Overflow Traffic from a Trunk Group with Balking*

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A stream of telephone calls is submitted to a group of trunks, the first-choice group, according to a recurrent process. We allow balking on this trunk group; i.e., if a call finds k of the first-choice trunks busy it may be served, with probability p_k, or may fail to be served, with probability q_k. A call which fails to receive immediate service on the first-choice trunk group is submitted to a second-choice trunk group, the overflow group. We also allow balking on the overflow group. Calls which fail to receive immediate service on the overflow group are lost to the system. Holding times have negative-exponential distribution.

We give methods for finding the joint distributions of numbers of busy trunks on the first-choice and overflow groups, at overflow instants (i.e., instants at which calls are submitted to the overflow group), at arrival instants, and at arbitrary instants. We consider the transient as well as the limiting distributions (and demonstrate the existence of the limiting distributions).

The methods developed are illustrated by several examples. Numerical results are given for the blocking in the particular case that the first-choice group constitutes a random slip, while the overflow group is full-access (common).

I. INTRODUCTION

1.1 *Balking and Overflow Traffic*

A telephone call is submitted to a group of m trunks. This call may fail to occupy a trunk, even though not all m trunks are busy. There may be a number of reasons for such a failure, e.g.: the calling line may not have access to any *idle* trunks, some equipment other than the

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trunk itself may be required to complete a connection and this equipment may be busy, or the m trunks may be merely first-stage links in a connecting network and there may be no free path through this network. Whatever the cause of the failure, we shall say that the submitted call *balks* (although the word is perhaps more appropriate in queueing theory applications). In this paper we shall restrict ourselves to the case in which the probability of balking depends only on the number of busy trunks: if an arriving call finds k trunks busy, it is served, with probability p_k, or balks with probability q_k ($p_k + q_k = 1$). If all trunks are busy, an arriving call cannot be served, and therefore $q_m = 1$. Thus we subsume blocking under the term balking.

The traffic which overflows from a trunk group with balking has different characteristics from that which overflows from a *full-access* group. [By a full-access trunk group we mean one for which $q_k = 0$ ($k < m$), $q_m = 1$.] Suppose *recurrent* traffic is submitted to a full-access group (when we refer to recurrent input traffic we mean that the intervals between arriving calls are independent, identically distributed random variables). Suppose further that the holding times of calls have negative-exponential distribution. Then, as Conny Palm[1] has shown, the overflow traffic is also recurrent. This is not the case for traffic overflowing from a trunk group with balking.

The traffic which balks on the first-choice group may be submitted to an overflow group of, say, M trunks. There may also be balking on the overflow group. Now L. Takács[2] has treated in detail the process of numbers of busy trunks in a trunk group with balking to which a recurrent stream of calls of negative-exponential holding times is submitted. Thus, if the first-choice group is full-access, we know how to describe what goes on on the overflow group. However, if the first-choice group is not full-access, the stream of calls submitted to the overflow group is not recurrent, and therefore further analysis is required to describe the process of numbers of busy trunks on the overflow group. We attempt to treat this problem in the present paper; in so doing, we are led to consider the joint distribution of numbers of busy trunks on the first-choice and overflow groups, which is also of interest in itself.

1.2 *Mathematical Description of the Problem, and Some Notation*

Calls are submitted to a group of m trunks, the first-choice group, at successive instants $\tau_1, \tau_2, \cdots, \tau_n, \cdots$. The interarrival times, $\theta_n = \tau_n - \tau_{n-1}$ ($n = 2, 3, 4, \cdots$), are independent, identically distributed random variables with common distribution function

$$P\{\theta_n \leqq x\} = F(x),$$

and we specify further that $P\{\tau_1 \leqq x\} = F(x)$. We assume that the $\{\theta_n\}$ are not *lattice variables* (i.e., that the interarrival times are not confined to multiples of a constant), that $F(0) = 0$ and that

$$0 < \alpha < \infty,$$

where

$$\alpha = \int_0^\infty x dF(x)$$

is the mean interarrival time.

Note that the class of recurrent inputs just described includes, among others: Poisson arrivals, equally spaced arrivals, and, as previously remarked, arrivals which are themselves overflows from a full-access trunk group to which a Poisson process of calls with negative-exponential holding time is submitted.

If the nth call receives service, then its holding time is a random variable, χ_n. The $\{\chi_n\}$ are independent and identically distributed, with common distribution function

$$P\{\chi_n \leqq x\} = \begin{cases} 1 - e^{-x} & \text{for } x \geqq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

and are independent of the arrival process $\{\tau_n\}$.

Note that we are measuring time in units of the mean holding time; thus $a = 1/\alpha$ is the submitted traffic in erlangs.

An arriving call which finds k trunks of the first-choice group busy is served with probability p_k, or balks with probability q_k. We have

$$p_k + q_k = 1 \qquad (k = 0, 1, \cdots, m)$$

$$q_m = 1.$$

A call which balks on the first-choice group is immediately submitted to a second group of M trunks, the *overflow group* (we allow the case $M = \infty$). We denote the sequence of instants at which calls are submitted to the overflow group by $\{T_N\}$ $(N = 1, 2, 3, \cdots)$. If such a call finds K trunks of the overflow group busy, it is served, with probability G_K, or balks, with probability H_K. We have

$$G_K + H_K = 1 \qquad (K = 0, 1, \cdots, M)$$

$$H_M = 1 \qquad (\text{if } M < \infty).$$

We make the following plausible restriction on the balking probabilities

$$p_k > 0 \quad \text{for} \quad k < m$$
$$G_K > 0 \quad \text{for} \quad K < M.$$

A call which balks on the overflow group is said to be *blocked*. It immediately disappears from the system and is not resubmitted; i.e., lost calls are cleared.

We now define the following random variables:

$$\begin{aligned}
\xi(t) &= \text{number of busy trunks on first-choice group at time } t \\
\xi_n &= \xi(\tau_n -) \\
\xi_n^{\,o} &= \xi(T_N -) \text{ (the superscript ``}o\text{'' means ``overflow''.)} \\
\Xi(t) &= \text{number of busy trunks on overflow group at time } t \\
\Xi_n &= \Xi(\tau_n -) \\
\Xi_N^{\,o} &= \Xi(T_N -).
\end{aligned}$$

We also define the following probabilities, which it will be our object to determine:

$$P\{\xi_N^{\,o} = k,\ \Xi_N^{\,o} = K\} = P^o(k,K,N)$$
$$\lim_{N\to\infty} P^o(k,K,N) = P^o(k,K)$$
$$P\{\xi_n = k,\ \Xi_n = K\} = P(k,K,n)$$
$$\lim_{n\to\infty} P(k,K,n) = P(k,K)$$
$$P\{\xi(t) = k,\ \Xi(t) = K\} = P(k,K,t)$$
$$\lim_{t\to\infty} P(k,K,t) = P^*(k,K).$$

When one of the variables k or K in one of these probabilities is not written, it is understood to be summed over, e.g.

$$P(k,t) = \sum_{K=0}^{M} P(k,K,t).$$

A quantity of particular interest in applications is the blocking

$$B = \sum_{k=0}^{m} \sum_{K=0}^{M} q_k H_K P(k,K).$$

We shall also have occasion to refer to the blocking on the first-choice group

$$b = \sum_{k=0}^{m} q_k P(k).$$

Further notation will be introduced as it is needed. The notation will as far as possible conform to that of Takács.[2] We shall, when possible, use lower-case letters to refer to the first-choice group and the corresponding capital letters for the overflow group. Equations of Ref. 2 will be denoted by a T: e.g., "(T44)." We note here only the following definitions:

$$\varphi(s) = \int_0^\infty e^{-sx}\, dF(x)$$

$$C_r = \prod_{j=1}^{r} \frac{\varphi(j)}{1 - \varphi(j)} \qquad (C_0 = 1)$$

$$C_r(s) = \prod_{j=0}^{r} \frac{\varphi(s+j)}{1 - \varphi(s+j)} \qquad (C_{-1}(s) \equiv 1).$$

1.3 *Previous Results*

Let us denote the interoverflow times by $\Theta_N = T_N - T_{N-1}$. As we have mentioned, if the first-choice group is full-access, the $\{\Theta_N\}$ are independent and identically distributed. In this case let us denote their common distribution function by

$$G(x) = P\{\Theta_N \leqq x\}$$

with Laplace-Stieltjes transform

$$\gamma(s) = \int_0^\infty e^{-sx}\, dG(x).$$

Takács[3] solves a recurrence of Palm[1] to obtain

$$\gamma(s) = \frac{\displaystyle\sum_{r=0}^{m} \binom{m}{r} \frac{1}{C_{r-1}(s)}}{\displaystyle\sum_{r=0}^{m+1} \binom{m+1}{r} \frac{1}{C_{r-1}(s)}}. \tag{1}$$

A. Descloux[4] gives convenient recurrence formulas for calculating $\gamma(s)$ and the moments of $G(x)$ in the case of Poisson input, i.e., when

$$F(x) = \begin{cases} 1 - e^{-ax} & (x \geqq 0) \\ 0 & (x < 0) \end{cases}.$$

Some results exist for $P(k,K)$ in the case of Poisson input [for which, and only for which, as we shall see, $P^*(k,K) = P(k,K)$]. The first of these is due to L. Kosten.[5] He considers a full-access first-choice group and an infinite full-access overflow group. Let us denote binomial moments with respect to the overflow group by

$$U(k,R) = \sum_{K=R}^{M} \binom{K}{R} P(k,K).$$

Then Kosten finds

$$U(k,R) = C_0^R(a) \frac{C_0^m(a) C_R^k(a)}{C_R^m(a) C_{R+1}^m(a)}. \tag{2}$$

(See also the appendix by J. Riordan to a paper of R. I. Wilkinson.[6]) The polynomials in (2) are defined by

$$C_R^k(a) = \sum_{j=0}^{k} \binom{j+R-1}{j} \frac{a^{k-j}}{(k-j)!} \tag{3}$$

so that $C_0^k(a) = a^k/k!$, if we agree that $\binom{-1}{0} = 1$. J. Riordan (Ref. 7, p. 120) remarks that these polynomials are closely related to the Poisson-Charlier polynomials $C_n(x,a)$; in fact

$$C_R^k(a) = C_k(-R,a).$$

E. Brockmeyer,[8] N. Bech,[9] and K. Lundkvist[10] consider a problem which differs from Kosten's only in that M is finite ($G_M = 0$). Brockmeyer finds

$$P(k,K) = \sum_{S=0}^{M-K} (-1)^S Y_{S+K} \binom{S+K}{K} C_{K+S}^{k-S}(a) \tag{4}$$

where

$$Y_S = \sum_{J=S}^{M} (-1)^{J-S} \binom{J-1}{S-1} a_J \qquad (S = 1, 2, \cdots, M)$$

$$Y_0 = \frac{1}{C_1^{m+M}(a)}$$

$$a_J = \frac{1}{C_1^{m+M}(a)} \cdot \frac{1}{C_J^m(a)} \sum_{L=J}^{M} \binom{L-1}{J-1} C_0^{m+L}(a).$$

We do not consider here more complicated trunking situations (graded multiples, alternate routing arrangements in which the overflow group is at the same time the first-choice group for other sources of traffic). See, however, Wilkinson,[6] and R. Syski (Ref. 11, chapters 7, 8, 10).

Takács[2] gives, for arbitrary q_k, methods of finding $P(k,n)$, $P(k)$, $P(k,t)$, and $P^*(k)$. Thus in what follows we shall take the attitude that everything we need concerning the first-choice group only is, in principle, known.

1.4 *An Example*

This paper grew out of the following problem, in which both balking and overflow are involved. Subscriber lines are connected to the m trunks of the first-choice group in such a way that each line has access to only γ of them. We refer to a particular set of γ trunks as the access pattern for a particular line or group of lines. Equal traffic is submitted to each of the $\binom{m}{\gamma}$ possible access patterns. When a connection is made, any idle trunk in the subscriber's access pattern is equally likely to be selected. This arrangement is referred to as a *random slip*, or *Erlang's ideal grade*. It is easy to see that the balking probabilities are

$$q_k = 0, \quad \text{for} \quad 0 \leqq k < \gamma, \quad \text{and}$$

$$q_k = \frac{\binom{k}{\gamma}}{\binom{m}{\gamma}}, \qquad \text{for} \quad \gamma \leqq k \leqq m.$$

Traffic which balks on the first-choice group is submitted to a full-access overflow group of M trunks. If a call is blocked on the overflow group, it is lost.

