Optical Maser Oscillators and Noise

By EUGENE I. GORDON

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The transmission line matrix formalism so useful for describing the transfer properties of microwave networks is extended to the electromagnetic fields associated with optical masers. The spontaneous emission noise of the optical maser is examined and shown to be amenable to a thermal description. Taking the point of view, well accepted at microwave frequencies, that a weakly nonlinear oscillator is a saturated amplifier of noise, the power and linewidth of the noise radiation emitted by the optical maser is calculated using the transmission line formalism. The significant parameters for any optical maser are shown to be the frequency, the single-pass gain of the maser medium, the effective mirror reflectivity and the population ratio. The pre-oscillation characteristics of the maser are examined and the reason for the extremely sharp oscillation threshold of the gas masers is discussed. Some observations concerning semiconductor optical masers are also made.

I. INTRODUCTION

This paper represents an attempt to describe the optical maser or laser from a microwave circuit point of view and is largely tutorial, since many of the results obtained from a circuit viewpoint are already known. The generality of the method of approach enlarges their area of validity, however.

Many of the people working on optical masers who do not have a background in microwave theory and techniques may find a fresh point of view. In particular, they may find a very modest introduction to an extremely well developed store of computational techniques which are applicable to optical masers. This may save them the trouble of inventing their own.

On the other hand, those who have previously been working in the field of microwaves may find that the analogies between optical masers and more conventional microwave devices are more cogent than they had appreciated. Finally, it is hoped that some of the distinctions between oscillating and pre-oscillating or subthreshold masers will be clarified.

II. THE CONVENTIONAL OSCILLATOR

Excluding strongly nonlinear oscillators with periodic but non-sinusoidal waveforms, it is often stated that an oscillator is a device having an internal gain which exceeds its total losses. Supposedly, noise triggers it off and it then continues to put out oscillatory power at a level determined only by saturation effects. The steady-state saturation level is defined as that for which the internal gain just equals the loss. An extensive discussion of microwave oscillators, based on this point of view, is given by Slater.[1]

Although this point of view often constitutes a good working definition of a feedback oscillator, it is incomplete in that it neglects the continuing presence of the noise. As a result, when the internal gain of the oscillator exactly equals the losses, so that the effective lifetime of a photon in the feedback loop is infinite, the noise power output of the oscillator must increase without limit. Similarly, the associated line-width of the noise output must be zero. Since this situation is physically unrealizable, it is clear that the noise must be taken into account and that the steady-state gain could never exactly equal the loss and must always saturate at a slightly lower value. As a result, there could be no continuing, self-sustained oscillation which starts from noise.

From these considerations, it appears that a better, but still incomplete, description of an oscillator would be to say that a steady-state regenerative oscillator is a feedback amplifier driven to saturation by a noise input. Internally produced noise is usually the driving force; however, an additional source may be the noise entering through the output port. The external gain of the amplifier, that is, the gain experienced by any input signal, is usually extremely large unless the amplifier is saturated by the input signal. Since the amplification is obtained regeneratively, that is, by the use of feedback, the bandwidth of the gain is limited; the higher the gain the more limited it is. The amplified, narrow-band noise output is the output signal of the oscillator. When the large gain can be obtained without regeneration, the noise output need not be narrow-band.

This concept of the oscillator as a saturated amplifier of noise is not new and is well known in the microwave art. More recently, this concept has been employed by Gordon, Zeiger and Townes[2] in their treatment of the microwave maser oscillator, and by Wagner and Birnbaum,[3] Schawlow and Townes,[4] Shimoda,[5] Blaquier[6] and Fleck[7] in their treatments of the optical maser oscillator.

While this definition of an oscillator is somewhat more satisfying, it implies that an oscillator is merely an extremely narrow-band filter with gain. As a result, the statistical properties of the noise input should be preserved, except for the spectral narrowing. For example, a filtered Gaussian noise input[8] would remain Gaussian. Since a narrow-band Gaussian noise process shows amplitude fluctuations with a time constant approximately the inverse of its spectral range,[8] the output of a filter with gain should exhibit this property also. The fact that a true oscillator does not indicates that a correlating mechanism is operative.

Gain saturation is one mechanism that operates to eliminate fluctuations in the output intensity. The action is similar to that of a limiter. In an optical maser, gain saturation arises almost entirely from depletion of the population inversion rather than from any nonlinearity in the stimulated emission process. To a very high degree, the output waveform is sinusoidal and the saturation depends only upon the time-average power, the time average extending over a time long compared to the period of the output waveform but short compared to any relaxation mechanism or pumping rate.

Suppose now that the filtered output has a spectral range $\Delta\nu$ so that the input noise has power fluctuations with a time constant approximately $\Delta\nu^{-1}$. If the gain of the maser medium has a relaxation time $\tau_g < \Delta\nu^{-1}$, then the input fluctuations will be virtually absent in the output. On the other hand, if $\tau_g > \Delta\nu^{-1}$ the input fluctuations will appear in the output.

Thus, for a true oscillator

$$\tau_g \Delta\nu < 1 \tag{1a}$$

while for an amplifying filter

$$\tau_g \Delta\nu > 1. \tag{1b}$$

As will be seen later, for a weak optical maser $\Delta\nu$ can be quite large, while for a strong maser $\Delta\nu$ becomes vanishingly small. The cavity bandwidth $\Delta\nu_c$ represents an upper limit for $\Delta\nu$. For example, in a gas maser $\Delta\nu_c \approx 10^6$ cps and $\tau_g \approx 10^{-6}$, so that for unsaturated gains less than the loss, $\Delta\nu \approx \Delta\nu_c$, $\Delta\nu\tau_g \approx 1$, and the device acts like an amplifying filter. When the unsaturated gain exceeds the loss, $\Delta\nu$ becomes much less than $\Delta\nu_c$ and the device becomes an oscillator.

In the semiconductor maser, the lowest reported value is $\Delta\nu \approx 3 \times 10^9$

cps, corresponding to 0.1Å. Unless $\tau_g < 3 \times 10^{-10}$, it is quite probable that the device acts like an amplifying filter rather than a true oscillator.

The interested reader will find a short discussion of the effect of noise on oscillators in a book by van der Ziel.[9] His discussion indicates a method of approach to obtaining a quantitative solution to a very complicated nonlinear problem. However, the concept of the amplifying filter probably yields a good first approximation to the spectral width of the oscillator output.

In the following sections, the foregoing concepts will be exploited to exhibit the circuit formalism and to study linewidth and threshold behavior. The basic results are applicable to any uniformly pumped, single-mode maser or any multimode maser for which nonlinear mixing or coupling of modes is not significant. Some remarks concerning the lack of extremely narrow linewidths in the semiconductor maser will also be made.

III. SINGLE MODE REPRESENTATION FOR AN OPTICAL MASER

The electromagnetic fields associated with the optical maser are very close to being plane waves. To a very good approximation, each mode of the electromagnetic field can be represented by field quantities, $E(z)$ and $I(z)$. These quantities can be normalized to have the dimensions of voltage and current, respectively. The relationship between $E(z)$ and $I(z)$ is obtained by specifying the value of the function $Z(z) = E(z)/I(z)$ at some point z. In the content of this paper, a mode is one member of a complete set of transverse eigenfunctions which are appropriate for the geometry in question. No orthogonality with respect to the z coordinate is implied.

If the various modes of the electromagnetic field are uncoupled and E and I are represented as complex quantities, then a linear relationship of the form

$$\begin{vmatrix} E(z_1) \\ I(z_1) \end{vmatrix} = \begin{vmatrix} A & Z_0 B \\ Z_0^{-1} C & D \end{vmatrix} \begin{vmatrix} E(z_2) \\ I(z_2) \end{vmatrix} \tag{2a}$$

or

$$\boldsymbol{\Psi}(z_1) = \mathbf{T}(z_1, z_2) \cdot \boldsymbol{\Psi}(z_2). \tag{2b}$$

can exist for each mode.[10,11] The input side is taken at z_1 and the output side at z_2, as in Fig. 1. The quantity Z_0 is the characteristic impedance associated with the transmission medium. The directions of E and I are defined so that when the phase difference between E and I falls in

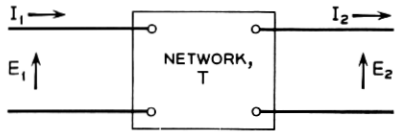

Fig. 1 — Generalized linear two-port network for single mode representation.

the first and fourth quadrant, power is flowing in the direction $z_1 \rightarrow z_2$, while for the second and third quadrant, the direction $z_2 \rightarrow z_1$. The choice of directions makes it possible to determine the result of cascading a number of sections by writing

$$\mathbf{\Psi}(z_1) = \mathbf{T}(z_1, z_2) \cdot \mathbf{T}(z_2, z_3) \cdots \mathbf{T}(z_{n-1}, z_n) \cdot \mathbf{\Psi}(z_n). \tag{3}$$

The complex quantities A, B, C and D are dimensionless and are independent of time in the steady state. In general, they are functions of frequency. For convenience, the characteristic impedance of the transmission medium will be taken as unity, i.e., $Z_0 = 1$. The transmission medium between planes z_n and z_{n+1} will be referred to as a "network." The properties of the network are described uniquely by the quantities A, B, C and D. General relations among these quantities can be determined by specifying the transfer properties of the network. These are reviewed in detail in Appendix A and are described below.† For example, a matched network which produces no reflection from side z_n when terminated by the characteristic impedance on side z_{n+1} can be written

$$\begin{vmatrix} A & B \\ B & A \end{vmatrix} \tag{4}$$

in which A and B are, in general, independent complex parameters.

