because in the former we have made use of additional information regarding the distribution of reflectors.

Table I — Identical, Equally Spaced Reflectors

N = number of reflectors
Gain (db) = $20 \log_{10} e^{N\alpha l} \equiv 20 \log_{10} e^{\alpha L_N}$ = one-way gain of active line in db
$| c |_{max}$ = maximum value of $| c |$ for stability, as determined in Section II, Ref. 1
Bound on $| c |$ — (98) = maximum value of $| c |$ for which stability is guaranteed by (98)
Bound on $| c |$ — (9) or (62) = maximum value of $| c |$ for which stability is guaranteed by (9) or (62).

Case (Sec. II, Ref. 1)	N	Gain, db	$\| c \|_{max}$ (Sec. II, Ref. 1)	Bound on $\| c \|$ (98)	Bound on $\| c \|$ (9) or (62)
(*i*)	30	30	0.00860	0.00590	0.00149
(*ii*)	300	30	0.000860	0.000710	0.000149
(*iii*)	50	5	0.0650†	0.01105	0.0130

† Note that for this case in Ref. 1 the high-gain approximation given there was inappropriate, so that this result was obtained by use of a computer.

Finally, we consider the application of the above stability conditions to some of the problems involving random reflectors studied in Ref. 1. The stability of the various deterministic cases discussed above in the present section has been treated exactly here or in Ref. 1 without using the new results of the present paper; these cases have been discussed in the present section both to show that any possible improvement in these general stability conditions must be quite small, and to provide partial confirmation of these results. However, the application of (9) and (98) to cases involving random reflectors provides the principal motivation for the present analysis, since no other information whatever is available regarding stability in these cases.

Let us consider the example of the first part of Section IV, Ref. 1, in which the average normalized loss and the rms loss fluctuation were determined for an amplifier with reflections having identical magnitude but random spacing. The following parameters were chosen for this illustration:

$$
\begin{aligned}
l_k &= \text{spacing between } (k-1)\text{th and } k\text{th reflectors [(3) and Fig. 1]} \\
l_0 &= \langle l_k \rangle, \text{ average value of } l_k \text{, independent of } k \\
c_k &= \text{magnitude of } k\text{th reflection coefficient [(3) and Fig. 1]} \\
c_k &= c_0 \text{; all reflectors identical, } c_0 > 0 \\
N &= 30, \text{ number of sections} \\
20 \log_{10} e^{N\alpha l_0} &= 30 \text{ db, nominal total gain} \\
20 \log_{10} e^{\alpha l_0} &= 1 \text{ db, nominal gain per section.}
\end{aligned} \tag{122}
$$

The following assumptions were made in these calculations of Ref. 1:

(a) l_k is always a large number of wavelengths;

$$\beta l_k \gg 2\pi, \qquad \beta l_0 \gg 2\pi. \tag{123}$$

(b) The distribution of the l_k about their mean l_0 is very narrow with respect to the mean, but wide compared to $2\pi/\beta$; further, this distribution is smooth and symmetrical about l_0.

The probability density for l_k did not have to be further specified for the calculation of average loss and rms loss fluctuation in Ref. 1. (Note however that in the calculations of Ref. 1 for the covariance of the loss, the specific form of the probability density for l_k must be known, and was assumed to be Gaussian in Ref. 1.) The average loss and the rms loss fluctuation for the amplifier of (122) were given in Fig. 9 of Ref. 1 versus c_0, the magnitude of the reflections. These curves were shown dotted for $c_0 > 0.00860$, because it was known that instability is possible in this range, in particular for $l_k = l_0$, i.e., equally spaced reflectors [see Section II, Ref. 1 and case (i), Table I]. However it was noted that this was only a symbolic reminder of the unsolved question of stability; these results are valid for small enough c_0, but how small was not known from the results of Ref. 1.

We illustrate the utility of the results of the present paper by applying them to this problem; these results provide useful information concerning stability in this case, and of course in many similar problems. For convenience we make one further assumption in addition to those mentioned following (122):

(c) The distribution of l_k about its mean l_0 is strictly bounded; in particular

$$| l_k - l_0 | \leqq \nu l_0 ; \tag{124}$$

further, we assume for convenience that

$$\nu < 1. \tag{125}$$

ν is in (124) the upper bound on the fractional deviation in spacing from its average value; the restriction of (125) requires that $l_k \geqq 0$, and so prevents the order of the reflectors from being altered. In practical cases we will be interested in small values of ν,

$$\nu \ll 1. \tag{126}$$

We determine upper bounds on the reflector magnitude c_0 that guarantee stability, as a function of ν, the maximum fractional deviation in spacing between reflectors. For $\nu = 0$ the reflectors are equally spaced; Ref. 1 or Table I shows that stability is guaranteed if

$$c_0 < 0.00860, \qquad \nu = 0. \tag{127}$$

Next, the bound of (9) guarantees stability independently of the particular distribution of reflectors. Since however the total length may vary somewhat, we must in (9) set

$$L_N \equiv L_{30} = 30l_0(1 + \nu), \tag{128}$$

yielding

$$c_0 < 0.00149(0.03162)^{\nu} \tag{129}$$

as a sufficient stability condition.