Such an arrangement may be economically desirable. The average traffic carried per trunk (for a given blocking probability, B) is less than for a full-access group of $m + M$ trunks, but the traffic per crosspoint is greater. Knowing the costs of trunks and of crosspoints, and given $m + M$ and the desired value of B, one wishes to select γ and m so as to minimize the cost per unit of carried traffic. We shall give some numerical results for this arrangment.

II. THE STATE OF THE SYSTEM AT OVERFLOW INSTANTS

2.1 *Transient Behaviour*

Unless the first-choice group is full-access, the overflow process $\{T_N\}$ is not recurrent and the sequence $\{\Xi_N^o\}$ is not a Markov chain. However, the sequence of pairs of random variables $\{\xi_N^o, \Xi_N^o\}$ is a homogeneous Markov chain. This may be seen as follows. Suppose we know that $\xi(T_N-) = k$ and $\Xi(T_N-) = K$. T_N is an arrival instant; because the arrival process is recurrent and independent of the holding times, the history of the system before T_N has no effect on the epochs of future arrivals. T_N is an overflow instant; whether or not the overflowing call is accepted by the overflow group depends only on the value of K. Because of the exponential distribution of holding times, the stochastic behaviour of the system after T_N is independent of the ages of calls in progress at T_N. Thus the values of $\xi(T_N-)$ and $\Xi(T_N-)$ determine the whole future stochastic behaviour of the system. Therefore we are led first to a consideration of the probabilities $P^o(k,K,N)$.

If $\xi(t) = k$, $\Xi(t) = K$, then we say that at time t the system is in the state (k,K). The values of $\xi_N{}^o$ are limited to those k for which $q_k > 0$. We denote the set of such integers k by $\mathcal{A}$. As initial conditions we take $\xi(0+) = i$, $\Xi(0+) = I < \infty$. (It is not required that $i \in \mathcal{A}$.) Under these initial conditions, we seek $P^o(k,K,N)$ for $k \in \mathcal{A}$; $K = 0, 1, 2, \cdots$; $N = 1, 2, 3, \cdots$.

Let us now define the following quantities:

$$\begin{aligned} G_{jk}(x) &= P\{\xi_{N+1}{}^o = k,\ \Theta_{N+1} \leqq x \mid \xi(T_N+) = j\} \\ &= P\{\xi_{N+1}{}^o = k,\ \Theta_{N+1} \leqq x \mid \xi_N = j\} \\ &= P\{\xi_1{}^o = k,\ T_1 \leqq x \mid \xi(0+) = j\} \end{aligned}$$

with Laplace-Stieltjes transform

$$\gamma_{jk}(s) = \int_0^\infty e^{-sx}\, dG_{jk}(x)$$

$$U^o(k,R,N) = \sum_{K=R}^{M} \binom{K}{R} P^o(k,K,N) \qquad (R = 0, 1, \cdots, M)$$

$$V^o(k,R,N) = \sum_{K=R}^{M} \binom{K}{R} G_K P^o(k,K,N) \qquad (R = 0, 1, \cdots, M)$$

$$V^o(k,-1,N) = 0.$$

We may now state:

Theorem 1: The distribution $P^o(k,K,N)$ is uniquely determined by the binomial moments $U^o(k,R,N)$; the latter are determined by

$$U^o(k,R,1) = \binom{I}{R} \gamma_{ik}(R) \tag{5}$$

$$U^o(k,R,N+1) = \sum_{j \in \mathcal{A}} \gamma_{jk}(R)[U^o(j,R,N) + V^o(j,R-1,N)]. \tag{6}$$

Proof: The transition probabilities for the homogeneous Markov chain $\{\xi_N{}^o, \Xi_N{}^o\}$ are given by

$$p^o(j,J;k,K) = P\{\xi_{N+1}{}^o = k,\ \Xi_{N+1}{}^o = K \mid \xi_N{}^o = j,\ \Xi_N{}^o = J\}$$
$$= \int_0^\infty P\ \{\Xi_{N+1}{}^o = K \mid \Xi_N{}^o = J, \Theta_{N+1} = x\}\ dG_{jk}(x).$$

It is easy to see that

$$P\{\Xi_{N+1}{}^o = K \mid \Xi_N{}^o = J,\ \Theta_{N+1} = x\}$$
$$= G_J \binom{J+1}{K} e^{-xK}(1-e^{-x})^{J+1-K} + H_J \binom{J}{K} e^{-xK}(1-e^{-x})^{J-K}.$$

Thus

$$p^o(j, J; k, K) = \int_0^\infty dG_{jk}(x) \left[G_J\binom{J+1}{K} e^{-xK}(1-e^{-x})^{J+1-K} + H_J\binom{J}{K} e^{-xK}(1-e^{-x})^{J-K}\right]. \tag{7}$$

Now

$$P^o(k,K,N+1) = \sum_{j=0}^{m} \sum_{J=0}^{M} p^o(j,J;k,K)P^o(j,J,N). \tag{8}$$

Substituting (7) in (8), and taking the Rth binomial moment with respect to the overflow group, we obtain

$$U^o(k, R, N+1) = \sum_{j\in\alpha} \sum_{J=0}^{M} \int_0^\infty dG_{jk}(x) \left[G_J\binom{J+1}{R} + H_J\binom{J}{R}\right] e^{-xR}\ P^o(j, J, N)$$
$$= \sum_{j\in\alpha} \sum_{J=0}^{M} \gamma_{jk}(R) \left[\binom{J}{R} + G_J\binom{J}{R-1}\right] P^o(j, J, N)$$
$$= \sum_{j\in\alpha} \gamma_{jk}(R)\ [U^o(j, R, N) + V^o(j, R-1, N)],$$

which is (6).

For $N = 1$, we have

$$P^o(k, K, 1) = \int_0^\infty dG_{ik}(x) \binom{I}{K} e^{-xK}(1 - e^{-x})^{I-K}$$

so that

$$U^o(k, R, 1) = \int_0^\infty dG_{ik}(x) \binom{I}{R} e^{-Rx} = \binom{I}{R} \gamma_{ik}(R),$$

which is (5).

From the definition of $U^o(k,R,N)$, we have

$$\sum_{R=K}^{M} (-1)^{R-K} \binom{R}{K} U^o(k, R, N) = \sum_{R=K}^{M} (-1)^{R-K} \binom{R}{K} \sum_{J=R}^{M} \binom{J}{R} P^o(k, J, N). \tag{9}$$

Now, for any finite N the double series on the right contains a finite number of terms, even if $M = \infty$. This is so because

$$P^o(k,J,N) = 0 \quad \text{for} \quad k + J \geqq i + I + N,$$

and we have assumed $I < \infty$.

Thus the double series can be rearranged, and one obtains readily that the binomial moments determine the probabilities according to

$$P^o(k, K, N) = \sum_{R=K}^{M} (-1)^{R-K} \binom{R}{K} U^o(k, R, N). \tag{10}$$

In (5) and (6), the quantities $\gamma_{jk}(R)$ occur as coefficients. We regard these coefficients as known because they can be expressed in terms of certain quantities determined by Takács.[2] Let

$$M_{ik}(x) = \mathbf{E}\,\{\text{number of } \tau_n \text{ in } (o,x] \text{ for which } \xi_n = k \mid \xi(0+) = i\},$$

with Laplace-Stieltjes transform

$$\mu_{ik}(s) = \int_0^\infty e^{-sx}\, dM_{ik}(x).$$

Takács gives a method for finding the $\mu_{ik}(s)$ [(T70), in which, however, the index i is implicit]. The way in which the quantities $\mu_{jk}(R)$ determine the $\gamma_{jk}(R)$ is expressed in the following lemma (in which, it is to be noted, values of the indices j,k, etc. are no longer restricted to the set $\mathfrak{A}$).

Lemma 1: Define $M_{ik}{}^o(x) = \mathbf{E}\,\{$*number of* T_N *in* $(0,x]$ *for which* $\xi_N{}^o =$

$k \mid \xi(0+) = i\}$, with Laplace-Stieltjes transform

$$\mu_{ik}^{o}(s) = \int_0^\infty e^{-sx}\, dM_{ik}^{o}(x).$$

Let $\mu^{o,R}$ be the square matrix with elements $\mu_{jk}^{o}(R)$; $j,k = 0, 1, \cdots, m$.
Let γ^R be the square matrix with elements $\gamma_{jk}(R)$; $j,k = 0, 1, \cdots, m$. Then, for $R = 1, 2, \cdots$,

$$\gamma^R = \mu^{o,R}(E + \mu^{o,R})^{-1} \tag{11}$$

where E is the $(m+1)$ by $(m+1)$ unit matrix.

Since, obviously

$$\mu_{jk}^{o}(R) = q_k \mu_{jk}(R), \tag{12}$$

(11) provides the desired relation between the $\gamma_{jk}(R)$ and the $\mu_{jk}(R)$.

Proof: We shall first show that

$$\mu_{jk}^{o}(R) = \gamma_{jk}(R) + \sum_{l=0}^{m} \gamma_{jl}(R)\mu_{lk}^{o}(R) \tag{13}$$

for $R = 1, 2, \cdots$.

Suppose $\xi(0+) = j$, and consider a given R-tuple of trunks on the overflow group which are all busy at $t = 0+$. If $T_1 = x$, the probability that the overflow at T_1 will find this R-tuple still busy is e^{-Rx}.

Thus

$$\gamma_{jk}(R) = \int_0^\infty e^{-Rx}\, dG_{jk}(x)$$

is the probability that this R-tuple is still busy at T_1 *and* that $\xi(T_1-) = k$.

Again, if this R-tuple remains busy just until $t = x$, the expected number of overflows *from* k to find it busy is $M_{jk}^{o}(x)$. Therefore the unconditional expectation of the number of overflows from k to find it busy is

$$\int_0^\infty M_{jk}^{o}(x)\, d(1 - e^{-Rx}) = \int_0^\infty e^{-Rx}\, dM_{jk}^{o}(x) = \mu_{jk}^{o}(R).$$

Denote (temporarily) by $[\mu_{jk}^{o}(R) \mid l]$ the expected number of overflows from k to find this R-tuple still busy, on the condition that $\xi(T_1-) = l$ and the R-tuple is still busy at $t = T_1-$.

Then, by the principle of total expectation,

$$\mu_{jk}^{o}(R) = \sum_{l=0}^{m} [\mu_{jk}^{o}(R) \mid l]\gamma_{jl}(R). \tag{14}$$

Now because of the exponential holding-time distribution

$$[\mu_{jk}^{o}(R) \mid l] = \mu_{lk}^{o}(R) \quad \text{for} \quad l \neq k \tag{15}$$

and

$$[\mu_{jk}^{o}(R) \mid k] = 1 + \mu_{kk}^{o}(R). \tag{16}$$

Substituting (15) and (16) into (14), we obtain (13). Equation (13) may be written

$$\mu^{o,R} = \gamma^{R} + \gamma^{R}\mu^{o,R}. \tag{17}$$

Thus, to prove the lemma, it remains to show that $(E + \mu^{o,R})$ is nonsingular.

From (17)

$$(E - \gamma^{R})\mu^{o,R} = \gamma^{R}.$$

Therefore

$$(E - \gamma^{R})\cdot(E + \mu^{o,R}) = E$$

$$\det\,(E - \gamma^{R}) \cdot \det\,(E + \mu^{o,R}) = 1.$$

Since clearly both $\det\,(E - \gamma^{R})$ and $\det\,(E + \mu^{o,R})$ are finite (for $R > 0$), it follows that $\det\,(E - \gamma^{R}) \neq 0$ and $\det\,(E + \mu^{o,R}) \neq 0$, which completes the proof of the lemma.

We note, for later use, that we have also shown that

$$\mu^{o,R} = (E - \gamma^{R})^{-1}\gamma^{R}. \tag{18}$$

We need a separate method for finding $\gamma_{jk}(0)$, the above argument being invalid because $\mu_{jk}^{o}(0) = \infty$ for all $k \in \mathfrak{A}$.

We notice that $\gamma_{jk}(0) = G_{jk}(\infty) = P\{\xi(T_1-) = k \mid \xi(0+) = j\}$.