A reciprocal network is characterized by the relation $AD - BC = 1$. A reactive network has the transformation

$$\begin{vmatrix} \alpha & j\beta \\ j\gamma & \delta \end{vmatrix} \tag{5}$$

in which the four independent parameters α, β, γ and δ are real. A reciprocal reactive network, in addition, has $\alpha\delta + \beta\gamma = 1$, so that only three of the parameters are independent. A matched reciprocal reactive network has $\alpha = \delta$ and $\beta = \gamma$, so that only one parameter is independent. The network of this type can be written

† The Appendix is included because there is no single convenient reference.

$$\begin{vmatrix} \cos \varphi & j \sin \varphi \\ j \sin \varphi & \cos \varphi \end{vmatrix} \tag{6}$$

in which the real parameter φ is the phase shift from z_n to z_{n+1}. A length of transmission medium is an example of a matched reciprocal reactive network. For this case, $\varphi = 2\pi\nu(z_{n+1} - z_n)/c'$, in which c' is the phase velocity of the radiation.

The most general unmatched reciprocal reactive network, which has three independent parameters, can always be characterized as

$$\begin{vmatrix} \alpha & j\beta \\ j\gamma & \delta \end{vmatrix} = \begin{vmatrix} \cos \varphi_1 & j \sin \varphi_1 \\ j \sin \varphi_1 & \cos \varphi_1 \end{vmatrix} \begin{vmatrix} N & 0 \\ 0 & N^{-1} \end{vmatrix} \begin{vmatrix} \cos \varphi_2 & j \sin \varphi_2 \\ j \sin \varphi_2 & \cos \varphi_2 \end{vmatrix}. \tag{7}$$

The network

$$\begin{vmatrix} N & 0 \\ 0 & N^{-1} \end{vmatrix} \tag{8}$$

is known as an ideal transformer of turns ratio N. Thus, the most general unmatched reactive reciprocal network, aside from phase shifts φ_1 and φ_2, is an ideal transformer.

A resistive network produces no phase shift and can be characterized as having A, B, C and D all real. A matched reciprocal resistive network has only one independent parameter and can be written

$$\begin{vmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{vmatrix}. \tag{9}$$

The matched reciprocal resistive network is known as an attenuator. It is shown in Appendix A that the power attenuation is given by $\exp -2\theta$. The parameter θ is referred to as the attenuator line length. The most general matched reciprocal network (resistance plus reactance) has two independent variables and can be written

$$\begin{vmatrix} \cos (\varphi - j\theta) & j \sin (\varphi - j\theta) \\ j \sin (\varphi - j\theta) & \cos (\varphi - j\theta) \end{vmatrix} = \begin{vmatrix} \cos \varphi & j \sin \varphi \\ j \sin \varphi & \cos \varphi \end{vmatrix} \begin{vmatrix} \cosh \theta & \sinh \theta \\ \sinh \theta & \cosh \theta \end{vmatrix}. \tag{10}$$

Therefore, the most general matched reciprocal network consists of an attenuator with phase shift. Note that the attenuation and phase shift commute. As a result, a matched network with distributed attenuation and phase shift can be lumped into a network with attenuation in cascade with a network having only phase shift.

The transmission factor or transmissivity of the network, denoted as L, is shown in Appendix A to have the value

$$L = \frac{4}{| A + B + C + D |^2}. \tag{11}$$

For a matched reactive reciprocal network as in (6), $L = 1$. The factor L is also known as the gain of the network. The transmission factor equals the ratio of power transmitted to power incident when the network is preceded and followed by matched terminations.

The reflection factor or reflectivity of a network is given by

$$R = \tfrac{1}{4} L \, | A + B - C - D |^2. \tag{12}$$

The reflection factor is the ratio of power reflected to power incident when the input and output terminals are matched. For a matched network, $(A = D, B = C)$, $R = 0$.

The noise generated in a network can be represented by suitably chosen current and voltage generators i and e at the input to the network as in Fig. 2.[12] The network itself can then be considered as noiseless. For example, for a series resistor r, the noise generators are appropriately $i = 0$ and $| e | = [4p(\nu)d\nu r]^{\frac{1}{2}}$ in which

$$p(\nu)\,d\nu = \frac{h\nu d\nu}{\exp h\nu/kT - 1} \tag{13}$$

is the thermally generated noise power in a frequency range $d\nu$ centered at frequency ν when the resistor is at temperature T.[13] The phase of e is a random variable. For a shunt resistor, r, the noise generators are $e = 0$ and $| i | = [4p(\nu)d\nu/r]^{\frac{1}{2}}$. For reactive networks, $e = i = 0$.

For an arbitrary network containing lossy elements, the appropriate values of e and i can be expressed in terms of the components of the transformation matrix, A, B, C and D. Since an arbitrary passive network can always be matched by suitable use of transformers and line lengths, placed on either side of the network, the arbitrary network can

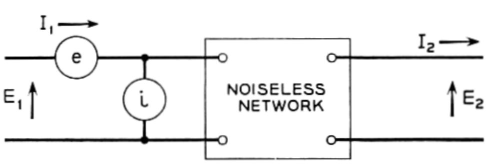

Fig. 2 — Equivalent external current and voltage generators for noiseless network representation.

always be made to appear matched and resistive. It follows that an arbitrary lossy network can always be represented as a resistive matched network imbedded in reactive networks.[14] The reactive networks on either side of the imbedded network are the inverse in reverse order of the networks required for matching the original arbitrary network.

It follows, too, that if the appropriate values of e and i can be determined for any resistive matched network, then the values can be determined for the imbedded network. These values can then be transformed through the reactive networks at the input side of the imbedded network to represent the appropriate values at the input side of the original network.

Thus, it is only necessary to have a general formula for e and i pertinent to a resistive matched network. In Appendix B it is shown that the appropriate values of e and i for any matched resistive network are given by[15]

$$\begin{aligned} |e|^2 &= |i|^2 = AB4p(\nu)d\nu \\ ei^* &= e^*i = \tfrac{1}{2}(A^2 + B^2 - 1)4p(\nu)d\nu. \end{aligned} \tag{14}$$

The values of e and i without the factor $4p(\nu)d\nu$ will be referred to as the normalized values. The quantities, $|e|^2$, $|i|^2$ and ei^* commute with a phase shift network (a length of transmission line). A value e following a transformer of turns ratio N becomes, at the input side of the transformer, Ne, while a current i becomes i/N. Thus, the procedure in finding the values of e and i preceding any given network, $\mathbf{T}$, is to find the appropriate reactive transformations of the form

$$\mathbf{T} = \mathbf{T}(\varphi_1)\cdot\mathbf{T}(N)\cdot T(\varphi_2)\cdot\mathbf{T}(r)\cdot\mathbf{T}(\varphi_3)\cdot\mathbf{T}(M)\cdot\mathbf{T}(\varphi_4) \tag{15}$$

in which $\mathbf{T}(r)$ is the imbedded matched resistive network, and find the values of e and i appropriate to $\mathbf{T}(r)$ using (14). The values of e' and i' appropriate to the input side of $\mathbf{T}$ are $e' = Ne$ and $i' = i/N$.

In Appendix B, it is shown that the noise power into a matched load following any given network at uniform temperature T (considering only the noise arising from the given network and ignoring the noise originating in the matched loads at the input and output side) is given by

$$dP = L\,|e + i|^2 p(\nu)d\nu \tag{16}$$

in which L is the transmission or gain factor for the network, given by (11), and e and i are the normalized noise generators at the input to the network. For example, for an attenuator with transmission factor

L, (16) yields

$$dP = (1 - L)p(\nu)d\nu \tag{17}$$

as is shown in Appendix B.

Since the noise parameters commute with a matched phase shift network, attenuation and phase shift can be lumped with respect to noise properties as well as transfer properties.

IV. THE OPTICAL MASER

The maser medium has the property that light passing through the medium once is amplified by a factor $G_1(\nu)$. In addition, the light undergoes a phase shift, $\varphi(\nu)$. Thus, the matched maser medium can be characterized as an attenuator with a transmission factor G_1 in cascade with a matched reciprocal phase shift network.

Since the spontaneous emission from the maser medium can be considered as thermal noise,[16] the spontaneous emission power radiated into a given mode by a uniform maser medium should be given by

$$dP(\nu) = [1 - G_1(\nu)]p(\nu)d\nu \tag{18}$$

as follows from (17). Since $p(\nu) = h\nu/(\exp h\nu/kT - 1)$, the question naturally arises as to what temperature to associate with the maser medium. In particular, one wonders whether a noise formula like (18), which is valid for passive networks in thermal equilibrium, can be used when the maser medium is active, i.e., when $G_1 > 1$. Normally, the radiation temperature of the uniform maser medium is defined by the Boltzmann factor[16]

$$n_2(\nu)/n_1(\nu) = \exp -h\nu/kT \tag{19}$$

in which n_2 and n_1 are the densities of upper- and lower-state atoms, respectively. Using (19) as the definition of maser temperature, it follows that (18) is precisely correct for a uniform medium.