Finally, we apply the bound of (98) to this example. We set

$$R = \frac{\tanh^{-1} c_0}{l_0}, \tag{130}$$

$$h = (1 + 60\nu)l_0 \tag{131}$$

and make use of (128) in (98) to obtain a sufficient stability condition.

The sufficient stability conditions of (127), (129), and (98) are plotted in Fig. 6; the result of (129) is identified as originating from (9), and that of (127) from Section II of Ref. 1. The curves of Fig. 6 have been plotted out to fractional spacing variations ν of 10 per cent; over this region the stability condition of (98) is superior to that of (9). However the bound of (9) [i.e., (129)] will be superior to that of (98) for large enough ν. Note that the factor $(0.03162)^{\nu}$ in (129) arises from the fact that the total length and hence the total gain is subject to statistical fluctuation [a similar factor occurs in using (98) for the problem]; in the range of probable interest, i.e., for very small fractional spacing fluctuations ν, this numerical factor will be close to 1. The fact that the limit of the bound of (98) as $\nu \to 0$ is substantially below the maximum value of c_0 given by (127) is due to the fact that the nominal gain per section in the example of (122) is 1 db, which is not too small;

as the gain per section decreases these two quantities will approach each other, as indicated above.

These results, plotted on Fig. 6, show that the range of c_0 over which the calculations of Section IV of Ref. 1 are guaranteed to be valid. If the maximum fractional variation in the spacing between reflectors is very small, then the results plotted on Fig. 9 of Ref. 1 are valid for c_0 up to approximately 0.00590.

The stability conditions of (9) and (98) may be applied to a variety of similar problems. In the above example we have found the maximum value of c_0 for which stability is guaranteed, i.e., for which the probability of oscillation is zero, as a function of the maximum departure of the spacing between reflectors from its average value. The results of (9) and (98) may also be used to determine an upper bound on the probability of oscillation in similar problems where no absolute guarantee of

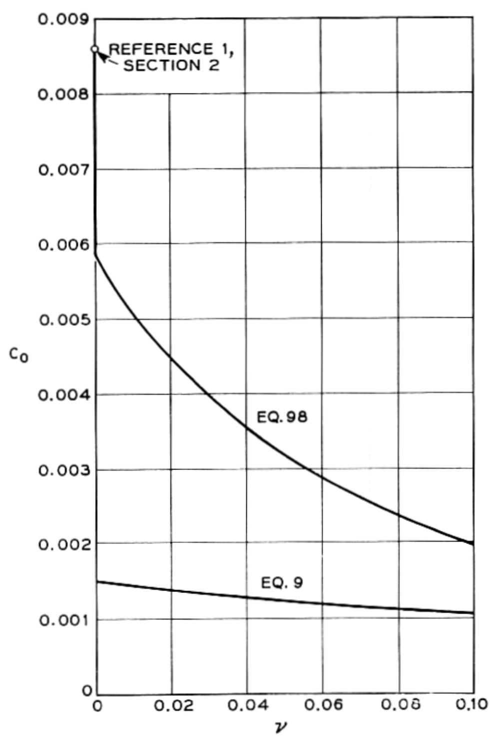

Fig. 6 — Bounds on magnitude of coupling coefficient to guarantee stability for amplifier of (122), with reflectors of identical magnitude and nominally equal spacing.

stability can be given, e.g., perhaps in cases where the probability distribution for the spacing deviations is not strictly bounded.

The main emphasis of the present paper has been on the discrete case; the continuous case was introduced only as an intermediate step leading to the desired results. However, it is clear that related problems with continuous reflection may be studied for stability using the general results derived above.

Finally, the present calculations have assumed for definiteness a rather special model; i.e., the forward and backward gains have been assumed equal and a particular form has been taken for the matrix of the discrete reflectors. These assumptions are not essential to the analysis; similar results can be derived for many related cases of interest, such as systems using isolators to partially attenuate the backward waves, etc.

VII. ACKNOWLEDGMENT

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APPENDIX

Summation of the Series $S = \sum_{n=0}^{\infty} \frac{(z + \delta n)^n}{n!}$

The summation of (97) was initially performed by a method suggested by S. O. Rice, employing contour integration; this method is straightforward but lengthy. A much shorter analysis presented by the unknown referee is given here. It has been shown that[6]

$$e^{ax} = 1 + \sum_{n=1}^{\infty} \frac{a(a - nb)^{n-1}}{n!} y^n \tag{132}$$

where

$$y = xe^{bx} \quad \text{and} \quad | yb | < (1/e). \tag{133}$$

Differentiate (132) with respect to y and then set $y = 1$ to obtain

$$\frac{e^{(a-b)x}}{1 + bx} = \sum_{n=0}^{\infty} \frac{[(a - b) - nb]^n}{n!} \tag{134}$$

where

$$x = e^{-bx} \quad \text{and} \quad | b | < (1/e). \tag{135}$$

Finally, set

$$a = z - \delta, \qquad b = -\delta, \qquad x = r_1 \tag{136}$$

to obtain

$$\sum_{n=0}^{\infty} \frac{(z + \delta n)^n}{n!} = \frac{e^{r_1 z}}{1 - \delta r_1}, \qquad 0 \leqq \delta < \frac{1}{e}$$

where r_1 is given by (137)

$$r_1 = e^{\delta r_1}, \qquad r_1 < e.$$

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