The quantities $\gamma_{jk}(0)$ are determined by the following system of equations:

$$\begin{aligned}\gamma_{jk}(0) = q_k \int_0^\infty dF(x) \binom{j}{k} e^{-kx}(1 - e^{-x})^{j-k} + \sum_{l=0}^{m} p_l\, \gamma_{l+1,k}(0)\,\cdot \\ \cdot \int_0^\infty dF(x) \binom{j}{l} e^{-lx}(1 - e^{-x})^{j-l} \qquad (j, k = 0, 1, \cdots, m).\end{aligned} \tag{19}$$

This may be seen as follows:

The event $\{\xi(T_1-) = k\}$ can occur in these mutually exclusive ways:

(*i*) the first arrival after $t = 0$ encounters k busy trunks on the first-choice group, with probability

$$\int_0^\infty dF(x) \binom{j}{k} e^{-kx}(1 - e^{-x})^{j-k},$$

and overflows, with probability q_k;

(*ii*) the first arrival after $t = 0$ encounters l busy trunks and does not overflow [so that $\xi(T_1+) = l + 1$]; the next overflow following this occurrence is from k [probability $\gamma_{l+1,k}(0)$].

For each k, (19) is a set of linear equations in the $\gamma_{jk}(0)$. These equations determine the $\gamma_{jk}(0)$ uniquely if the coefficient matrix is nonsingular (for each k). Call this matrix $A^{(k)}$. If we can show that $|A_{jj}^{(k)}| > \sum_{l \neq j} A_{jl}^{(k)}$ for each j, it will follow from the theorem of Lévy-Hadamard-Gerschgorin (Ref. 12, p. 79) that $\det A^{(k)} \neq 0$. That is, we want to show that

$$\sum_{l=0}^{m} p_l \int_0^\infty dF(x) \binom{j}{l} e^{-lx}(1 - e^{-x})^{j-l} < 1. \tag{20}$$

The left side of (20) is evidently strictly less than

$$\sum_{l=0}^{m} \int_0^\infty dF(x) \binom{j}{l} e^{-lx}(1 - e^{-x})^{j-l} = 1, \quad \text{for each } j, \text{ Q.E.D.}$$

Equations (5) and (6) may be solved, in some cases, by means of generating functions.

Let

$$U^o(k,R,w) = \sum_{N=1}^{\infty} U^o(k,R,N)w^N$$

$$V^o(k,R,w) = \sum_{N=1}^{\infty} V^o(k,R,N)w^N$$

Note that it follows from (10) that

$$\sum_{N=1}^{\infty} P^o(k,K,N)w^N = \sum_{R=K}^{M} (-1)^{R-K} \binom{R}{K} U^o(k,R,w). \tag{21}$$

From (5) and (6) we obtain

$$U^o(k,R,w) = w\left\{\binom{I}{R}\gamma_{ik}(R) + \sum_{j \in \mathfrak{a}} \gamma_{jk}(R)[U^o(j,R,w) + V^o(j,R-1,w)]\right\}. \tag{22}$$

We illustrate the use of (22) by a simple example.

Example 1:

If the first-choice group is full-access (the only element of $\mathcal{A}$ is m), then $U^o(k,R,N)$ and $V^o(k,R,N)$ vanish except for $k = m$. For simplicity, we assume that $i = m$; then the only relevant element of the matrix γ^R is $\gamma_{mm}(R)$, and (22) becomes:

$$U^o(m,R,w) = w\,\gamma_{mm}(R)\left[\binom{I}{R} + U^o(m,R,w) + V^o(m,R-1,w)\right],$$

whence

$$U^o(m, R, w) = \frac{w\,\gamma_{mm}(R)}{1 - w\,\gamma_{mm}(R)}\left[\binom{I}{R} + V^o(m, R - 1, w)\right]. \quad (23)$$

$\gamma_{mm}(s)$ is the Laplace-Stieltjes transform of the interoverflow-time distribution, i.e., it is just the function $\gamma(s)$ given by (1). Thus (23) is exactly equivalent to (T32), and merely serves to illustrate our remark (Section 1.1) that if the first-choice group is full-access, we can use the methods of Ref. 2 to describe the behaviour of the sequence $\{\Xi_N{}^o\}$.

2.2 *The Limiting Distribution* $P^o(k,K)$

Theorem 2: The quantities $P^o(k,K) = \lim_{N\to\infty} P^o(k,K,N)$ *exist, are strictly positive, form a probability distribution independent of the initial state, and are uniquely determined by the binomial moments* $U^o(k,R) = \sum_{K=R}^{M}\binom{K}{R}P^o(k,K)$; *the latter are determined by*

$$U^o(k,R) = q_k \sum_{j\in\mathcal{A}} \mu_{jk}(R)V^o(j,R-1) \qquad (R = 1, 2, \cdots, M) \quad (24)$$

and

$$U^o(k, 0) = \frac{q_k P(k)}{b} \quad (25)$$

where

$$V^o(k,R) = \sum_{K=R}^{M}\binom{K}{R} G_K P^o(k,K).$$

Proof: We first show the existence of the limiting distribution.

In this section, we use theorems given in Feller,[13] chapter 15, sections 5 and 6.

The Markov chain $\{\xi_N{}^o, \Xi_N{}^o\}$ is evidently irreducible (since $p_k > 0$ for $k < m$) and aperiodic. Therefore $\lim_{N\to\infty} P^o(k,K,N)$ exists. Since it is irreducible, the chain has either all transient, all recurrent null, or all recurrent non-null states.

If a state (k,K) is transient or recurrent null, then $\lim_{N\to\infty} P(k,K,N) = 0$.

Therefore, to show that all states are recurrent non-null it will suffice to show that for *some* state (k,K), $\lim_{N\to\infty} P^o(k,K,N) > 0$. It will then follow that this is so for all states, and that $\sum_{k\in\mathfrak{a}} P^o(k,K) = 1$. We look at the state (0,0):

To see that $\lim_{N\to\infty} P^o(0,0,N) > 0$, we compare our system (with arbitrary balking probabilities) to the special system for which $m = 0$, $M = \infty$, $H_K = 0$ (always assuming the same input process). For this special system, write $P\{\Xi_N{}^o = K\} = \tilde{P}^o(K,N)$, and take as initial condition: $\Xi(0+) = i + I$.

It is clear that for any system with $M = \infty$, and with the same initial condition,

$$P^o(0,0,N) \geqq \tilde{P}^o(0,N),$$

for each N, whence

$$\lim_{N\to\infty} P^o(0,0,N) \geqq \lim_{N\to\infty} \tilde{P}^o(0,N).$$

But it is known[3] that $\lim_{N\to\infty} \tilde{P}^o(0,N) > 0$; thus

$$\lim_{N\to\infty} P^o(0,0,N) = P^o(0,0) > 0$$

and all states are recurrent non-null. Hence, since the chain is also irreducible and aperiodic, it is ergodic.

We now know also that a unique stationary distribution exists and that it coincides with the limiting distribution. From (6), we must have

$$U^o(k,R) = \sum_{j\in\mathfrak{a}} \gamma_{jk}(R)[U^o(j,R) + V^o(j,R-1)]. \tag{26}$$

Denote by $U^{o,R}$ the row-vector with components $U^o(k,R)$, $0 \leqq k \leqq m$.

Then (26) may be written

$$U^{o,R} = (U^{o,R} + V^{o,R-1})\gamma^R.$$

Thus, from (18),

$$U^{o,R} = V^{o,R-1}\mu^{o,R}. \tag{27}$$

Writing out (27) in components, and using (12), we obtain (24).

We now prove (25). Denote by $C^{(n)}$ the event that the nth arrival overflows. Thus,

$$b = \lim_{n\to\infty} P\{C^{(n)}\}.$$

Now,

$$P^o(k,K) = \lim_{N\to\infty} P\{\xi_N{}^o = k,\ \Xi_N{}^o = K\} = \lim_{n\to\infty} P\{\xi_n = k,\ \Xi_n = K \mid C^{(n)}\}$$

$$= \lim_{n\to\infty} \frac{P\{\xi_n = k, \Xi_n = K\}\ P\ \{C^{(n)} \mid \xi_n = k, \Xi_n = K\}}{P\{C^{(n)}\}}.$$

But

$$P\{C^{(n)} \mid \xi_n = k,\ \Xi_n = K\} = P\{C^{(n)} \mid \xi_n = k\} = q_k .$$

Therefore

$$P^o(k, K) = \frac{q_k P(k, K)}{b} \tag{28}$$

and

$$U^o(k, 0) = \sum_{K=0}^{M} P^o(k, K) = \frac{q_k \sum_{K=0}^{M} P(k, K)}{b}$$

$$= \frac{q_k\ P(k)}{b}, \text{ Q.E.D.}$$

To complete the proof of Theorem 2, it remains to show that the binomial moments $U^o(k,R)$ uniquely determine the probabilities $P^o(k,K)$. This proof will be easier after we have discussed the stationary distribution at arrival moments, $P(k,K)$, and we therefore defer it until then.

It is sometimes convenient to work with the double binomial moments

$$B^o(r, R) = \sum_{k=r}^{m} \binom{k}{r} U^o(k, R)$$

$$C^o(r, R) = \sum_{k=r}^{m} \binom{k}{r} V^o(k,R).$$

In terms of these, (24) and (25) of Theorem 2 become

$$B^o(r,R) = \sum_{j=0}^{m} [f_{jr}(R) - g_{jr}(R)]C^o(j,R-1) \qquad (29)$$

$$(R = 1, 2, \cdots, M)$$

$$B^o(r, 0) = \frac{1}{b}\sum_{k=r}^{m} \binom{k}{r} q_k\, P(k). \qquad (30)$$

Here we have used the following definitions: $f_{lr}(s)$ and $g_{lr}(s)$ are the lth differences of $\Phi_{0r}(s)$ and $\Psi_{0r}(s)$:

$$f_{lr}(s) = \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} \Phi_{jr}(s) \qquad (31)$$

$$g_{lr}(s) = \sum_{j=0}^{l} (-1)^{l-j} \binom{l}{j} \Psi_{jr}(s) \qquad (32)$$

where $\Phi_{jr}(s)$ and $\Psi_{jr}(s)$ are defined, following Takács [(T59), (T60)], by

$$\Phi_{jr}(s) = \sum_{k=r}^{m} \binom{k}{r} \mu_{jk}(s) \qquad (33)$$

$$\Psi_{jr}(s) = \sum_{k=r}^{m} \binom{k}{r} p_k\mu_{jk}(s) \qquad (34)$$

and must satisfy [(T61) and (T62)]

$$\Phi_{j0}(s) = \frac{\varphi(s)}{1-\varphi(s)} \qquad (35)$$

and

$$\frac{\Phi_{jr}(s)}{C_r(s)} = \frac{1}{C_{r-1}(s)}\left[\binom{j}{r} + \Psi_{j,r-1}(s)\right] \qquad (36)$$

as well as the relations in r implied by their definitions [see (T25)],

$$\Psi_{jr}(s) = \sum_{l=r}^{m} \binom{l}{r} (\Delta^{l-r} p_r)\Phi_{jl}(s). \qquad (37)$$

Examples of the application of the methods of this section will be found in Section V.

III. THE STATE OF THE SYSTEM AT ARRIVAL INSTANTS

3.1 *Transient Behaviour*

The sequence $\{\xi_n, \Xi_n\}$ is clearly a homogeneous Markov chain. We assume initial conditions $\xi(0+) = i$, $\Xi(0+) = I$, and seek the distribution $P(k,K,n)$. We no longer restrict our attention to states (k,K) for which $q_k > 0$, but consider all states (k,K), $0 \leqq k \leqq m \leqq \infty$, $0 \leqq K \leqq M \leqq \infty$.

We shall prove the following:

Theorem 3: The distribution $P(k,K,n)$ is uniquely determined by the double binomial moments

$$B(r,R,n) = \sum_{k=r}^{m} \sum_{K=R}^{M} \binom{k}{r} \binom{K}{R} P(k,K,n);$$

the latter are determined by

$$B(r,R,1) = \varphi_{r+R} \binom{i}{r} \binom{I}{R} \tag{38}$$

$$(r = 0, 1, \cdots, m; R = 0, 1, \cdots, M)$$

$$B(r,R,n+1) = \varphi_{r+R}[B(r,R,n) + D(r-1,R,n) + C(r,R-1,n) - E(r,R-1,n)] \tag{39}$$

$$(r = 0, 1, \cdots, m; R = 0, 1, \cdots, M; n = 1, 2, \cdots).$$

Here

$$C(r,R,n) = \sum_{k=r}^{m} \sum_{K=R}^{M} \binom{k}{r} \binom{K}{R} G_K P(k,K,n)$$

$$D(r,R,n) = \sum_{k=r}^{m} \sum_{K=R}^{M} \binom{k}{r} \binom{K}{R} p_k P(k,K,n)$$

$$E(r,R,n) = \sum_{k=r}^{m} \sum_{K=R}^{M} \binom{k}{r} \binom{K}{R} p_k G_K P(k,K,n)$$

and all these quantities are defined to be zero if $r < 0$ or $R < 0$.