To illustrate, one can write for the emission power, dP, into a given mode in a frequency range $d\nu$, along the z-axis

$$\partial dP/\partial z = h\nu w_i dP(n_2 - n_1) + \tfrac{1}{2}dw_s h\nu n_2 \tag{20}$$

in which w_i is the probability per unit intensity per unit time for stimulated emission into the mode and dw_s is the probability for spontaneous emission in either direction for the same mode into frequency range $d\nu$. The population densities for the upper and lower maser levels, n_2 and

n_1, are assumed constant with z.† Solving (20) subject to the initial condition $dP = 0$ at $z = 0$ yields

$$dP(z) = \frac{\frac{1}{2}(dw_s/w_i)(1 - G_1)}{(n_1/n_2) - 1} \tag{21}$$

in which

$$G_1 = \exp h\nu w_i(n_2 - n_1)z. \tag{22}$$

Since the probability for stimulated emission is related to the probability for spontaneous emission into frequency range $d\nu$ by

$$dw_s = 2w_i h\nu d\nu \tag{23}$$

for a given mode, (21) and (18) are identical.‡ It follows that (18) correctly accounts for the spontaneous noise into a single mode, so long as one writes $p(\nu) = h\nu/(n_1/n_2 - 1)$. It also indicates the applicability of the formalism described in the preceding section to maser media.

The fact that the maser temperature T, as defined by (19), and $p(\nu)$, as defined by (13), are negative should not distract the reader from the more significant fact that (18) or (21) correctly predicts the noise power emitted by the maser medium. This quantity is never negative, and varies smoothly as T goes from positive to negative values.

The noise output from the maser can be calculated using (16), and this will be the aim of the following analysis. It is worth noting that the only significant parameters characterizing the medium, assuming that the maser medium is uniform and matched, are the total single-pass gain, G_1, the population ratio and the phase shift through the medium. The effective attenuator line length, θ, is given by the expression $G_1 = \exp -2\theta$.

V. THE REGENERATIVE OPTICAL MASER OSCILLATOR

The mirrors forming the optical cavity, being reactive (except for a small absorption loss which will be neglected) and reciprocal, can be characterized as ideal transformers. The phase shift associated with the mirrors on the cavity side can be added to the single-pass phase shift of the maser medium. The mirror phase shift external to the cavity is not significant to the problem at hand.

† This implies no saturation or uniform saturation.

‡ Equation (23) is equivalent to the statement that the total rate of spontaneous emission is related to the total rate of induced emission per unit intensity by $dw_s = w_i 8\pi h\nu^3 d\nu/c^3$, since the total number of modes per unit volume per unit frequency interval is given by $8\pi\nu^2/c^3$. (See Ref. 16.)

Since the reflection factor for an ideal transformer is, from (12) and (8)

$$R = (N^2 - 1)^2/(N^2 + 1)^2 \tag{24}$$

it follows that the equivalent turns ratio for a mirror of reflectivity R is given by

$$N^2 = (1 + R^{\frac{1}{2}})/(1 - R^{\frac{1}{2}}). \tag{25}$$

The transmission factor L_m for the maser cavity with unequal mirrors of reflectivity R and R' is obtained by combining the cascade of transformer with turns ratio N, attenuator with loss parameter $\theta = -\frac{1}{2} \ln G_1$, transmission line with phase shift $\varphi = 2\pi\nu L/c' +$ constant, in which $2L/c'$ is the equivalent round-trip time,† and transformer with turns ratio M^{-1}, $M^2 = (1 + R'^{\frac{1}{2}})/(1 - R'^{\frac{1}{2}})$, as in Fig. 3. The noise power

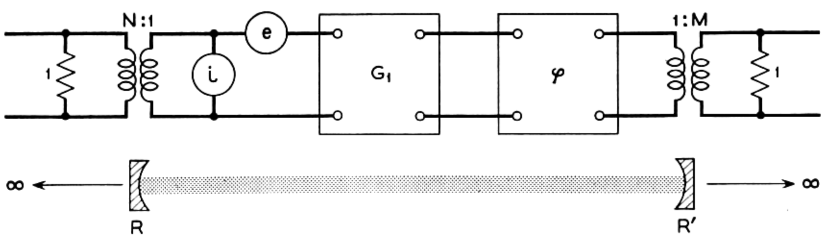

Fig. 3 — Equivalent circuit for maser cavity with mirrors of unequal reflectivity.

from one end of the maser is given by (16)

$$dP = L_m \,|\, Ne + N^{-1}i \,|^2 p(\nu)d\nu \tag{26}$$

where the normalized noise generators for the maser medium are given by

$$\begin{aligned} |\, e \,|^2 &= |\, i \,|^2 = \cosh\theta \sinh\theta \\ ei^* &= e^*i = \tfrac{1}{2}(\cosh^2\theta + \sinh^2\theta - 1) \end{aligned} \tag{27}$$

as follows from (14) and (9)

The cascade of mirror, maser medium and mirror takes the form

$$\begin{aligned} &\begin{vmatrix} N & 0 \\ 0 & N^{-1} \end{vmatrix} \cdot \begin{vmatrix} \cos(\varphi - j\theta) & j\sin(\varphi - j\theta) \\ j\sin(\varphi - j\theta) & \cos(\varphi - j\theta) \end{vmatrix} \cdot \begin{vmatrix} M^{-1} & 0 \\ 0 & M \end{vmatrix} \\ &\quad = \begin{vmatrix} (N/M)\cos(\varphi - j\theta) & NMj\sin(\varphi - j\theta) \\ (NM)^{-1}j\sin(\varphi - j\theta) & (M/N)\cos(\varphi - j\theta) \end{vmatrix}. \end{aligned} \tag{28}$$

† The phase velocity c' is a function of ν and G_1 by virtue of the anomalous dispersion of the maser medium.

Using (11), the transmission factor is given by

$$L_m = \frac{4}{|(N/M + M/N)\cos(\varphi - j\theta) + j(MN + [MN]^{-1})\sin(\varphi - j\theta)|^2} = \frac{1}{[(1 - R^{\frac{1}{2}}R'^{\frac{1}{2}}G_1)^2/G_1(1-R)(1-R')] + [4R^{\frac{1}{2}}R'^{\frac{1}{2}}/(1-R)(1-R')]\sin^2\varphi}. \tag{29}$$

Note that in the limit $R = R' = 0$ (no mirrors), $L_m = G_1$ as would be expected; while in the limit $R = R'$, $G_1 = 1$ (a transparent maser medium)

$$L_m = \frac{1}{1 + [4R/(1-R)^2]\sin^2\varphi} \tag{30}$$

which is the transmission factor for the Fabry-Perot or optical cavity surrounding the maser medium.[7,17] Equation (29), or versions of it with $R = R'$, has been derived before.[7] In these cases, however, it had been necessary to assume that the maser medium uniformly fills the region between the mirrors. No such restriction is necessary.

The noise power $|Ne + N^{-1}i|^2$ has the value, using (27)

$$\begin{aligned} |Ne + N^{-1}i|^2 &= (N^2 + N^{-2})|e|^2 + 2ei^* \\ &= (N^2 + N^{-2})\cosh\theta\sinh\theta \\ &\quad + (\cosh^2\theta + \sinh^2\theta - 1) \end{aligned} \tag{31}$$

which after some manipulation yields

$$|Ne + N^{-1}i|^2 = (1 - G_1)(1 + RG_1)/G_1(1 - R). \tag{32}$$

Combining (26), (29) and (32), the noise power in frequency range $d\nu$ leaving the maser cavity through the mirror R' can be written

$$dP = \frac{(1 + RG_1)(1 - R')(1 - G_1)p(\nu)d\nu}{(1 - R^{\frac{1}{2}}R'^{\frac{1}{2}}G_1)^2 + 4R^{\frac{1}{2}}R'^{\frac{1}{2}}G_1\sin^2\varphi}. \tag{33}$$

In the vicinity of a cavity resonance at frequency ν_0, the phase shift φ differs from some multiple of π by an amount $\Delta\varphi = 2\pi(\nu - \nu_0)L/c'$. The free spectral range of the cavity mode at frequency ν_0 is the range $\nu_0 - c'/4L \leqq \nu \leqq \nu_0 + c'/4L$ as illustrated in Fig. 4; the mode spacing is $c'/2L$. Thus, the total noise power leaving the cavity through R', associated with one cavity mode, is obtained by integrating dP over the free spectral range of the mode, yielding

$$P = \int_{\nu_0 - c'/4L}^{\nu_0 + c'/4L} \frac{p(\nu) C_1 \, d\nu}{C_2 + C_3 \sin^2 2\pi(\nu - \nu_0)L/c'} . \tag{34}$$

With the substitution $\varphi = 2\pi(\nu - \nu_0)L/c'$, (34) can be written

$$P = p(\nu_0) \frac{c'}{2\pi L} \int_{-\pi/2}^{+\pi/2} \frac{C_1 \, d\varphi}{C_2 + C_3 \sin^2 \varphi} \tag{35}$$

in which the quantities C_1, C_2 and C_3 are given by

$$C_1 = [(1 + RG_1)(1 - R')(1 - G_1)],$$

$$C_2 = [1 - R^{\frac{1}{2}} R'^{\frac{1}{2}} G_1]^2$$

and

$$C_3 = 4R^{\frac{1}{2}} R'^{\frac{1}{2}} G_1 .$$

Since the integrand is large only in a very small range of frequencies near ν_0, it can safely be assumed that $p(\nu)$, G_1 and R are constant with frequency and have their values at $\nu = \nu_0$.