Proof: If the arrival at τ_n finds the system in the state (j,J), it may either get on the first-choice group, with probability p_j, or balk on the first-choice group with probability q_j; in the latter case, it may get on the overflow group, with probability G_J, or balk there too, with probability H_J. Thus the transition probabilities are given by

$$p(j,J;k,K) = P\{\xi_{n+1} = k,\ \Xi_{n+1} = K \mid \xi_n = j,\ \Xi_n = J\}$$

$$= \int_0^\infty dF(x) \left\{ p_j \binom{j+1}{k} e^{-xk}(1-e^{-x})^{j+1-k} \binom{J}{K} e^{-xK}(1-e^{-x})^{J-K} + q_j \binom{j}{k} e^{-xk}(1-e^{-x})^{j-k} \left[G_J \binom{J+1}{K} e^{-xK}(1-e^{-x})^{J+1-K} + H_J \binom{J}{K} e^{-xK}(1-e^{-x})^{J-K} \right] \right\}. \tag{40}$$

Now

$$P(k,K,n+1) = \sum_{j=0}^{m} \sum_{J=0}^{M} p(j,J;k,K)\, P(j,J,n). \tag{41}$$

Substituting (40) in (41) and taking binomial moments with respect to both the first-choice and overflow groups, we obtain:

$$\begin{aligned} B(r,R,n+1) = \varphi_{r+R} \sum_{j=0}^{m} \sum_{J=0}^{M} \Big\{ & p_j \binom{j+1}{r} \binom{J}{R} \\ & + q_j \binom{j}{r} \Big[G_J \binom{J+1}{R} + H_J \binom{J}{R} \Big] \Big\} P(j,J,n). \end{aligned} \tag{42}$$

Note that the quantity in braces in (42) is

$$\left\{ \binom{j}{r}\binom{J}{R} + p_j \binom{j}{r-1}\binom{J}{R} + q_j G_J \binom{j}{r}\binom{J}{R-1} \right\}. \tag{43}$$

Substituting (43) in (42), we obtain (39).
For $n = 1$, we have

$$P(k,K,1) = \int_0^\infty dF(x) \binom{i}{k} e^{-xk}(1-e^{-x})^{i-k} \binom{I}{K} e^{-xK}(1-e^{-x})^{I-K};$$

taking binomial moments with respect to both trunk groups, we obtain (38).

From the double binomial moments, one obtains the probabilities $P(k,K,n)$ by using:

$$U(k,R,n) = \sum_{r=k}^{m} (-1)^{r-k} \binom{r}{k} B(r,R,n) \tag{44}$$

and

$$P(k,K,n) = \sum_{R=K}^{M} (-1)^{R-K} \binom{R}{K} U(k,R,n). \tag{45}$$

Clearly $P(k,K,n) = 0$ for $k + K \geqq i + I + n$; it follows that the sums in (44) and (45) contain a finite number of terms for finite n, even if $M = \infty$, and there are no problems about convergence.

Equations (38) and (39) may be solved, in some cases, by means of generating functions; we give an example.

Example 2:

We consider the simplest possible case, in which

$$q_k = 0 \qquad (k = 0, 1, \cdots, m-1)$$

$$q_m = 1$$

$$M = \infty$$

$$H_K = 0 \qquad (K = 0, 1, 2, \cdots).$$

In this case,

$$C(r,R,n) = B(r,R,n), \tag{46}$$

$$E(r,R,n) = D(r,R,n), \tag{47}$$

and

$$D(r,R,n) = B(r,R,n) - \binom{m}{r} B(m,R,n). \tag{48}$$

Substituting (46), (47), and (48) in (39), we get

$$B(r,R,n+1) = \varphi_{r+R}[B(r,R,n) + B(r-1,R,n) - \binom{m}{r-1} B(m,R,n) + \binom{m}{r} B(m,R-1,n)]. \tag{49}$$

Let

$$B(r,R,w) = \sum_{n=1}^{\infty} B(r,R,n)w^n.$$

From (38) and (49):

$$B(r,R,w) = \frac{w\varphi_{r+R}}{1 - w\varphi_{r+R}}\left[\binom{i}{r}\binom{I}{R} + B(r-1,R,w) - \binom{m}{r-1} B(m,R,w) + \binom{m}{r} B(m,R-1,w)\right]. \tag{50}$$

The solution of (50) is

$$B(r,R,w) = \Gamma_{r+R}(w)\left\{\frac{\sum_{j=r}^{m}\binom{m}{j}\frac{1}{\Gamma_{j+R}(w)}}{\sum_{j=0}^{m}\binom{m}{j}\frac{1}{\Gamma_{j+R}(w)}}\cdot\sum_{S=0}^{R}\binom{I}{S}\sum_{j=0}^{i}\binom{i}{j}\frac{1}{\Gamma_{j+S-1}(w)} - \frac{\sum_{j=r+1}^{m}\binom{m}{j}\frac{1}{\Gamma_{j+R-1}(w)}}{\sum_{j=0}^{m}\binom{m}{j}\frac{1}{\Gamma_{j+R-1}(w)}}\cdot\sum_{S=0}^{R-1}\binom{I}{S}\sum_{j=0}^{i}\binom{i}{j}\frac{1}{\Gamma_{j+S-1}(w)} - \binom{I}{R}\sum_{j=r+1}^{m}\binom{i}{j}\frac{1}{\Gamma_{j+R-1}(w)}\right\}$$

where we have defined

$$\Gamma_r(w) = \prod_{j=0}^{r} \frac{w\varphi_j}{1 - w\varphi_j}, \qquad (r = 0, 1, 2, \cdots)$$

$$\Gamma_{-1}(w) \equiv 1.$$

3.2 *The Limiting Distribution* $P(k,K)$

Theorem 4: The quantities $P(k,K) = \lim_{n\to\infty} P(k,K,n)$ *exist, are strictly positive, form a probability distribution independent of the initial state, and are uniquely determined by the double binomial moments* $B(r,R) = \sum_{k=r}^{m} \binom{k}{r} U(k,R)$, *where* $U(k,R) = \sum_{K=R}^{M} \binom{K}{R} P(k,K)$; *the* $B(r,R)$ *are given by*

$$B(r,R) = bC_{r+R}\left[\sum_{j=r}^{m} \frac{B^o(j,R)}{C_{j+R}} - \sum_{j=r+1}^{m} \frac{C^o(j,R-1)}{C_{j+R-1}}\right] \tag{51}$$

$$(r = 0, 1, \cdots, m;\ R = 0, 1, \cdots, M).$$

Here

$$C^o(r,R) = \sum_{k=r}^{m} \binom{k}{r}\binom{K}{R} G_K P^o(k,K).$$

Proof: That the limits $P(k,K)$ exist and are independent of the initial state again follows from the fact that the Markov chain $\{\xi_n, \Xi_n\}$ $(n = 1, 2, \cdots)$ is irreducible and aperiodic. To show that the $P(k,K)$ are strictly positive and form a probability distribution, we must show that there exists some state (k,K) such that $P(k,K) > 0$. This can be done by a method similar to that used in the proof of Theorem 2; we omit the argument. It follows that a unique stationary distribution exists and that it coincides with the limiting distribution. We express this stationary distribution in terms of the stationary distribution $P^o(k,K)$ in the following way:

Consider the arrival which occurs at τ_n (under equilibrium conditions).

It either overflows, with probability b, or does not, with probability $(1 - b)$.

If it overflows, the probability that it encountered the state (j,J) is $P^o(j,J)$.

If it does not overflow, let us denote the probability that it encountered the state (j,J) by $P^{\emptyset}(j,J)$.

We note that

$$P(j,J) = bP^o(j,J) + (1 - b)P^{\emptyset}(j,J). \tag{52}$$

Suppose that $\theta_{n+1} = x$.

If the arrival at τ_n encountered the state (j,J) and overflowed, the probability that the arrival at τ_{n+1} encounters the state (k,K) is:

$$\binom{j}{k} e^{-xk}(1 - e^{-x})^{j-k}\left[G_J \binom{J+1}{K} e^{-xK}(1 - e^{-x})^{J+1-K} + H_J \binom{J}{K} e^{-xK}(1 - e^{-x})^{J-K}\right] = \alpha(x), \text{ say.} \tag{53}$$

If the arrival at τ_n encountered the state (j,J) and did not overflow, the probability that the arrival at τ_{n+1} encounters the state (k,K) is:

$$\binom{j+1}{k} e^{-xk}(1 - e^{-x})^{j+1-k} \binom{J}{K} e^{-xK}(1 - e^{-x})^{J-K} = \beta(x), \text{ say.} \tag{54}$$

Taking account of both these possibilities, and removing the condition on θ_{n+1},

$$P(k,K) = \sum_{j=0}^{m} \sum_{J=0}^{M} \int_0^\infty dF(x)[bP^o(j,J)\alpha(x) + (1 - b)P^{\not o}(j,J)\beta(x)].$$

Using (52),

$$P(k,K) = \sum_{j=0}^{m} \sum_{J=0}^{M} \int_0^\infty dF(x)\{bP^o(j,J)[\alpha(x) - \beta(x)] + P(j,J)\beta(x)\}.$$

Taking binomial moments with respect to both trunk groups, and using (53) and (54),

$$\begin{aligned} B(r,R) &= \varphi_{r+R} \sum_{j=0}^{m} \sum_{J=0}^{M} \left\{ bP^o(j,J) \left[\binom{j}{r} \left(G_J \binom{J+1}{R} + H_J \binom{J}{R} \right) - \binom{j+1}{r} \binom{J}{R} \right] + P(j,J) \binom{j+1}{r} \binom{J}{R} \right\} \\ &= \varphi_{r+R}\{B(r,R) + B(r-1,R) + b[C^o(r,R-1) - B^o(r-1,R)]\}. \end{aligned} \tag{55}$$

The solution of (55) is

$$\frac{B(r,R)}{C_{r+R}} = \frac{B(m,R)}{C_{m+R}} + b\left[\sum_{j=r}^{m-1} \frac{B^o(j,R)}{C_{j+R}} - \sum_{j=r+1}^{m} \frac{C^o(j,R-1)}{C_{j+R-1}}\right]. \tag{56}$$

Now note that, from (28),

$$bB^o(m,R) = B(m,R). \tag{57}$$

Substituting (57) in (56), we obtain (51).

To complete the proof of Theorem 4, it remains to show that the double binomial moments $B(r,R)$ uniquely determine the probabilities $P(k,K)$. It is clear that the $B(r,R)$ uniquely determine the $U(k,R)$ through the equation

$$U(k,R) = \sum_{r=k}^{m} (-1)^{r-k} \binom{r}{k} B(r,R) \tag{58}$$

because m is finite. Thus we must show that

$$P(k,K) = \sum_{R=K}^{M} (-1)^{R-K} \binom{R}{K} U(k,R) \tag{59}$$

when M is infinite; it will suffice to show that the series on the right converges absolutely.

From (39) we have

$$B(0,R) = \frac{\varphi_R}{1 - \varphi_R} [C(0,R-1) - E(0, R-1)]. \tag{60}$$

Now,

$$C(0,R) - E(0,R) = \sum_{k=r}^{m} \sum_{K=R}^{M} \binom{k}{r} \binom{K}{R} q_k G_K P(k,K) \leqq B(0,R). \tag{61}$$

Therefore,

$$B(0,R) \leqq \frac{\varphi_R}{1 - \varphi_R} B(0,R-1). \tag{62}$$

Now

$$\lim_{R\to\infty} \varphi_R = \lim_{s\to\infty} \varphi(s) = F(0+) = 0$$

whence

$$\lim_{R\to\infty} \frac{\varphi_R}{1 - \varphi_R} = 0.$$

Thus

$$\lim_{R\to\infty} \frac{B(0,R)}{B(0,R-1)} = 0. \tag{63}$$

Equation (63) is sufficient to insure that

$$\sum_{R=K}^{M} \binom{R}{K} B(0,R)$$

converges.

Consider for simplicity the case $m = 1$. Then we have

$$B(0,R) = U(0,R) + U(1,R). \tag{64}$$

At least one of the statements

$$\lim_{R\to\infty} \frac{U(0,R)}{U(0,R-1)} = 0 \tag{65}$$

$$\lim_{R\to\infty} \frac{U(1,R)}{U(1,R-1)} = 0 \tag{66}$$

must be true, for if both failed to be true, then for some $\epsilon > 0$ there would be terms for which

$$\frac{U(0,R)}{U(0,R-1)} > \epsilon$$

$$\frac{U(1,R)}{U(1,R-1)} > \epsilon$$

for arbitrarily large R; it would follow that for arbitrarily large R

$$\frac{B(0,R)}{B(0,R-1)} = \frac{U(0,R) + U(1,R)}{U(0,R-1) + U(1,R-1)} > \epsilon$$

which contradicts (63).

Say (65) is true. Then the series

$$\sum_{R=K}^{M} \binom{R}{K} U(0,R)$$

converges; thus

$$\sum_{R=K}^{M} \binom{R}{K} U(1,R) = \sum_{R=K}^{M} \binom{R}{K} B(0,R) - \sum_{R=K}^{M} \binom{R}{K} U(0,R)$$

converges, and this proves (59) for $m = 1$. The generalization to arbitrary m is straightforward.

Corollary: We can now easily complete the proof of Theorem 2 by remarking that [using (28)]

$$\begin{aligned} bU^{o}(k,R) &= b \sum_{J=R}^{M} \binom{J}{R} P^{o}(k,J) \\ &= \sum_{J=R}^{M} \binom{J}{R} q_k P(k,J) \leqq \sum_{J=R}^{M} \binom{J}{R} P(k,J) = U(k,R) \end{aligned}$$

so that the series

$$P^{o}(k,K) = \sum_{R=K}^{M} (-1)^{R-K} \binom{R}{K} U^{o}(k,R)$$

converges absolutely, Q.E.D.

We again defer examples to Section V.

IV. THE STATE OF THE SYSTEM AT ANY TIME

4.1 *Transient Behaviour*

Let

$$B(r,R,t) = \sum_{k=r}^{m} \sum_{K=R}^{M} \binom{k}{r} \binom{K}{R} P(k,K,t)$$

with *Laplace* transform

$$\beta(r,R,s) = \int_0^\infty e^{-st} B(r,R,t) dt.$$

Let $M_{ik}{}^{IK}(t)$ be the expected number of arrivals in $(0,t]$ to encounter k trunks busy on the first-choice group and K on the overflow group, on the condition that $\xi(0+) = i$, $\Xi(0+) = I$, with Laplace-Stieltjes transform

$$\mu_{ik}{}^{IK}(s) = \int_0^\infty e^{-sx}\, dM_{ik}{}^{IK}(x).$$

We also define several kinds of double binomial moments:

$$\Phi_{ir}{}^{IR}(s) = \sum_{k=r}^{m} \sum_{K=R}^{M} \binom{k}{r} \binom{K}{R} \mu_{ik}{}^{IK}(s)$$

$$X_{ir}{}^{IR}(s) = \sum_{k=r}^{m} \sum_{K=R}^{M} \binom{k}{r} \binom{K}{R} G_K \mu_{ik}{}^{IK}(s)$$

$$\Psi_{ir}{}^{IR}(s) = \sum_{k=r}^{m} \sum_{K=R}^{M} \binom{k}{r} \binom{K}{R} p_k \mu_{ik}{}^{IK}(s)$$

$$Y_{ir}{}^{IR}(s) = \sum_{k=r}^{m} \sum_{K=R}^{M} \binom{k}{r} \binom{K}{R} p_k G_K \mu_{ik}{}^{IK}(s).$$

Theorem 5:

$$\Phi_{ir}{}^{IR}(s) = \frac{\varphi(s+r+R)}{1-\varphi(s+r+R)} \cdot \left[\binom{i}{r} \binom{I}{R} + \Psi_{i,r-1}{}^{IR}(s) + X_{ir}{}^{I,R-1}(s) - Y_{ir}{}^{I,R-1}(s) \right]. \quad (67)$$

Proof: Consider a certain set of r first-choice trunks and a certain set of R overflow trunks. We shall call the union of these two sets an (r,R)-tuple of trunks, and if the r first-choice trunks and the R overflow trunks are all busy at time t, we shall say that this particular (r,R)-tuple of trunks is busy at time t. Thus, when the system is in the state (k,K), the number of busy (r,R)-tuples is $\binom{k}{r}\binom{K}{R}$. Let us make the convention that there is always one busy $(0,0)$-tuple. The expected number of busy (r,R)-tuples at time t is evidently $B(r,R,t)$.

Let us now calculate the expected total number of encounters between arriving calls and busy (r,R)-tuples in the interval $(0,t]$. Denote this expectation by $E_{ir}^{IR}(t)$.

If the nth arrival occurs in $(0,t]$, and if $(\xi_n = k,\ \Xi_n = K)$, then the nth arrival encounters $\binom{k}{r}\binom{K}{R}$ busy (r,R)-tuples. Thus

$$E_{ir}^{IR}(t) = \sum_{n=1}^{\infty}\sum_{k=r}^{m}\sum_{K=R}^{M}\binom{k}{r}\binom{K}{R}\int_0^{\infty} dP\{\tau_n \leqq u,\ \xi_n = k,\ \Xi_n = K\}.$$

But

$$\sum_{n=1}^{\infty} P\{\tau_n \leqq u,\ \xi_n = k,\ \Xi_n = K\} = M_{ik}^{IK}(u). \tag{68}$$

Therefore

$$E_{ir}^{IR}(t) = \sum_{k=r}^{m}\sum_{K=R}^{M}\binom{k}{r}\binom{K}{R} M_{ik}^{IK}(t)$$

with Laplace-Stieltjes transform

$$\epsilon_{ir}^{IR}(s) = \Phi_{ir}^{IR}(s). \tag{69}$$

But $\epsilon_{ir}^{IR}(s)$ can be found in another way. If $(\xi_n = k,\ \Xi_n = K)$, then at time τ_n+, the system is in the state $(k+1,K)$ with probability p_k, the state $(k,K+1)$ with probability q_kG_K, or the state (k,K) with probability q_kH_K. Thus the expected number of busy (r,R)-tuples at time τ_n+, under the stated condition, is

$$p_k\binom{k+1}{r}\binom{K}{R} + q_k\binom{k}{r}\left[G_K\binom{K+1}{R} + H_K\binom{K}{R}\right]$$
$$= \binom{k}{r}\binom{K}{R} + p_k\binom{k}{r-1}\binom{K}{R} + q_kG_K\binom{k}{r}\binom{K}{R-1},$$

and the expected number of busy (r,R)-tuples *created* by the nth arrival, under the stated condition, is

$$p_k \binom{k}{r-1}\binom{K}{R} + (1 - p_k)G_K \binom{k}{r}\binom{K}{R-1}.$$

Now the probability that the life of a busy (r,R)-tuple will be longer than x is $\exp(-(r + R)x)$. Thus the expected number of encounters between arriving calls and created (r,R)-tuples in the interval $(0,t]$ is:

$$\sum_{n=1}^{\infty} \sum_{k=r-1}^{m} \sum_{K=R-1}^{M} \int_0^t dP\{\tau_n \leqq u, \xi_n = k, \Xi_n = K\} \cdot \left[p_k \binom{k}{r-1}\binom{K}{R} + (1 - p_k)G_K \binom{k}{r}\binom{K}{R-1}\right] \cdot \int_0^{t-u} e^{-(r+R)x}\, dM(x) \tag{70}$$

where $M(x)$ is the expected number of arrivals in an interval of length x, when there was an arrival at the start of the interval. $M(x)$ has Laplace-Stieltjes transform

$$\mu(s) = \frac{\varphi(s)}{1 - \varphi(s)}.$$

Equation (70) is a convolution. Recalling (68), we see that (70) has Laplace-Stieltjes transform,

$$\sum_{k=r-1}^{m} \sum_{K=R-1}^{M} \left[p_k \binom{k}{r-1}\binom{K}{R} + (1 - p_k)G_K \binom{k}{r}\binom{K}{R-1}\right] \mu_{ik}{}^{IK}(s)\mu(s + r + R). \tag{71}$$

We must not forget the (r,R)-tuples which were busy initially; the expected number of encounters between arriving calls and these is

$$\binom{i}{r}\binom{I}{R} \sum_{n=1}^{\infty} \int_0^t dP\{\tau_n \leqq u\} e^{-(r+R)u} = \binom{i}{r}\binom{I}{R} \int_0^t dM(u) e^{-(r+R)u}$$

with Laplace-Stieltjes transform

$$\binom{i}{r}\binom{I}{R} \mu(s + r + R). \tag{72}$$

Adding (71) and (72) we get

$$\epsilon_{ir}{}^{IR}(s) = \frac{\varphi(s + r + R)}{1 - \varphi(s + r + R)} \cdot \left[\binom{i}{r}\binom{I}{R} + \Psi_{i,r-1}{}^{IR}(s) + X_{ir}{}^{I,R-1}(s) - Y_{ir}{}^{I,R-1}(s)\right]. \tag{73}$$

Now comparing (69) and (73), we obtain (67).

Theorem 6: The distribution $P(k,K,t)$ $(t > 0)$ *is determined by*

$$\beta(r,R,s) = \frac{1 - \varphi(s + r + R)}{\varphi(s + r + R)} \cdot \frac{1}{s + r + R} \Phi_{ir}{}^{IR}(s). \qquad (74)$$

Proof: We have

$$\begin{aligned} P(k,K,t) = &\binom{i}{k} e^{-tk}(1 - e^{-t})^{i-k} \binom{I}{K} e^{-tK}(1 - e^{-t})^{I-K}[1 - F(t)] \\ &+ \sum_{n=1}^{\infty} \sum_{j=0}^{m} \sum_{J=0}^{M} \int_0^t dP\{\xi_n = j, \Xi_n = J, \tau_n \leqq u\} \\ &\cdot \left\{ p_j \binom{j+1}{k} e^{-(t-u)k}(1 - e^{-(t-u)})^{j+1-k} \binom{J}{K} \right. \\ &\cdot e^{-(t-u)K}(1 - e^{-(t-u)})^{J-K} + q_j \binom{j}{k} e^{-(t-u)k} \\ &\cdot (1 - e^{-(t-u)})^{j-k} \left[G_J \binom{J+1}{K} e^{-(t-u)K} \right. \\ &\cdot (1 - e^{-(t-u)})^{J+1-K} + H_J \binom{J}{K} e^{-(t-u)K} \\ &\left. \left. \cdot (1 - e^{-(t-u)})^{J-K} \right] \right\} [1 - F(t - u)]. \end{aligned} \qquad (75)$$

This may be seen as follows: either no calls arrive in the interval $(0,t]$, or the last call to arrive in that interval is the nth ($n = 1, 2, \cdots$), i.e. the nth call arrives at time u and no calls arrive in the interval $(u,t]$. If this call encounters the state (j,J) it may get on the first-choice group (probability p_j), the overflow group (probability q_jG_J), or neither (probability q_jH_J). Then enough calls must end in the interval $(u,t]$ so that the state at time t is (k,K).

From (75), and keeping in mind (68),

$$\begin{aligned} B(r,R,t) = &\binom{i}{r}\binom{I}{R} e^{-t(r+R)}[1 - F(t)] + \sum_{j=0}^{m} \sum_{J=0}^{M} \int_0^t dM_{ij}{}^{IJ}(u) \\ &\cdot e^{-(t-u)(r+R)} \left\{ \binom{j}{r}\binom{J}{R} + p_j \binom{j}{r-1}\binom{J}{R} + q_jG_J \binom{j}{r} \right. \\ &\left. \cdot \binom{J}{R-1} \right\} [1 - F(t - u)], \end{aligned}$$

and taking the Laplace transform,

$$\beta(r,R,s) = \frac{1-\varphi(s+r+R)}{s+r+R}\left[\binom{i}{r}\binom{I}{R} + \Phi_{ir}{}^{IR}(s) + \Psi_{i,r-1}{}^{IR}(s) + X_{ir}{}^{I,R-1}(s) - Y_{ir}{}^{I,R-1}(s)\right]. \tag{76}$$

From (76) and (67) we obtain (74).