With this approximation the integral has the value $\pi C_1/[C_2^2 + C_2 C_3]^{\frac{1}{2}}$ (see Ref. 18), yielding

$$P(R') = [(1 + RG_1)(1 - R')/(1 - RR'G_1^2)] \\ (1 - G_1) p(\nu_0)(c'/2L). \tag{36}$$

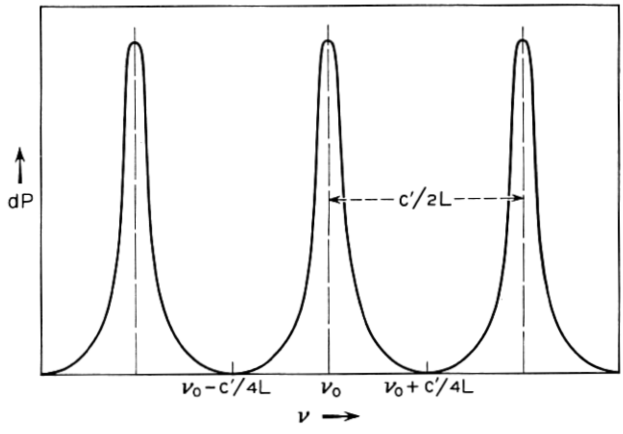

Fig. 4 — Free spectral range of cavity mode at frequency ν_0.

From (18) it may be noted that $[1 - G_1(\nu_0)]p(\nu_0)c'/2L$ is the spontaneous emission power that would be emitted by the maser medium, in the absence of the cavity, into the spectral range $c'/2L$ if the gain were constant over that range. When $G_1 > 1$, the spontaneous emission is enhanced by stimulated emission. The noise power

$$[1 - G_1(\nu_0)]p(\nu_0)c'/2L$$

will be denoted as $P_s(\nu_0)$. The noise power leaving the cavity through mirror R' is, therefore

$$P(R') = P_s(1 + RG_1)(1 - R')/(1 - RR'G_1^2). \tag{37}$$

The power leaving the cavity through mirror R is given by (37) with R and R' interchanged. The total power leaving the cavity is

$$P_t = 2P_s[1 - RR'G_1 + \tfrac{1}{2}(G_1 - 1)(R + R')]/[1 - RR'G_1^2] \tag{38}$$

so that the cavity enhances the spontaneous emission by the factor following $2P_s$.

It should be noted that the integration of (36), which leads to (37) and (38), is valid even if $G_1(\nu)$ varies over the range $c'/2L$, so long as the frequency range over which G_1 has a significant variation is large compared to the spectral range of the noise power. This will always be so, providing the natural linewidth of the transition is large compared to the spectral range of the noise, independent of whether the transition is homogeneously or inhomogeneously broadened. Equations (37) and (38) are valid even when the gain profile is saturated, providing the maser medium is uniformly saturated with respect to the axial direction. In general, the saturation will tend to be uniform, assuming uniform pumping, since the power in the cavity tends to be uniform with length. For very high-gain maser media the latter statement is not valid.

It should be noted that (38) contains an implication which is not immediately obvious. In the limit $n_1 = n_2$ with $G_1 = 1$ and P_s finite, (38) states that $P_t = 2P_s$ independent of R or R'. Thus the total spontaneous emission noise power leaving the optical cavity is independent of its bandpass. The spectral distribution of the noise is altered, however, to correspond to the bandpass.†

The preceding results can be used to determine the linewidth of the oscillator. The spectral range or bandwidth of the noise power is obtained from (33) by determining the frequencies at which dP falls to

† In the limit $c'/2L$ very much less than the inhomogeneously broadened linewidth, (38) implies that the total spontaneous emission power into all modes is independent of the mirror reflectivity since $G_1 = 1$ over the entire line.

half its value at $\nu = \nu_0$. This occurs at frequencies $\nu_{\frac{1}{2}}$ defined by

$$4R^{\frac{1}{2}}R'^{\frac{1}{2}}G_1 \sin^2 2\pi(\nu_{\frac{1}{2}} - \nu_0)L/c' = (1 - R^{\frac{1}{2}}R'^{\frac{1}{2}}G_1)^2 \tag{39}$$

yielding a spectral width $\Delta\nu = 2(\nu_{\frac{1}{2}} - \nu_0)$ given by

$$\begin{aligned}\Delta\nu &= (c'/L\pi) \sin^{-1} \tfrac{1}{2}(1 - R^{\frac{1}{2}}R'^{\frac{1}{2}}G_1)/(RR'G_1^2)^{\frac{1}{4}} \\ &\approx (c'/2L\pi)(1 - R^{\frac{1}{2}}R'^{\frac{1}{2}}G_1)/(RR'G_1^2)^{\frac{1}{4}}.\end{aligned} \tag{40}$$

Equation (40) can be rewritten by substituting for $1 - (RR')^{\frac{1}{2}}G_1$ the value given by (38)

$$\begin{aligned}1 - (RR'G_1^2)^{\frac{1}{2}} = 2(P_s/P_t)[1 - RR'G_1 + \tfrac{1}{2}(G_1 - 1)(R + R')]/ \\ [1 + (RR'G_1^2)^{\frac{1}{2}}].\end{aligned} \tag{41}$$

The resulting expression is a function of the single-pass gain G_1. For the strong oscillator the gain saturates at a value such that $RR'G_1^2$ differs from unity by a vanishingly small amount. Thus it is expedient, in the expression for $\Delta\nu$ found by substituting (41), to write for the saturated single-pass gain

$$G_1 \equiv [1 - (1 - (RR'G_1^2)^{\frac{1}{2}})]/(RR')^{\frac{1}{2}} \tag{42}$$

and then to replace $1 - (RR'G_1^2)^{\frac{1}{2}}$ by its approximate value obtained from (41). In the reiteration, the approximate value can be written

$$1 - (RR'G_1^2)^{\frac{1}{2}} \approx \frac{(R^{\frac{1}{2}} + R'^{\frac{1}{2}})^2}{4(RR')^{\frac{1}{2}}} \frac{(1 - (RR')^{\frac{1}{2}})}{(RR')^{\frac{1}{4}}} \frac{2\pi h\nu\Delta\nu_c}{(1 - n_1/n_2)P_t} \tag{43}$$

which follows by substituting in (41) the value $G_1 = (RR')^{-\frac{1}{2}}$ and replacing P_s by its value $(1 - G_1)(c'/2L)h\nu/(n_1/n_2 - 1)$. The cavity bandwidth $\Delta\nu_c$ is given by (40) with $G_1 = 1$.

Combining (40–43) and performing the necessary algebra yields

$$\begin{aligned}\Delta\nu = 2\pi(h\nu/P_t)(\Delta\nu_c)^2(1 - n_1/n_2)^{-1}z[1 + (\pi h\nu\Delta\nu_c/P_t) \\ \cdot(1 - n_1/n_2)^{-1} \cdot (- z[x + 3] + 1 + x)/x^{\frac{1}{2}}]\end{aligned} \tag{44}$$

in which

$$z = (R^{\frac{1}{2}} + R'^{\frac{1}{2}})^2/4(RR')^{\frac{1}{2}} \tag{45}$$

is a term which is identically unity when $R = R'$ and remains close to unity even for R and R' differing by a factor of ten, and the quantity $x = (RR')^{\frac{1}{2}}$.

When $h\nu\Delta\nu_c/P_t \ll 1$ the linewidth is

$$\Delta\nu \approx 2\pi \frac{h\nu}{P_t} (\Delta\nu_c)^2 (1 - n_1/n_2)^{-1} \tag{46}$$

and differs from the well-known Schawlow-Townes[4] formula by a factor of two. The correction term $(1 - n_1/n_2)^{-1}$ approaches unity for the ideal maser ($n_1/n_2 = 0$). This term has also been found by Shimoda.[5] It should be noted that n_1/n_2 is the saturated value and in general may be very difficult to evaluate. An approximate value may be found by setting the single-pass gain given by (22) equal to the saturated gain given by (42). Extracting the appropriate value of n_1/n_2 requires a knowledge of the transition probability, the time constants for the upper and lower states and the mirror reflectivities. The degeneracy of the upper and lower levels should also be taken into account.

For the 6328 Å gas maser, typical values for a one-meter-long discharge are $P_t \approx 10^{-3}$ watts/mode, $\Delta\nu_c \approx 10^6$ cps and $n_1/n_2 = 0.98$, yielding a linewidth of 10^{-1} cps.

For the semiconductor optical maser, (46) predicts a linewidth of approximately 5×10^8 cps, taking $L = 0.06$ cm and $P_s/P_t \approx 10^{-2}$. The latter numbers are taken from the data of Quist et al.[19] The corresponding wavelength range is 10^{-2} Å. This estimate is probably a conservative lower limit because of the neglect of internal losses in the derivation and the fact that the internal losses are significant in the actual device. The lowest observed linewidths are about 10^{-1} Å. The relatively large linewidths arise from very short spontaneous emission lifetimes ($<10^{-9}$ sec). The relatively large amount of enhanced spontaneous emission available produces saturation at small values of P_t/P_s.