It remains to show that the double binomial moments uniquely determine the probabilities $P(k,K,t)$. As in the proof of Theorem 4, it will suffice to show that for all $t > 0$

$$\lim_{R\to\infty} \frac{B(0,R,t)}{B(0,R-1,t)} = 0. \tag{77}$$

From (67), for $R > I$,

$$\Phi_{i0}{}^{IR}(s) \leqq \frac{\varphi(s+R)}{1-\varphi(s+R)}\,\Phi_{i0}{}^{I,R-1}(s). \tag{78}$$

But, *for all* $s > 0$,

$$\lim_{R\to\infty} \frac{\varphi(s+R)}{1-\varphi(s+R)} = 0.$$

Therefore

$$\lim_{R\to\infty} \frac{\Phi_{i0}{}^{IR}(s)}{\Phi_{i0}{}^{I,R-1}(s)} = 0. \tag{79}$$

Now from (74),

$$\frac{\beta(0,R,s)}{\beta(0,R-1,s)} = \frac{1-\varphi(s+R)}{\varphi(s+R)}\,\frac{\varphi(s+R-1)}{1-\varphi(s+R-1)} \cdot \frac{s+R-1}{s+R}\,\frac{\Phi_{i0}{}^{IR}(s)}{\Phi_{i0}{}^{I,R-1}(s)}$$

and so

$$\lim_{R\to\infty} \frac{\beta(0,R,s)}{\beta(0,R-1,s)} = \lim_{R\to\infty} \frac{\Phi_{i0}{}^{IR}(s)}{\Phi_{i0}{}^{I,R-1}(s)} = 0, \tag{80}$$

since

$$\lim_{s\to\infty} \frac{\varphi(s)}{\varphi(s-1)} = 1.$$

From (80), and the inversion formula for the Laplace transform, the result (77) follows.

Example 3:

Consider the case

$$q_k = 0 \qquad (k = 0, 1, \cdots, m-1)$$
$$q_m = 1$$
$$M = \infty$$
$$H_K = H, G_K = G \ (G + H = 1) \qquad (K = 0, 1, 2, \cdots).$$

This example may be of some practical interest. It represents a situation in which some equipment, other than a free trunk, is needed to set up a connection on the overflow group. If this equipment is serving a large number of trunk groups, the chance of its being idle may be substantially independent of the situation on the particular overflow group being considered here, and may be represented by a constant, G.

In this case we have

$$X_{ir}^{IR}(s) = G\Phi_{ir}^{IR}(s)$$
$$Y_{ir}^{IR}(s) = G\Psi_{ir}^{IR}(s)$$

and

$$\Psi_{ir}^{IR}(s) = \Phi_{ir}^{IR}(s) - \binom{m}{r}\Phi_{im}^{IR}(s).$$

Equation (67) becomes

$$\Phi_{ir}^{IR}(s) = \frac{\varphi(s+r+R)}{1-\varphi(s+r+R)}\left\{\binom{i}{r}\binom{I}{R} + \Phi_{i,r-1}^{IR}(s) - \binom{m}{r-1}\Phi_{im}^{IR}(s) + G\binom{m}{r}\Phi_{im}^{I,R-1}(s)\right\}. \tag{81}$$

The solution of (81) is:

$$\Phi_{ir}^{IR}(s) = C_{r+R}(s)\left\{\frac{\sum_{j=r}^{m}\binom{m}{j}\frac{1}{C_{j+R}(s)}}{\sum_{j=0}^{m}\binom{m}{j}\frac{1}{G^R C_{j+R}(s)}}\cdot\sum_{S=0}^{R}\binom{I}{S}\sum_{j=0}^{i}\binom{i}{j}\frac{1}{G^S C_{j+S-1}(s)} - \frac{\sum_{j=r+1}^{m}\binom{m}{j}\frac{1}{C_{j+R-1}(s)}}{\sum_{j=0}^{m}\binom{m}{j}\frac{1}{G^R C_{j+R-1}(s)}}\cdot\sum_{S=0}^{R-1}\binom{I}{S}\sum_{j=0}^{i}\binom{i}{j}\frac{1}{G^S C_{j+S-1}(s)} - \binom{I}{R}\sum_{j=r+1}^{m}\binom{i}{j}\frac{1}{C_{j+R-1}(s)}\right\}. \tag{82}$$

The expression for $\beta(r,R,s)$ can now be obtained from (82), using (74).

4.2 *The Limiting Distribution* $P^*(k,K)$

Theorem 7: The quantities $P^*(k,K)$ *exist, are strictly positive, form a probability distribution, are independent of the initial state, and are uniquely determined by the double binomial moments*

$$B^*(r,R) = \sum_{k=r}^{m} \sum_{K=R}^{M} \binom{k}{r} \binom{K}{R} P^*(k,K);$$

the latter satisfy

$$B^*(r,R) = \frac{a}{r+R} \frac{1-\varphi_{r+R}}{\varphi_{r+R}} B(r,R), \qquad \text{for} \quad r+R>0 \qquad (83)$$

$$B^*(0,0) = 1.$$

Proof: To prove the existence, we consider the limit of (75) as $t \to \infty$. Clearly the first term goes to zero, and we have

$$\begin{aligned} P^*(k,K) = \lim_{t\to\infty} \sum_{J=0}^{M} \int_0^t \sum_{j=0}^{m} dM_{ij}{}^{IJ}(u) \\ \cdot \Bigg\{ p_j \binom{j+1}{k} e^{-(t-u)k} (1 - e^{-(t-u)})^{j+1-k} \binom{J}{K} \\ \cdot e^{-(t-u)K} (1 - e^{-(t-u)})^{J-K} + q_j \binom{j}{k} \\ \cdot e^{-(t-u)k} (1 - e^{-(t-u)})^{j-k} \Bigg[G_J \binom{J+1}{K} \\ \cdot e^{-(t-u)K} (1 - e^{-(t-u)})^{J+1-K} + H_J \binom{J}{K} \\ \cdot e^{-(t-u)K} (1 - e^{-(t-u)})^{J-K} \Bigg] \Bigg\} [1 - F(t-u)]. \end{aligned} \qquad (84)$$

It follows from Smith's "fundamental theorem,"[14] the assumption that $F(x)$ is not a lattice distribution, and the fact that $P(j,J) > 0$ for all j and J, that the limit in (84) exists and is given by

$$
\begin{aligned}
P^*(k,K) = \sum_{j=0}^{m} \sum_{J=0}^{M} \frac{P(j,J)}{\alpha} \int_0^\infty du[1 - F(u)] \\
\cdot \left\{ p_j \binom{j+1}{k} e^{-uk}(1 - e^{-u})^{j+1-k} \right. \\
\cdot \binom{J}{K} e^{-uK}(1 - e^{-u})^{J-K} + q_j \binom{j}{k} e^{-uK}(1 - e^{-u})^{j-k} \\
\cdot \left[G_J \binom{J+1}{K} \cdot e^{-uK}(1 - e^{-u})^{J+1-K} \right. \\
\left. \left. + H_J \binom{J}{K} e^{-uk}(1 - e^{-u})^{J-K} \right] \right\}.
\end{aligned} \tag{85}
$$

It is clear from (85) that $P^*(k,K) > 0$ for all (k,K), since the integrands are all strictly positive. (Note also that we have assumed $\alpha > 0$.) The dependence on (i,I) has disappeared, and it is easy to show from (85) that

$$\sum_{k=0}^{m} \sum_{K=0}^{M} P^*(k,K) = 1.$$

Thus $B^*(0,0) = 1$. To show (83), we take a different tack:

Consider any state (k,K). Transitions into the state (k,K) are of four types:

$$
\begin{aligned}
&(k-1,K) \rightarrow (k,K) && \text{(type } a\text{)} \\
&(k,K-1) \rightarrow (k,K) && \text{(type } b\text{)} \\
&(k+1,K) \rightarrow (k,K) && \text{(type } c\text{)} \\
&(k,K+1) \rightarrow (k,K) && \text{(type } d\text{)}.
\end{aligned}
$$

Transitions out of the state (k,K) are also of four types:

$$
\begin{aligned}
&(k,K) \rightarrow (k-1,K) && \text{(type } a'\text{)} \\
&(k,K) \rightarrow (k,K-1) && \text{(type } b'\text{)} \\
&(k,K) \rightarrow (k+1,K) && \text{(type } c'\text{)} \\
&(k,K) \rightarrow (k,K+1) && \text{(type } d'\text{)}.
\end{aligned}
$$

Denote by $N_y(t)$ the expected number of transitions of type y in the interval $(0,t]$.

If we consider the process only at times when the state (k,K) exists, transitions of type (a') form a Poisson process of density k, and transitions of type (b') form a Poisson process of density K. Thus,

$$N_{a'}(t) = k \int_0^t P(k, K, t)dt \tag{86a'}$$

$$N_{b'}(t) = K \int_0^t P(k, K, t)dt. \tag{86b'}$$

Similarly,

$$N_c(t) = (k + 1) \int_0^t P(k + 1, K, t)dt \tag{86c}$$

$$N_d(t) = (K + 1) \int_0^t P(k, K + 1, t)dt. \tag{86d}$$

Now $\{\xi_n = k, \Xi_n = K\}$ is a recurrent event, with mean recurrence time $[\alpha/P(k,K)] > 0$. Thus, from the "elementary renewal theorem,"[15]

$$\lim_{t\to\infty} \frac{M_{ik}{}^{IK}(t)}{t} = \frac{P(k,K)}{\alpha}.$$

But clearly,

$$N_{d'}(t) = q_k G_K M_{ik}{}^{IK}(t),$$

so that

$$\lim_{t\to\infty} \frac{N_{d'}(t)}{t} = \frac{q_k G_K P(k,K)}{\alpha} = \frac{b G_K P^o(k,K)}{\alpha}. \tag{86d'}$$

Similarly,

$$\lim_{t\to\infty} \frac{N_b(t)}{t} = \frac{G_{K-1} b P^o(k, K - 1)}{\alpha} \tag{86b}$$

$$\lim_{t\to\infty} \frac{N_{c'}(t)}{t} = \frac{p_k P(k,K)}{\alpha} = \frac{P(k,K) - bP^o(k,K)}{\alpha} \tag{86c'}$$

$$\lim_{t\to\infty} \frac{N_a(t)}{t} = \frac{P(k - 1, K) - bP^o(k - 1, K)}{\alpha}. \tag{86a}$$

We now notice that in any interval $(0,t]$, the number of transitions out of the state (k,K) can differ from the number of transitions into the state (k,K) by at most 1. From this remark, and all the equations (86), it follows that

$$\begin{aligned}(k + K)P^*(k,K) &+ aP(k,K) - abH_K P^o(k,K)\\ &= ab[G_{K-1}P^o(k,K - 1) - P^o(k - 1,K)] + aP(k - 1,K) \qquad (87)\\ &\quad + (k + 1)P^*(k + 1,K) + (K + 1)P^*(k,K + 1).\end{aligned}$$

Taking the double binomial moment of (87), one obtains

$$(r + R)B^*(r,R) = a\Big\{B(r-1,R) - \binom{m+1}{r} B(m,R) + b\Big[C^o(r,R-1) - B^o(r-1,R) + \binom{m+1}{r} B^o(m,R)\Big]\Big\}. \tag{88}$$

We now note that, according to (51),

$$a\Big[B(r-1,R) - \binom{m+1}{r} B(m,R)\Big] = abC_{r+R-1}\Big[\sum_{j=r-1}^{m} \frac{B^o(j,R)}{C_{j+R}} - \sum_{j=r}^{m} \frac{C^o(j,R-1)}{C_{j+R-1}}\Big] - ab\binom{m+1}{r} B^o(m,R). \tag{89}$$

Putting (89) into (88), we obtain (83).

It is now easy to see that the $B^*(r,R)$ determine the $P^*(k,K)$. For from (83)

$$\lim_{R\to\infty} \frac{B^*(0,R)}{B^*(0,R-1)} = \lim_{R\to\infty} \frac{r+R-1}{r+R} \frac{\varphi_{R-1}}{\varphi_R} \frac{B(0,R)}{B(0,R-1)} = \lim_{R\to\infty} \frac{B(0,R)}{B(0,R-1)} = 0.$$

Corollary: For Poisson input, $P^*(k,K) = P(k,K)$.