As has just been shown, it is possible to write formal expressions for the power output and spectral width of the noise emitted in a given mode. Since the power, and hence the spectral width, depend on the saturated single-pass gain, it is necessary to take the dynamic properties of the maser medium into account. This is illustrated graphically in Fig. 5, in which the curve of P vs G_1 as represented by (38) is shown. Also shown are a curve of $\Delta\nu$ and three dashed curves representing the dynamic properties of the maser medium. The latter curves may be found by solving the rate equations for the maser medium to obtain a relation between the power taken from the maser medium and the single-pass gain. When large power is taken from the maser medium, G_1 must approach unity, since the population inversion must approach zero. When the noise power taken from the maser medium approaches zero, the single-pass gain approaches its small signal value G_{10}. The curve representing the dynamics of the maser medium intersects the curve representing the static characteristics of the cavity at some value $G_1 < (1/RR')^{\frac{1}{2}}$. This is the operating point of the maser. The value of G_1 at the operating point determines the value of $\Delta\nu$.

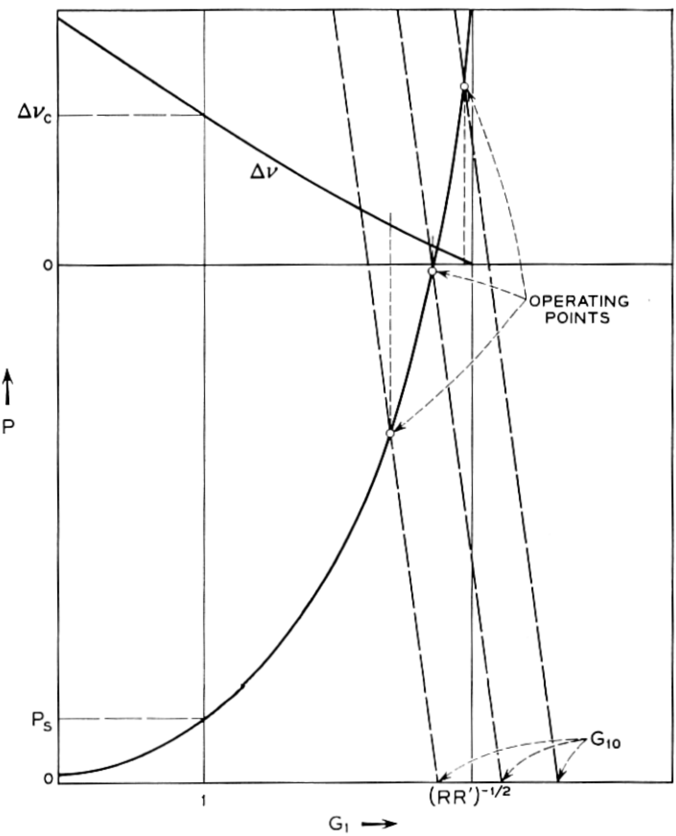

Fig. 5 — Noise power vs single-pass gain as represented by (38).

For a gas maser with small single-pass gain, the dynamic properties of the maser medium can be shown to be controlled by an equation of the form[20]

$$G_1 \approx 1 + k_0 l/(1 + \kappa P_c/P_s) \tag{47}$$

in which k_0 is the small signal gain parameter for the mode in question, l is the active length of the medium, κ/P_s is a saturation parameter dependent on the Einstein A and B coefficients for the maser levels and P_c is the power in the cavity. In most maser media $\kappa \ll 1$, so that G_1 varies quite slowly with P_c/P_s. It should be noted that P_s is essentially constant and can be evaluated for $G_1 = 1$.

The gain parameter k_0 varies with discharge current. For low currents, k_0 is proportional to current.[20] Thus, as the discharge current is increased from zero, the output power increases. From Fig. 5 it would be expected that as the discharge current approaches a value for which $1 + k_0 l$ approaches $(RR')^{-\frac{1}{2}}$, the output power would show an extremely sharp rise with current. This can be illustrated by writing for G_1 as a function of discharge current I, taking $R = R'$ for convenience,

$$G_1 \approx 1 + [(1 - R)/R](I/I_0)/(1 + \kappa P_c/P_s) \tag{48}$$

in which I_0 is the current for which $G_1 R = 1$ in the absence of saturation. From (38), taking $P_c = P_t/2(1 - R)$

$$P_c/P_s = 1/(1 - RG_1). \tag{49}$$

Solving (48) and (49) for P_c/P_s yields

$$P_c/P_s = [\Delta I/I_0 + \kappa + \sqrt{(\Delta I/I_0 + \kappa)^2 + 4\kappa}]/2\kappa \tag{50}$$

in which $\Delta I = I - I_0$. Note that when $(\Delta I/I_0) + \kappa = 0$, $P_c/P_s = \kappa^{-\frac{1}{2}}$. When $(\Delta I/I_0) + \kappa$ has the value $\kappa^{\frac{1}{2}}(f - f^{-1})$, P_c/P_s has the value $f\kappa^{-\frac{1}{2}}$. Therefore, the change in P_c/P_s by a factor f is larger than the change in $\Delta I/I_0$ of $\kappa^{\frac{1}{2}}(f - f^{-1})$ by a factor $\kappa^{-\frac{1}{2}}$. Thus, when $\kappa \ll 1$ the power output shows a very sharp threshold with current. The value of κ is of order 10^{-10} for the 6328 Å gas maser.

VI. THE NONREGENERATIVE OPTICAL MASER OSCILLATOR

Some maser media have such large single-pass gain that spontaneous emission originating at one end can be amplified sufficiently to saturate the maser medium at the opposite end. For example, the 3.39 μ transition in neon, in a helium-neon gas maser, has gains of order 50 db/meter in small-bore tubing.[21] The saturated output power is in the 1–10 mw range. Xenon at 3.5 μ has even larger gain.[22] Under these circumstances the maser can behave as a saturated oscillator without an optical cavity. The extreme line narrowing characteristic of the cavity oscillator will be lacking, but the power levels and directionality will be comparable.

This type of oscillator is characterized by a peak output always at the line center, independent of temperature and any physical dimension, line narrowing over the inhomogeneously broadened line and an extremely stable output. The power per steradian per cycle will be much greater than conventional light sources, and so these structurally simple oscillators may be very useful as frequency standards and for calibration purposes.

Some idea of the line narrowing possible with this type of oscillator can be obtained by reference to (18), which describes the noise power emanating from one end of the unsaturated maser medium in one mode with no mirrors at either end, while (33) with $R' = 0$ gives the noise power when one end has a mirror, but the other does not

$$dP = [1 + RG_1(\nu)][1 - G_1(\nu)]p(\nu)d\nu. \tag{51}$$

When $R = 0$, (51) properly yields (18); however, when $R = 1$

$$dP = [1 - G_1^2(\nu)]p(\nu)d\nu. \tag{52}$$

Since G_1^2 is the gain of a maser of twice the length of a maser of gain G_1, the perfectly reflecting mirror placed at one end serves to double the effective length of the maser medium.

Assuming no saturation, so that the gain can be written

$$G_1(\nu) = \exp k(\nu)l \tag{53}$$

in which l is the effective length of the medium, and assuming further that the gain parameter $k(\nu)$ has a Doppler profile

$$k(\nu) = k_0 \exp - [2(\nu - \nu_0)/\Delta\nu_D]^2 \ln 2 \tag{54}$$

in which ν_0 is the frequency of the line center and $\Delta\nu_D$ is the full Doppler width, the spectral width at half power of the spontaneous emission $\Delta\nu$ can be determined by writing

$$1 - \exp [k_0 l \exp - [\Delta\nu/\Delta\nu_D]^2 \ln 2] = \tfrac{1}{2}[1 - \exp k_0 l]. \tag{55}$$

In the limit $k_0 l \gg 1$, (55) can be solved for $\Delta\nu/\Delta\nu_D$ to yield

$$\Delta\nu/\Delta\nu_D = (k_0 l)^{-\frac{1}{2}}. \tag{56}$$

The 3.39 μ maser is saturated by its own spontaneous emission by gains of order 80 db, so that the maximum possible value of $k_0 l$ before saturation is about 20, corresponding to $l \approx 2$ meters. This yields $\Delta\nu \approx \Delta\nu_0/4.5$. The Doppler width at 3.39 μ is about 300 mc, yielding $\Delta\nu \approx 70$ mc. It would be possible to decrease this value somewhat by using several lengths of maser media separated by attenuators to prevent saturation. The slow dependence on l exhibited by (56) indicates the impracticality of this scheme in achieving any more than a factor of four decrease in linewidth unless k_0 can be increased drastically. Fortunately, there is evidence that this can be done, and the nonregenerative oscillator may turn out to be an extremely interesting device. In addition, it can be tuned by application of magnetic fields.

VII. CONCLUSION

The output properties of an optical maser oscillator have been derived subject to the supposition that the oscillator is a saturated amplifier of spontaneous emission noise. The most significant new results concern an expression for the linewidth of the oscillating maser which differs from the commonly accepted value. Some techniques, well-known in the microwave art and used only in a limited way in optics (stratified media), have been generalized to apply to the transmission and noise properties of optical masers.

VIII. ACKNOWLEDGMENT

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APPENDIX A

Network Representations

The basic network of interest will be a linear two-port network which will be represented schematically by Fig. 1. The major concern here will be the relationships between the quantities I_1 and E_1 at port 1 and I_2 and E_2 at port 2. The linear relationship among these quantities will be represented by the transformation matrix

$$\begin{vmatrix} E_1 \\ I_1 \end{vmatrix} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} \cdot \begin{vmatrix} E_2 \\ I_2 \end{vmatrix} \tag{57a}$$

which will be abbreviated by

$$\mathbf{\Psi}_1 = \mathbf{T} \cdot \mathbf{\Psi}_2 \tag{57b}$$

in which

$$\mathbf{\Psi}_i = \begin{vmatrix} E_i \\ I_i \end{vmatrix} \qquad i = 1, 2, \cdots \tag{58}$$

and

$$\mathbf{T} = \begin{vmatrix} A & B \\ C & D \end{vmatrix}. \tag{59}$$

Note that the direction of positive current flow is defined to be into the network at port 1 and out of the network at port 2. This choice simplifies the discussion of cascaded networks. It is clear that two networks in cascade can be represented by $\mathbf{\Psi}_1 = \mathbf{T}_1 \cdot \mathbf{\Psi}_2 = \mathbf{T}_1 \cdot \mathbf{T}_2 \cdot \mathbf{\Psi}_3$. The product $\mathbf{T}_1 \cdot \mathbf{T}_2$ follows the usual rules of matrix multiplication. In general, $\mathbf{T}_1 \cdot \mathbf{T}_2 \neq \mathbf{T}_2 \cdot \mathbf{T}_1$; that is, the two networks do not commute. For a cascade of n networks

$$\mathbf{\Psi}_1 = \mathbf{T}_1 \cdot \mathbf{T}_2 \cdots \mathbf{T}_n \cdot \mathbf{\Psi}_{n+1}. \tag{60}$$

The determinant of the transformation will be a quantity of interest and will be defined as $\Delta = AD - CB$. The inverse of (57) is defined by

$$\mathbf{\Psi}_2 = \mathbf{T}^{-1} \cdot \mathbf{\Psi}_1 \tag{61}$$

in which

$$\mathbf{T}^{-1} = \frac{1}{\Delta} \begin{vmatrix} D & -B \\ -C & A \end{vmatrix} \tag{62}$$

is the inverse transformation matrix. Reference will sometimes be made to the exchange network $\underset{\sim}{\mathbf{T}}$, which is merely the network $\mathbf{T}$ with its terminals exchanged so that port 2 becomes port 1 and vice versa. The components of $\underset{\sim}{\mathbf{T}}$ are obtained by writing $\mathbf{\Psi}_2 = \mathbf{T}^{-1} \cdot \mathbf{\Psi}_1$ and noting that for the exchange network the quantities E_2, I_2 become E_1', $-I_1'$ and E_1, I_1 become E_2', $-I_2'$. As a result, the correct description is given by

$$\begin{vmatrix} E_1' \\ I_1' \end{vmatrix} = \frac{1}{\Delta} \begin{vmatrix} D & B \\ C & A \end{vmatrix} \cdot \begin{vmatrix} E_2' \\ I_2' \end{vmatrix} \tag{63}$$

and

$$\underset{\sim}{\mathbf{T}} = \frac{1}{\Delta} \begin{vmatrix} D & B \\ C & A \end{vmatrix}. \tag{64}$$

The net power *into* port 1 is given by

$$P_1 = \tfrac{1}{2}(E_1 I_1^* + E_1^* I_1) = \tfrac{1}{2} \overline{\mathbf{\Psi}}_1 \cdot \mathbf{\Sigma} \cdot \mathbf{\Psi}_1 \tag{65}$$

in which $\overline{\mathbf{\Psi}}$ is the complex transpose of $\mathbf{\Psi}$ and

$$\Sigma = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}. \tag{66}$$

Likewise, the net power *out* of port 2 is given by

$$P_2 = \tfrac{1}{2}\overline{\boldsymbol{\Psi}}_2 \cdot \boldsymbol{\Sigma} \cdot \boldsymbol{\Psi}_2 . \tag{67}$$

Other transformations will be introduced as they are needed. In the following, certain types of networks will be characterized in terms of relationships among the components of the transformation matrix, i.e., A, B, C and D. These networks will form the basic "tools" of the analyses.

A.1 *Matched Networks*

If one of the ports of the network is terminated in a matched load and it is then observed that the input impedance of the other port is matched to the line, the network is said to be matched. First, consider the input impedance of the network in Fig. 6. The line is assumed to have unit characteristic impedance. The quantities B and C are thus normalized with respect to the characteristic impedance and admit-

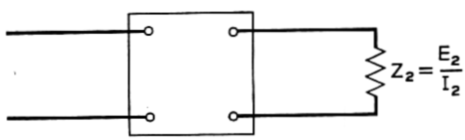

Fig. 6 — Terminating impedance for matched network.

tance of the line, respectively. Writing $E_1 = AE_2 + BI_2$, $I_1 = CE_2 + DI_2$ and

$$Z_{\text{input}(1)} = E_1/I_1 = (AZ_2 + B)/(CZ_2 + D) \tag{68}$$

in which $Z_2 = E_2/I_2$ is the terminating impedance. The network is then matched if $Z_{\text{input}(1)} = Z_2 = 1$, yielding the requirement $A + B = C + D$. Likewise, the exchange network must also be matched, requiring $D + B = C + A$. The two requirements yield the relations $A = D$, $B = C$, and for a matched network

$$\mathbf{T} = \begin{vmatrix} A & B \\ B & A \end{vmatrix}. \tag{69}$$

A.2 *Reciprocal Networks*

A reciprocal network has the same transmission properties in either direction. It follows that for a reciprocal network $\mathbf{T}$ and $\underset{\sim}{\mathbf{T}}$ must have the same transmission factor and phase shift, which from (84) and (72) implies $\Delta = 1$. The simplest nonreciprocal network can be written

$$\mathbf{T}_a = \begin{vmatrix} \Delta^{\frac{1}{2}} & 0 \\ 0 & \Delta^{\frac{1}{2}} \end{vmatrix} \tag{70}$$

with $\Delta \neq 1$; and all nonreciprocal networks can be written

$$\mathbf{T} = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = \begin{vmatrix} \Delta^{\frac{1}{2}} & 0 \\ 0 & \Delta^{\frac{1}{2}} \end{vmatrix} \cdot \begin{vmatrix} A/\Delta^{\frac{1}{2}} & B/\Delta^{\frac{1}{2}} \\ C/\Delta^{\frac{1}{2}} & D/\Delta^{\frac{1}{2}} \end{vmatrix} = \mathbf{T}_a \cdot \mathbf{T}_r . \tag{71}$$

The reciprocal network $\mathbf{T}_r$ is known as the reduced network; all the nonreciprocity resides in $\mathbf{T}_a$. The network $\mathbf{T}_a$, known as the abstracted network, commutes with all networks.

In general, Δ is complex and can be written $\Delta = |\Delta| \exp j\, 2\varphi$. Since

$$\mathbf{T}_a = \begin{vmatrix} |\Delta|^{\frac{1}{2}} \exp j\varphi & 0 \\ 0 & |\Delta|^{\frac{1}{2}} \exp j\varphi \end{vmatrix} \tag{72}$$

and the exchange network

$$\underset{\sim}{\mathbf{T}}_a = \begin{vmatrix} |\Delta|^{-\frac{1}{2}} \exp -j\varphi & 0 \\ 0 & |\Delta|^{-\frac{1}{2}} \exp -j\varphi \end{vmatrix} \tag{73}$$

it follows that argument φ is the nonreciprocal phase shift.

A.3 *Reactive Networks*

Since a reactive network is lossless, it would be expected that $P_2 = P_1$. It follows that $\bar{\mathbf{\Psi}}_2 \Sigma \mathbf{\Psi}_2 = \bar{\mathbf{\Psi}}_2 \bar{\mathbf{T}} \Sigma \mathbf{T} \mathbf{\Psi}_2$, so that for a reactive network $\bar{\mathbf{T}} \Sigma \mathbf{T} = \Sigma$. Performing the indicated matrix multiplication yields

$$\begin{vmatrix} A^*C + AC^* & A^*D + C^*B \\ AD^* + CB^* & BD^* + B^*D \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}. \tag{74}$$

It follows that A^*C and BD^* are imaginary and $A^*D + C^*B = 1$. Note that multiplication of $A^*D + C^*B = 1$ by CB^* yields $CB^* = (A^*C)(B^*D) + |C|^2 |B|^2$, which is real. Likewise,

$$AD^* = |A|^2 |D|^2 + (AC^*)(BD^*)$$

is real. If the reactive network is reciprocal so that $\Delta = AD - BC = 1 = (AC^*)(D/C^*) - D^*B(C/D^*) = (A/B^*)DB^* - (B/A^*)A^*C$, it follows that both D/C^* and A/B^* are imaginary. Since AC^* is imaginary and C^*B is real, A/B is imaginary. Thus B/B^* is real, which can only occur when B is real or imaginary. It follows that B and C are real (or imaginary) while A and D are imaginary (or real). The question of which set to choose real is decided by noting that the idemfactor (no network) is a reactive reciprocal network. Thus, the appropriate choice is A, D real and B, C imaginary, and the reactive reciprocal network can be written

$$\begin{vmatrix} \alpha & j\beta \\ j\gamma & \delta \end{vmatrix} \tag{75}$$

for which $\alpha\delta + \beta\gamma = 1$. If the reactive reciprocal network is also matched so that $\alpha = \delta$ and $\beta = \gamma$, the network can be written

$$\begin{vmatrix} \cos \varphi & j \sin \varphi \\ j \sin \varphi & \cos \varphi \end{vmatrix} \tag{76}$$

in which the parameter φ is known as the angular length or phase shift. Note that the input impedance for this network is

$$Z_{\text{input}(1)} = \frac{Z_2 + j \tan \varphi}{1 + Z_2 j \tan \varphi} \tag{77}$$

which is recognizable as the impedance transformation for a transmission line of angular length φ and unit characteristic impedance.