Proof: For Poisson input, $F(x) = 1 - e^{-ax}$, $0 < a < \infty$; $a = 1/\alpha$. Thus

$$\varphi(s) = \frac{a}{a+s}, \qquad \varphi_r = \frac{a}{a+r}$$

$$B^*(r,R) = \frac{a}{r+R} \frac{r+R}{a} B(r,R) = B(r,R),$$

and since the double binomial moments determine the probabilities uniquely, the result follows.

Examples will be found in the next section.

V. EXAMPLES FOR THE STATIONARY PROCESS

5.1 *Categories of Examples*

In this section we will try to find the stationary binomial moments $B^o(r,R)$, $B(r,R)$, and $B^*(r,R)$ for certain special cases, or categories of cases. In the easiest cases we will succeed in finding explicit expressions for all these moments; in a harder case we will find explicit expressions only when $R = 1$ or $R = 2$; in the most complicated example (the random slip with overflow group, mentioned in Section I), the treatment is numerical, and only the results for the over-all blocking, B, are reported.

If the first-choice group is full-access, the situation is particularly simple, since overflow can only occur if $\xi_n = m$; the vector equations (24) for $U^o(k,R)$ then become scalar, and $B^o(r,R) = \binom{m}{r} U^o(m,R)$.

If the balking on the first-choice group is arbitrary, but the overflow group is infinite with no balking, or with constant balking probability, as in Example 3 above, some simplification occurs. For then,

$$V^o(k,R) = GU^o(k,R)$$

and hence (24) becomes a recurrence relation, although the quantities it relates are vectors. In such a case it is straightforward to find the first few moments of the distribution on the overflow group.

In cases in which neither of the above simplifications occur, the form of the balking probabilities may still be such as to facilitate calculation; an example of this is the random slip with overflow group.

5.2 *Full-Access First-Choice Group*

We suppose

$$q_k = 0 \qquad (k = 0, 1, \cdots, m - 1)$$

$$q_m = 1.$$

Equations (24) reduce to the single equation

$$U^o(m,R) = \mu_{mm}(R) V^o(m, R - 1) \tag{90}$$

and from (13),

$$\mu_{mm}(R) = \frac{\gamma(R)}{1 - \gamma(R)} \qquad (R = 1,2, \cdots).$$

$\gamma(R)$ is given by (1); it easily follows that

$$\mu_{mm}(R) = \frac{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j-1}(R)}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_j(R)}}. \tag{91}$$

Noting that, from the definitions,

$$C_j(R) = \frac{C_{j+R}}{C_{R-1}}, \tag{92}$$

(91) becomes

$$\mu_{mm}(R) = \frac{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+R-1}}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+R}}}. \tag{93}$$

We also know [from (25)] that

$$U^o(m,0) = \frac{P(m)}{b} = 1. \tag{94}$$

Example 4:

We now consider a slight generalization of the system considered by Brockmeyer (see Section I). Namely, let

$$\begin{aligned} q_k &= 0 \qquad (k = 0, 1, \cdots, m-1) \\ q_m &= 1 \\ H_K &= H \qquad (K = 0, 1, \cdots, M-1) \\ H_M &= 1. \end{aligned}$$

In this case we have

$$V^o(m,R) = G\left[U^o(m,R) - \binom{M}{R} U^o(m,M)\right].$$

Thus, from (90),

$$U^o(m,R) = \mu_{mm}(R)G\left[U^o(m,R-1) - \binom{M}{R-1} U^o(m,M)\right] \tag{95}$$

$$(R = 1, 2, \cdots, M).$$

The solution of (95) is

$$U^o(m,R) = \left[G^R \prod_{Q=1}^{R} \mu_{mm}(Q)\right] \frac{\sum_{J=R}^{M} \binom{M}{J} \left[G^J \prod_{Q=1}^{J} \mu_{mm}(Q)\right]^{-1}}{\sum_{J=0}^{M} \binom{M}{J} \left[G^J \prod_{Q=1}^{J} \mu_{mm}(Q)\right]^{-1}}. \tag{96}$$

Now, from (93),

$$\prod_{Q=1}^{R} \mu_{mm}(Q) = \frac{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_j}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+R}}} \qquad (R = 1, 2, \cdots). \tag{97}$$

Thus,

$$B^o(r,R) = \binom{m}{r} G^R \frac{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_j}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+R}}} \cdot \frac{\sum_{J=R}^{M} \frac{\binom{M}{J}}{G^J} \sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+J}}}{\sum_{J=0}^{M} \frac{\binom{M}{J}}{G^J} \sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+J}}}. \tag{98}$$

We notice [see (T54)] that

$$\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_j} = \frac{1}{b} = \frac{1}{P(m)}.$$

Thus, from (51),

$$B(r,R) = G^R C_{r+R} \frac{\sum_{J=R}^{M} \frac{\binom{M}{J}}{G^J} \sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+J}}}{\sum_{J=0}^{M} \frac{\binom{M}{J}}{G^J} \sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+J}}} \cdot \left\{ \frac{\sum_{j=r}^{m} \binom{m}{j} \frac{1}{C_{j+R}}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+R}}} - \frac{\sum_{j=r+1}^{m} \binom{m}{j} \frac{1}{C_{j+R-1}}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+R-1}}} \right\} \qquad (R = 1, 2, \cdots, M). \tag{99}$$

$B^*(r,R)$ follows from (83).

When $G = 1$, (99) is the generalization to recurrent input of Brockmeyer's result, (4). It can indeed be verified that (99), for Poisson input and for $G = 1$, agrees with (4).

For infinite full-access overflow group ($M = \infty$, $G = 1$), (99) becomes

$$B(r,R) = C_{r+R} \left\{ \frac{\sum_{j=r}^{m} \binom{m}{j} \frac{1}{C_{j+R}}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+R}}} - \frac{\sum_{j=r+1}^{m} \binom{m}{j} \frac{1}{C_{j+R-1}}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_{j+R-1}}} \right\}. \tag{100}$$

Equation (100) is the generalization to recurrent input of Kosten's result, (2). Again it can be verified that (100), for Poisson input, agrees with (2).

5.3 *Constant-Balking Overflow Group*

We suppose that $M = \infty$

$$G_K = G \qquad (K = 0, 1, 2, \cdots).$$

Then (24) becomes

$$U^o(k,R) = q_k G \sum_{j \in \mathcal{A}} \mu_{jk}(R) U^o(j,R-1) \qquad (R = 1, 2, 3, \cdots). \quad (101)$$

Example 5

Suppose further that

$$q_k = q \qquad (k = 0, 1, \cdots, m-1)$$

$$q_m = 1.$$

This might describe a system in which some auxiliary equipment is needed to set up a connection on the first-choice group, some other auxiliary equipment is needed to set up a connection on the overflow group, and the probability that the auxiliary equipment is idle is constant, but this probability is different for the two groups. This is a rather plausible system, except that the overflow group is infinite.

We note that the blocking for such a system is

$$B = \sum_{k=0}^{m} \sum_{K=0}^{\infty} q_k H_K P(k,K) = H[q + pP(m)].$$

It is easy to show by the methods of Ref. 2 that in this example

$$B(r,0) = p^r C_r \frac{\displaystyle\sum_{j=r}^{m} \binom{m}{j} \frac{1}{p^j C_j}}{\displaystyle\sum_{j=0}^{m} \binom{m}{j} \frac{1}{p^j C_j}} \qquad (102)$$

so that in particular

$$P(m) = B(m,0) = \frac{1}{\displaystyle\sum_{j=0}^{m} \binom{m}{j} \frac{1}{p^j C_j}}.$$

Thus,

$$B = H\left[q + \frac{p}{\displaystyle\sum_{j=0}^{m} \binom{m}{j} \frac{1}{p^j C_j}}\right].$$

Instead of (101), we use (29), which in our case becomes

$$B^o(r,R) = G \sum_{j=0}^{m} [f_{jr}(R) - g_{jr}(R)]B^o(j,R-1) \qquad (R = 1, 2, \cdots). \tag{103}$$

In this case we have, from (37),

$$\Psi_{jr}(s) = p\left[\Phi_{jr}(s) - \binom{m}{r}\Phi_{jm}(s)\right]. \tag{104}$$

We can solve (35), (36), and (104) to obtain

$$\Phi_{jr}(s) = \frac{p^r C_r(s)}{\sum_{l=0}^{m}\binom{m}{l}\frac{1}{p^l C_l(s)}}\left\{\left[\sum_{l=0}^{j}\binom{j}{l}\frac{1}{p^l C_{l-1}(s)}\right]\left[\sum_{l=r}^{m}\binom{m}{l}\frac{1}{p^l C_l(s)}\right] - \left[\sum_{l=0}^{m}\binom{m}{l}\frac{1}{p^l C_l(s)}\right]\cdot\left[\sum_{l=r+1}^{m}\binom{j}{l}\frac{1}{p^l C_{l-1}(s)}\right]\right\}.$$

It follows from (31) that

$$f_{lr}(s) = p^r C_r(s)\left[\frac{\sum_{k=r}^{m}\binom{m}{k}\frac{1}{p^k C_k(s)}}{\sum_{k=0}^{m}\binom{m}{k}\frac{1}{p^k C_k(s)}}\cdot\frac{1}{p^l C_{l-1}(s)} - \begin{cases}\frac{1}{p^l C_{l-1}(s)} & \text{if} \quad l > r\\ 0 & \text{if} \quad l \leqq r\end{cases}\right]. \tag{105}$$

From (105), $f_{lr}(s) - g_{lr}(s)$ can easily be calculated by observing that in this example

$$f_{lr}(s) - g_{lr}(s) = qf_{lr}(s) + p\binom{m}{r}f_{lm}(s).$$

Then, from (103) one obtains

$$B^o(r,R) = G\left\{\frac{p\binom{m}{r} + qp^r C_r(R)\sum_{k=r}^{m}\binom{m}{k}\frac{1}{p^k C_k(R)}}{\sum_{k=0}^{m}\binom{m}{k}\frac{1}{p^k C_k(R)}}\cdot\sum_{j=0}^{m}\frac{B^o(j,R-1)}{p^j C_{j-1}(R)} - qp^r C_r(R)\sum_{j=r+1}^{m}\frac{B^o(j,R-1)}{p^j C_{j-1}(R)}\right\}. \tag{106}$$

Noting that, from (30) and (102),

$$B^o(r,0) = \frac{qB(r,0) + p\binom{m}{r}B(m,0)}{q + pB(m,0)} \tag{107}$$

$$= \left[qp^rC_r \sum_{k=r}^{m} \binom{m}{k} \frac{1}{p^kC_k} + p\binom{m}{r}\right]\left[q\sum_{k=0}^{m}\binom{m}{k}\frac{1}{p^kC_k} + p\right]^{-1},$$

we can use (106) to find $B^o(r,1)$, $B^o(r,2)$, etc., and in particular, the first and second moments of the distribution on the overflow group only, at overflow instants, $B^o(0,1)$, $B^o(0,2)$. The formulas are long; we quote only:

$$B^o(0,1) = G\left\{qC_1 + \frac{p}{\sum_{k=0}^{m}\binom{m}{k}\frac{1}{p^kC_{k+1}}} \cdot \frac{\sum_{k=0}^{m}(1+kq)\binom{m}{k}\frac{1}{p^kC_k}}{p + q\sum_{k=0}^{m}\binom{m}{k}\frac{1}{p^kC_k}}\right\}. \tag{108}$$

5.4 *Other Cases*

Once $B^o(r,R)$ is known, it is straightforward to determine $B(r,R)$ and $B^*(r,R)$, using (51) and (83) respectively. [If $B^o(r,R)$ is known, $C^o(r,R)$ can be determined, for use in (51), from the relation, which follows from their definitions:

$$C^o(r,R) = \sum_{J=R}^{M}\binom{J}{R}(\Delta^{J-R}G_R)B^o(r,J); \tag{109}$$

see (T45).] The problem is thus to determine $B^o(r,R)$, from (29) and (30), or equivalently to determine $U^o(k,R)$ from (24) and (25). We consider the latter method.

To use (24) and (25), one must first of all determine $\mu_{jk}(R)$ for all relevant j, k, and R [say, from (T70)], as well as $P(k)$ [say, from (T44) and (T45)]. Then the $V^o(k,R)$ must be expressed in terms of the $U^o(k,R)$; in general $V^o(k,R)$ can be expressed in terms of the $U^o(k,J)$, with $J \geqq R$, by a relation analogous to (109):

$$V^o(k,R) = \sum_{J=R}^{M}\binom{J}{R}(\Delta^{J-R}G_R)U^o(k,J). \tag{110}$$

When (110) is substituted in (24), one obtains a set of simultaneous equations for the $U^o(k,R)$. Equation (25) serves as a boundary condition. If M is finite, (24) can be used to express $U^o(k,M-1)$, $U^o(k,M-2)$, $\cdots$, $U^o(k,0)$ successively in terms of $U^o(k,M)$, and (25) can then be used to determine $U^o(k,M)$.