Next, certain basic passive circuit elements of interest will be characterized.

A.4 *Series Impedance (Fig. 7)*

Since $I_1 = I_2$ and $E_1 = I_2Z + E_2$, the transfer matrix is given by

$$\mathbf{T} = \begin{vmatrix} 1 & Z \\ 0 & 1 \end{vmatrix}. \tag{78}$$

A.5 *Shunt Admittance (Fig. 8)*

For the shunt admittance, $E_1 = E_2$ and $I_1 = E_2Y + I_2$; thus

$$\mathbf{T} = \begin{vmatrix} 1 & 0 \\ Y & 1 \end{vmatrix}. \tag{79}$$

Both networks are reciprocal but not matched.

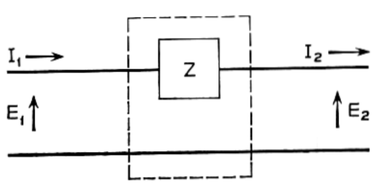

Fig. 7 — Series impedance.

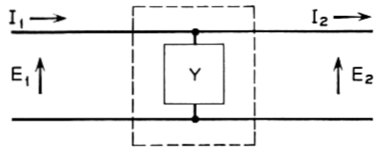

Fig. 8 — Shunt admittance.

A.6 *Ideal Transformer of Turns Ratio N:1 (Fig. 9)*

For the transformer $E_1 = NE_2$ and $I_1 = I_2/N_1$. Thus

$$T = \begin{vmatrix} N & 0 \\ 0 & N^{-1} \end{vmatrix}. \tag{80}$$

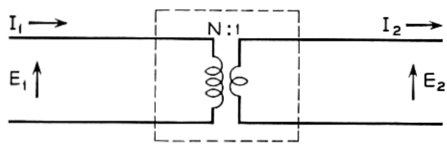

Fig. 9 — Ideal transformer.

A.7 *Network Transmissivity*

Consider a general two-port network which has a matched generator on side 1 and a matched load on side 2. The cascade of networks may be represented schematically as shown in Fig. 10, yielding

$$\begin{vmatrix} E_g \\ I_g \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} A & B \\ C & D \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} E_2 \\ 0 \end{vmatrix}. \tag{81}$$

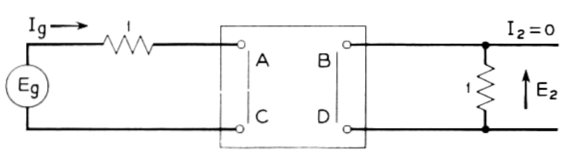

Fig. 10 — Circuit for two-port network with matched generator on side 1 and matched load on side 2.

Performing the indicated matrix multiplication yields

$$\begin{vmatrix} E_g \\ I_g \end{vmatrix} = \begin{vmatrix} A + B + C + D & B + D \\ C + D & D \end{vmatrix} \cdot \begin{vmatrix} E_2 \\ 0 \end{vmatrix}. \tag{82}$$

The network transmission factor is defined as the ratio

$$L = |E_2|^2 / |E_{20}|^2 \tag{83}$$

in which E_{20} is the voltage that would be measured if the network were absent. In the latter case, the appropriate two-port transformation must be the identity matrix, so that $A = D = 1$, $B = C = 0$. Hence

$$L = \frac{|E_g|^2}{|A + B + C + D|^2} \bigg/ \frac{|E_g|^2}{4} = \frac{4}{|A + B + C + D|^2}. \tag{84}$$

A transmission factor greater than one implies gain. The network transmission factor for a matched network is

$$L = 4|2A + 2B|^{-2} = |A + B|^{-2}. \tag{85}$$

A.8 *Network Reflectivity*

When a network is followed by a matched load, the input impedance to the network is given by $Z_{\text{input}} = (A + B)/(C + D)$. The power reflectivity, R, is given by

$$\begin{aligned} R = \left| \frac{(Z_{\text{input}} - 1)}{(Z_{\text{input}} + 1)} \right|^2 &= \frac{|A + B - C - D|^2}{|A + B + C + D|^2} \\ &= \tfrac{1}{4} L |A + B - C - D|^2 \end{aligned} \tag{86}$$

in which L is the transmission factor of the network. Note that for a matched network, $A = D$, $B = C$ and $R = 0$.

A.9 *Matched Attenuator*

An attenuator is a nonreactive reciprocal device. Thus, a matched attenuator must satisfy the conditions $A = D$, $B = C$ and $AD - BC = 1$, with A, B, C and D real. This is automatically satisfied by writing

$$\mathbf{T} = \begin{vmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{vmatrix}. \tag{87}$$

The parameter θ is referred to as the line length of the attenuator. The attenuator commutes with a matched reciprocal reactive network (a transmission line, for example) and the resultant network

$$\begin{aligned} &\begin{vmatrix} \cos\varphi & j\sin\varphi \\ j\sin\varphi & \cos\varphi \end{vmatrix} \begin{vmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{vmatrix} \\ &\qquad = \begin{vmatrix} \cos(\varphi - j\theta) & j\sin(\varphi - j\theta) \\ j\sin(\varphi - j\theta) & \cos(\varphi - j\theta) \end{vmatrix} \end{aligned} \tag{88}$$

is the representation for an attenuator with phase shift or angular length φ. From the fact that a lossy transmission line would have a phase shift of the form $\exp -j(\varphi - j\theta) = \exp -\theta \exp -j\varphi$, it would be expected that the transmission factor is given by $\exp -2\theta$. For an attenuator the transmission factor is obtained by using (85) and (87)

$$L = | \cosh \theta + \sinh \theta |^{-2} = \exp -2\theta. \tag{89}$$

Thus, the attenuator matrix is specified completely by knowledge of its transmission factor.

A.10 *Isolator*

An isolator is by definition nonreciprocal although it is matched. Thus, an isolator with forward transmission unity and reverse transmission $\exp -2\theta$ can be synthesized by cascading an attenuator of line length $\theta/2$, having a transmission $\exp -\theta$, with a nonreciprocal network with transmission $\exp +\theta$ in the forward direction and transmission $\exp -\theta$ in the backward direction. In its most simple form, the nonreciprocal network (70) is given by

$$\begin{vmatrix} \exp -\theta/2 & 0 \\ 0 & \exp -\theta/2 \end{vmatrix}. \tag{90}$$

Thus, the isolator can be represented by

$$\begin{aligned} \mathbf{T} &= \begin{vmatrix} \exp -\theta/2 & 0 \\ 0 & \exp -\theta/2 \end{vmatrix} \begin{vmatrix} \cosh \theta/2 & \sinh \theta/2 \\ \sinh \theta/2 & \cosh \theta/2 \end{vmatrix} \\ &= \begin{vmatrix} \exp -\theta/2 \cosh \theta/2 & \exp -\theta/2 \sinh \theta/2 \\ \exp -\theta/2 \sinh \theta/2 & \exp -\theta/2 \cosh \theta/2 \end{vmatrix}. \end{aligned} \tag{91}$$

and the transmission factor is unity in the forward direction and $\exp -2\theta$ in the backward direction. An ideal isolator is one for which the line length θ approaches infinity, yielding

$$\mathbf{T} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix}. \tag{92}$$

APPENDIX B

Noisy Networks

In the following, the source of noise will be considered to be spontaneous fluctuations which arise because of the thermal properties of the material. At extremely high frequencies, the thermal noise is more commonly called black-body radiation. The noise spectrum will be characterized by the following statement of Nyquist's theorem:[13] The noise power *available* per mode at frequency ν in a small frequency interval, $d\nu$, is given by

$$\begin{aligned} dP &= p(\nu)d\nu \\ p(\nu) &= h\nu(\exp h\nu/kT - 1)^{-1} \end{aligned} \tag{93}$$

for a passive circuit at temperature T. The constant k is Boltzmann's constant (1.38×10^{-23} joules/degree) and h is Planck's constant (6.6×10^{-34} joules-sec).

It is sometimes convenient to write the noise power in terms of equivalent rms voltage and current generators e and i as shown in Fig. 11. The internal impedance of the voltage generator is zero and for the current generator it is infinite, and one can write

$$|e| = (4rp(\nu)d\nu)^{\frac{1}{2}}, \qquad |i| = (4gp(\nu)d\nu)^{\frac{1}{2}}. \tag{94}$$

In the following, both e and i will have the units of $(\text{power})^{\frac{1}{2}}$ since r and g are normalized with respect to the line impedance and admittance.

A systematic method of handling the noise produced by a network will be developed next. First, the following theorem will be stated without proof: *Any passive noisy two-port network can be replaced by an equivalent noise-free network which has an added shunt current generator and series voltage generator at the input or output terminals which represent the noise contribution.*[12] Therefore, any passive noisy network can be replaced by the representation shown in Fig. 2, in which the network is now noise-free but the noise appears from equivalent voltage and current generators at one of the terminals. A proof for the theorem can be given, and although it is relatively simple, the proof is lengthy.

Next, a scheme for representing in a simple way the additional current and voltage appearing at the terminal will be described. The following technique is due to H. Seidel.[15] It is clear that one may always write

$$\begin{aligned} E_1 &= AE_2 + BI_2 + e \\ I_1 &= CE_2 + DI_2 + i. \end{aligned} \tag{95}$$

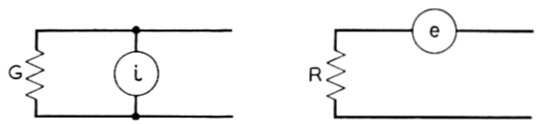

Fig. 11 — Noise-free representation of network with equivalent external voltage and current generators at one terminal.

The inclusion of the noise generators can be accomplished in an artificial but, as will be seen, highly useful way by writing

$$\begin{vmatrix} E_1 \\ I_1 \\ 1 \end{vmatrix} = \begin{vmatrix} A & B & e \\ C & D & i \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} E_2 \\ I_2 \\ 1 \end{vmatrix}. \tag{96}$$

This representation results from adding the trivial equation $1 = 0 + 0 + 1$ to the set in (95). No new information has been added, but the bookkeeping advantages afforded by this change will become apparent shortly.

Equation (96) will be the general representation for a passive noisy two-port network. The problem now is to learn how to characterize the noise quantities e and i in terms of the properties of the network represented by A, B, C and D. It will be seen that this can be done by comparing the network to some simple network whose properties are known.

First, it will be demonstrated that for any passive network, one may take A, B, C and D either all real or all imaginary. This is equivalent to saying that the network may always be taken to appear purely resistive. This is clear since the input impedance is $Z_{\text{in}} = (AZ_2 + B)$ $(CZ_2 + D)$. If Z_2 is real, then it would be expected that a resistive network will have Z_{in} real also. The proof follows from the fact that the input impedance can always be made real with a suitable length of transmission line. In addition, with an ideal transformer one may match the input impedance to the line. It follows, therefore, that for noise calculations, one only need consider matched networks ($A = D$, $B = C$) with the ratio A/B real.

First, consider a matched network which is cascaded with a transmission line of arbitrary length θ. The latter network has no loss and has no source of noise. In addition, the transmission line can change only the phase but not the magnitude of the noise current and voltage generators. Consequently, one expects that for a matched network

$$\begin{vmatrix} \cos\theta & j\sin\theta & 0 \\ j\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A & B & e \\ B & A & i \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} X & X & e\cos\theta + ij\cos\theta \\ X & X & ej\sin\theta + \ i\cos\theta \\ 0 & 0 & 1 \end{vmatrix}$$

has the same noise properties independent of θ. Thus, it is required that

$$|e|^2 = |e\cos\theta + i\sin\theta|^2 = |e|^2\cos^2\theta + |i|^2\sin^2\theta + e(ji)^* + e^*(ji)\sin\theta\cos\theta$$

or

$$(|e|^2 - |i|^2)\sin^2\theta + j(ei^* - e^*i)\sin\theta\cos\theta = 0.$$

Choosing $\theta = \pi/2$ yields

$$|e|^2 = |i|^2 \tag{97}$$

and it follows also that

$$ei^* = e^*i. \tag{98}$$

Next, the passive matched network at temperature T will be cascaded with a shunt conductance at the same temperature and the open-circuit noise voltage at the input terminals will be determined. The network representation for the shunt conductance in the new formalism follows from Nyquist's theorem as is shown in Fig. 11

$$|i'| = (4Gp(\nu)d\nu)^{\frac{1}{2}}$$

$$\begin{vmatrix} A & B & e \\ C & D & i \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ G & 1 & i' \\ 0 & 0 & 1 \end{vmatrix}. \tag{99}$$

Thus, the network to be studied is as shown in Fig. 12 with

$$\begin{vmatrix} E_1 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} A & B & e \\ B & A & i \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 \\ G & 1 & i' \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} E_2 \\ 0 \\ 1 \end{vmatrix}$$

$$= \begin{vmatrix} A + BG & B & Bi' + e \\ B + AG & A & Ai' + i \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} E_2 \\ 0 \\ 1 \end{vmatrix} \tag{100}$$

yielding

$$\begin{aligned} E_1 &= (A + BG)E_2 + Bi' + e \\ 0 &= (B + AG)E_2 + Ai' + i \end{aligned} \tag{101}$$

so that

$$\begin{aligned} E_2 &= -\frac{(Ai' + i)}{(B + AG)} \\ E_1 &= -\frac{(A + BG)}{B + AG}(Ai' + i) + Bi' + e. \end{aligned} \tag{102}$$

At the open-circuited terminal at which the noise voltage E_1 is measured, one must have from Nyquist's theorem, (94)

$$|E_1|^2 = 4Z_{\text{input}}p(\nu)d\nu \tag{103}$$

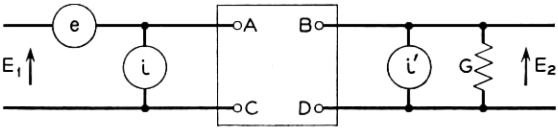

Fig. 12 — Passive matched network with shunt conductance for determination of open-circuit noise voltage.

since Z_{input} is resistive. Its value is

$$Z_{\text{input}} = \frac{A + BG}{B + AG}. \tag{104}$$

Hence, it is required that

$$\left(\frac{A + BG}{B + AG}\right) 4p(\nu)d\nu = \left| -\left(\frac{A + BG}{B + AG}\right)(Ai' + i) + Bi' + e \right|^2. \tag{105}$$

Remember also that $|e|^2 = |i|^2$ and ei^* is real. Since the noise arising from the shunt conductance G is independent of the noise produced by the network, cross terms like ei', i^*i' are zero, since their product represents a time average which must be zero. Solution of (105) yields uniquely the values[15]

$$\begin{aligned} |i|^2 &= |e|^2 = AB \\ ei^* &= \tfrac{1}{2}(A^2 + B^2 - 1) \end{aligned} \tag{106}$$

which are the desired relations measured in units of $4p(\nu)d\nu$. The veracity of (106) can be established by direct substitution into (105), which yields an identity independent of the value of G.

The equivalent values of e and i for an attenuator are given by

$$\begin{aligned} |e|^2 &= |i|^2 = AB = \cosh\theta \sinh\theta = \tfrac{1}{2}\sinh 2\theta \\ ei^* &= \tfrac{1}{2}(A^2 + B^2 - 1) = \tfrac{1}{2}(\cosh^2\theta + \sinh^2\theta - 1) = \sinh^2\theta. \end{aligned} \tag{107}$$

The phase angle between e and i^* is $\cos^{-1}\tanh\theta$. Since the phase angle is always positive, the implication is that the noise sources radiate more power toward the attenuator than away from it. This is reasonable, since the noise power leaving each end of the attenuator should be equal and the noise radiated toward the attenuator by the noise sources is attenuated before emerging from the output end. In the limit of large θ, the phase angle approaches zero.

The equivalent values for an isolator are given by

$$|e|^2 = |i|^2 = \exp -\theta \cosh \frac{\theta}{2} \sinh \frac{\theta}{2} = \tfrac{1}{2} \exp -\theta \sinh \theta$$
$$ei^* = \frac{1}{2}\left[\exp -\theta \left(\sinh^2 \frac{\theta}{2} + \cosh^2 \frac{\theta}{2}\right) - 1\right] = -|e|^2. \tag{108}$$

Therefore, the angle between e and i is always π, the implication being that the isolator radiates noise power away from the attenuator, i.e., only in the direction in which it attenuates.

The noise radiated into a matched resistor by any network is obtained by considering the network shown in Fig. 13

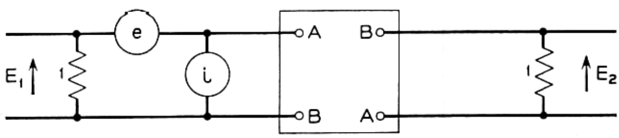

Fig. 13 — Noise radiated into matched resistor.

$$\begin{vmatrix} E_1 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} A & B & e \\ C & D & i \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} E_2 \\ 0 \\ 1 \end{vmatrix}$$
$$= \begin{vmatrix} A' & B' & e' \\ C' & D' & i' \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} E_2 \\ 0 \\ 1 \end{vmatrix}. \tag{109}$$

The noise contributed by the matched input and output resistors is neglected since this is additive. The primed quantities result from matrix multiplication. The noise output dP is evaluated by noting that

$$0 = C'E_2 + i' \qquad \text{and} \qquad dP = |E_2|^2 = |i'|^2/|C'|^2.$$

Performing the matrix multiplication yields $C' = A + B + C + D$ and $i' = e + i$. Thus, the output noise power is given by

$$dP = \frac{|e + i|^2 \, 4p(\nu)\, d\nu}{|A + B + C + D|^2} = L\,|e + i|^2\, p(\nu)\, d\nu \tag{110}$$

in which L is the network insertion loss [see (20)]. For a matched resistive network (attenuator)

$$|e + i|^2 = |e|^2 + |i|^2 + 2ei^* = (2AB + A^2 + B^2 - 1)$$
$$= [(A + B)^2 - 1] = L^{-1} - 1$$

which follows from (9) and (21). Thus,

$$dP = (1 - L)p(\nu)d\nu \tag{111}$$

is the noise power in frequency range $d\nu$ emanating from a matched resistive network. This result is well known in network theory. In optics, it is known as one form of Kirchhoff's law.

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