When the $U^o(k,R)$ are known, one finds the $B^o(r,R)$ by taking binomial moments, and then the $B(r,R)$ from (51). The probabilities $P(k,K)$ then follow by inverting the binomial moments, and the over-all blocking is determined by

$$B = \sum_{k=0}^{m} \sum_{K=0}^{M} q_k H_K P(k,K).$$

Example 6

We consider the system described in Section I

$$q_k = \binom{k}{\gamma} \Big/ \binom{m}{\gamma} \qquad (k = 0, 1, \cdots, m)$$

$$H_K = 0 \qquad (K = 0, 1, \cdots, M - 1)$$

$$H_M = 1.$$

The IBM 7090 computer at Murray Hill was programmed to find the blocking probability B for certain values of the parameters, namely:

$$m + M = 10$$

$$\gamma + M = 6.$$

The calculations were carried out for two kinds of input:

(*i*) Poisson

(*ii*) That sort of recurrent input which is itself the overflow from a group of m_0 trunks to which a Poisson stream of calls (with negative-exponential holding times) of mean intensity a_0 is submitted. Note that, since Poisson traffic is completely characterized by one parameter (its mean, in our case a_0), this sort of recurrent input is completely characterized by two parameters (a_0 and m_0).

Note also that this program allows one to calculate B for certain more complicated trunking arrangements, in the case of Poisson input, e.g., 2 common trunks overflowing to a random slip of 3 on 7 which in turn overflows to 1 common trunk. (This arrangement also involves a total of 10 trunks and 6 crosspoints per line.)

The results (blocking probability B as a function of input traffic a) are shown in Tables I and II and Fig. 1. The cases treated were $m_0 = 0$ (Poisson input, $a = a_0$) and $m_0 = 2$, in which case, of course,

$$a = \frac{a_0^{\,3}}{2} \Big/ \left(1 + a_0 + \frac{a_0^{\,2}}{2}\right);$$

γ was given the values 2,3,4,5,6. (Note that if $\gamma = 6$, then $M = 0$; there is no overflow group.)

TABLE I — RANDOM SLIP. BLOCKING AS A FUNCTION OF SUBMITTED TRAFFIC, FOR RECURRENT INPUT ($m_0 = 2$)

a_0 (call-hours)	a (call-hours)	Blocking, for the Configurations				
		2/6 + 4	3/7 + 3	4/8 + 2	5/9 + 1	6/10
1.0	0.2000	4.111×10^{-8}	2.711×10^{-8}	2.928×10^{-8}	5.785×10^{-8}	4.619×10^{-7}
1.5	0.4655	1.272×10^{-6}	8.990×10^{-7}	9.132×10^{-7}	1.425×10^{-6}	6.594×10^{-6}
2.0	0.8000	1.398×10^{-5}	1.039×10^{-5}	1.024×10^{-5}	1.290×10^{-5}	4.402×10^{-5}
2.5	1.179	8.454×10^{-5}	6.534×10^{-5}	6.338×10^{-5}	7.870×10^{-5}	1.891×10^{-4}
3.0	1.588	3.451×10^{-4}	2.757×10^{-4}	2.654×10^{-4}	3.101×10^{-4}	6.064×10^{-4}
3.5	2.018	1.066×10^{-3}	8.762×10^{-4}	8.407×10^{-4}	9.417×10^{-4}	1.575×10^{-3}
4.0	2.461	2.675×10^{-3}	2.253×10^{-3}	2.161×10^{-3}	2.348×10^{-3}	3.484×10^{-3}
4.5	2.916	5.713×10^{-3}	4.917×10^{-3}	4.727×10^{-3}	5.020×10^{-3}	6.792×10^{-3}
5.0	3.378	1.074×10^{-2}	9.421×10^{-3}	9.073×10^{-3}	9.486×10^{-3}	1.196×10^{-2}
5.5	3.847	1.823×10^{-2}	1.625×10^{-2}	1.570×10^{-2}	1.621×10^{-2}	1.935×10^{-2}
6.0	4.320	2.846×10^{-2}	2.573×10^{-2}	2.492×10^{-2}	2.552×10^{-2}	2.921×10^{-2}
6.5	4.797	4.149×10^{-2}	3.797×10^{-2}	3.688×10^{-2}	3.751×10^{-2}	4.158×10^{-2}
7.0	5.277	5.714×10^{-2}	5.286×10^{-2}	5.148×10^{-2}	5.208×10^{-2}	5.635×10^{-2}

TABLE II — RANDOM SLIP. BLOCKING AS A FUNCTION OF SUBMITTED TRAFFIC, FOR POISSON INPUT ($m_0 = 0$).

a_0 (call-hours)	a (call-hours)	Blocking, for the Configurations				
		2/6 + 4	3/7 + 3	4/8 + 2	5/9 + 1	6/10
1.0	1.0	1.010×10^{-6}	6.673×10^{-7}	6.771×10^{-7}	1.141×10^{-6}	6.407×10^{-6}
1.5	1.5	2.203×10^{-4}	1.573×10^{-5}	1.527×10^{-5}	2.099×10^{-5}	6.975×10^{-5}
2.0	2.0	1.750×10^{-4}	1.324×10^{-4}	1.265×10^{-4}	1.554×10^{-4}	3.664×10^{-4}
2.5	2.5	7.890×10^{-4}	6.250×10^{-4}	5.943×10^{-4}	6.815×10^{-4}	1.257×10^{-3}
3.0	3.0	2.474×10^{-3}	2.034×10^{-3}	1.935×10^{-3}	2.124×10^{-3}	3.307×10^{-3}
3.5	3.5	6.032×10^{-3}	5.112×10^{-3}	4.881×10^{-3}	5.204×10^{-3}	7.174×10^{-3}
4.0	4.0	1.225×10^{-2}	1.064×10^{-2}	1.020×10^{-2}	1.067×10^{-2}	1.347×10^{-2}
4.5	4.5	2.167×10^{-2}	1.922×10^{-2}	1.857×10^{-2}	1.909×10^{-2}	2.263×10^{-2}
5.0	5.0	3.449×10^{-2}	3.113×10^{-2}	3.010×10^{-2}	3.073×10^{-2}	3.481×10^{-2}
5.5	5.5	5.054×10^{-2}	4.627×10^{-2}	4.490×10^{-2}	4.553×10^{-2}	4.989×10^{-2}
6.0	6.0	6.938×10^{-2}	6.429×10^{-2}	6.259×10^{-2}	6.316×10^{-2}	6.755×10^{-2}
6.5	6.5	9.044×10^{-2}	8.464×10^{-2}	8.264×10^{-2}	8.309×10^{-2}	8.734×10^{-2}
7.0	7.0	1.131×10^{-1}	1.067×10^{-1}	1.045×10^{-1}	1.048×10^{-1}	1.087×10^{-1}

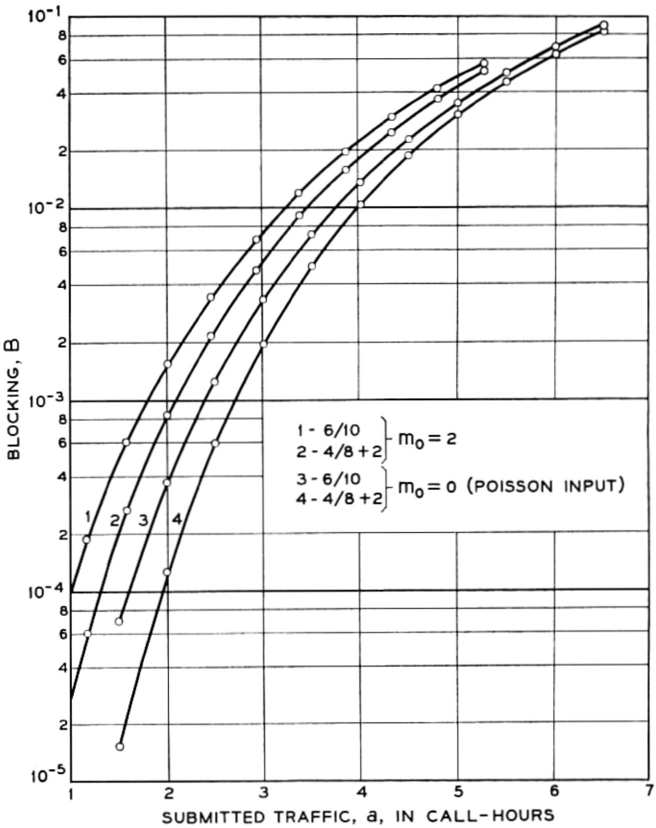

Fig. 1 — Blocking, B, vs submitted traffic, a.

Before commenting on the results, we mention parenthetically several special features introduced into the calculation by the special form of the balking probabilities and by the kind of input process considered in this example. First, as to finding the $P(k)$: (T44) and (T45) read, in our notation

$$B(r,0) = \frac{\varphi_r}{1 - \varphi_r} D(r - 1,0) \tag{111}$$

$$D(r,0) = \sum_{j=r}^{m} \binom{j}{r} (\Delta^{j-r} p_r) B(j,0). \tag{112}$$

In the present example,

$$\frac{\varphi_r}{1 - \varphi_r} = \frac{a_0}{r} \frac{C_r^{m_0}(a_0)}{C_{r+1}^{m_0}(a_0)} \qquad (r = 1, 2, \cdots) \tag{113}$$

and

$$\Delta^{j-r} p_r = -\frac{\binom{r}{j-\gamma}}{\binom{m}{\gamma}} \qquad (j = r + 1, r + 2, \cdots). \tag{114}$$

Also, since the overflow group is full-access (although finite), the relation (110) becomes

$$V^o(k,R) = U^o(k,R) - \binom{M}{R} U^o(k,M). \tag{115}$$

In Tables I and II and Fig. 1, we have used the notation $\gamma/m + M$ to describe a random-slip configuration in which each line has access to γ out of the m first-choice trunks and all the overflow trunks, except that the case $\gamma = 6$, $m = 10$, $M = 0$ is referred to as 6/10. The curves in Fig. 1 have been drawn, to avoid crowding, only for 4/8 + 2 and 6/10.

The following conclusions can be drawn from these results:

(*i*) The blocking is higher, for the same mean traffic, when $m_0 = 2$ than when $m_0 = 0$. This is consistent with the intuitive notion that overflow traffic is "peaky".

(*ii*) In a practical range of blocking ($B = 0.001$ or 0.01), 4/8 + 2 is the "best" arrangement and 6/10 is the "worst" of those considered, from the point of view of the traffic capacity of the system for a fixed blocking probability. It can be seen from the curves that if one wanted an arrangement using 6 crosspoints per line and 10 trunks, one would gain about 8 per cent (for $m_0 = 2$) or 6 per cent (for $m_0 = 0$) in traffic capacity at $B = 0.01$, by using the arrangement 4/8 + 2 instead of 6/10. At a blocking probability $B = 0.001$, these gains would be about 16 and 11 per cent respectively. Such increases in traffic capacity are not negligible; they seem to be larger for peaky traffic than for Poisson traffic.

(*iii*) For higher blockings ("overload" conditions), the advantage of 4/8 + 2 relative to 6/10 diminishes.

A study for a practical case would involve calculations of the blocking for other values of $\gamma + M$, a knowledge of the relative costs of trunks and crosspoints, and of course many other considerations, such as the relative costs of building and controlling 4/8 + 2 and 6/10 switches. Also, in such a study, one would want to keep in mind the approximations implicit in the model used in this paper. For example:

(*i*) In reality, blocked calls may wait or be resubmitted.

(*ii*) In reality, the number of traffic sources (lines) is finite, so that the arrival process after any instant is dependent on the number of trunks busy at that instant; thus the input is not, in reality, recurrent.

(*iii*) As a further result of the finiteness of the number of lines, the complete set of $\binom{m}{\gamma}$ access patterns required for a perfect random slip probably could not be used, and even if it could, equal traffic would not be submitted to each access pattern (so that the blocking experienced by different subscribers would be different).

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