On the Properties of Some Systems that Distort Signals — II

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In this paper we study the recoverability of square-integrable bandlimited signals (with arbitrary frequency bands) that are distorted by a frequency-selective time-variable nonlinear operator and subsequently are bandlimited to the original bands. The distortion operator characterizes a very general class of systems containing linear time-invariant elements and a single time-variable nonlinear element. The subsequent bandlimiting of the system's output signals can be thought of as being due to transmission through a channel that performs filtering.

Our principal result asserts that, under certain conditions that are satisfied by many realistic systems, it is possible to uniquely determine the bandlimited input to the system from a knowledge of the bandlimited version of the output, in spite of the intermediate distortion which generally produces signals that are not bandlimited to the original frequency bands. We show that the input signal can be determined by a stable iteration procedure in which the approximating functions converge to their limit at a rate that is at least geometric.

I. INTRODUCTION

In this paper we study the recoverability of square-integrable bandlimited signals (with arbitrary frequency bands) that are distorted by a frequency-selective time-variable nonlinear operator and subsequently are bandlimited to the original bands. The distortion operator characterizes a very general class of systems containing linear time-invariant elements and a single time-variable nonlinear element. The subsequent bandlimiting of the system's output signals can be thought of as being due to transmission through a channel that performs filtering.

Our principal result asserts that, under certain conditions that are satisfied by many realistic systems, it is possible to uniquely determine the bandlimited input to the system from a knowledge of the bandlimited version of the output, in spite of the intermediate distortion which generally produces signals that are not bandlimited to the original frequency bands. Of course the distortion operator is assumed to be known. We show that the input signal can be determined by a stable iteration procedure in which the approximating functions converge to their limit at a rate that is at least geometric. When the physical system consists of only a single nonlinear element, our result reduces to that of Landau and Miranker,[1] and Zames.[2]

In the electronic circuitry of a communication system, it is often the case that an ideally linear amplifier is supplied with an approximately bandlimited input signal and that the circuitry subsequent to the amplifier introduces approximate bandlimiting. Under the assumption that the bandlimiting is ideal, our results imply that in many cases it is possible to completely reverse the effect of nonlinear distortion that may be introduced by such an amplifier due to the malfunctioning of, for example, a transistor or its bias supply, even though, as is typically the case, the transistor may be in a feedback loop. Of course it is necessary to know the properties of the distorting circuit. Results of this type may be useful in situations in which received signals are recorded and the time delay introduced by the recovery scheme is not important. For example, it is conceivable that this type of result may be useful in improving the quality of distorted signals obtained from a transmitter in a space vehicle containing a television camera, in which the distortion is due to a faulty video amplifier.

Section II considers some mathematical preliminaries. In Section III we state our principal results after discussing in detail a mathematical model of the physical system to be considered which focuses attention on the influence of the time-variable nonlinear element. Sections IV and V are concerned with the proof of the results. In particular, Section V considers the rate of convergence and stability of the recovery procedure. Section VI is concerned with some results that relate to the necessity of the conditions introduced earlier.

II. PRELIMINARIES

It is assumed that the reader is familiar with the contraction-mapping fixed-point theorem stated in Part I.[3,4]

As in Part I, $\mathfrak{L}_2$ denotes the Hilbert space of complex-valued square-integrable functions with inner product

$$(f,g) = \int_{-\infty}^{\infty} f\bar{g}\, dt$$

in which $\underset{\sim}{g}$ is the complex conjugate of g. The norm of f [i.e., $(f,f)^{\frac{1}{2}}$] is denoted by $\| f \|$. The intersection of the space $\mathfrak{L}_2$ with the set of real-valued functions is denoted by $\mathfrak{L}_{2R}$.

We take as the definition of the Fourier transform of $f(t)$ in $\mathfrak{L}_2$:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$$

and consequently

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega.$$

With this definition, the Plancherel identity reads:

$$2\pi \int_{-\infty}^{\infty} f(t)\underset{\sim}{g}(t)\, dt = \int_{-\infty}^{\infty} F(\omega)\underset{\sim}{G}(\omega)\, d\omega.$$

As the notation above suggests, lower and upper case versions of a letter are used to denote, respectively, a function and its Fourier transform.

We shall be concerned with the following subspace of $\mathfrak{L}_{2R}$:

$$\mathfrak{B}(\Omega) = \{f(t) \mid f(t) \in \mathfrak{L}_{2R}\,; \quad F(\omega) = 0,\ \omega \notin \Omega\}$$

where Ω is a union of disjoint intervals. The measure of Ω is denoted by $\mu(\Omega)$, which, unless stated otherwise, is not assumed to be finite. In particular, Ω may be the entire real line.

The operator that projects an arbitrary element of $\mathfrak{L}_{2R}$ onto $\mathfrak{B}(\Omega)$ is denoted by **P**. In electrical engineering terms, **P** is an ideal filtering operation.

The symbols **I** and **O** denote, respectively, the identity operator and the null operator (i.e., $\mathbf{O}f = 0$ for all $f \in \mathfrak{L}_2$).

III. MATHEMATICAL DESCRIPTION OF THE PHYSICAL SYSTEM AND STATEMENT OF PRINCIPAL RESULTS

Consider a nonlinear time-variable element imbedded in a linear physical system. Let s_1 and s_2, respectively, denote the system's input and output signals, and let v and w, respectively denote the input and output signals associated with the nonlinear device, which is assumed to be characterized by the equation

$$w = \varphi(v,t) = \varphi[v], \tag{1}$$

where $\varphi(v,t)$ is a real-valued function of the real variables v and t.

It is assumed that v, w, $s_2 \in \mathfrak{L}_{2R}$, $s_1 \in \mathfrak{B}(\Omega)$, and that there exist well-defined linear operators $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$, with domain $\mathcal{B}(\Omega) \times \mathcal{L}_{2R}$, such that† $v = \mathbf{\Gamma}[s_1, w]$ and $s_2 = \mathbf{\Lambda}[s_1, w]$.

We shall be concerned throughout with the four linear operators **A**, **B**, **C**, and **D** derived from $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ in the following manner:

$$\begin{aligned} v = \mathbf{\Gamma}[s_1, w] &= \mathbf{\Gamma}[s_1, 0] + \mathbf{\Gamma}[0, w] \\ &= \mathbf{A}s_1 + \mathbf{C}w \end{aligned} \tag{2}$$

$$\begin{aligned} s_2 &= \mathbf{\Lambda}[s_1, w] = \mathbf{\Lambda}[s_1, 0] + \mathbf{\Lambda}[0, w] \\ &= \mathbf{D}s_1 + \mathbf{B}w. \end{aligned} \tag{3}$$

3.1 *Representation of the Operators* A, B, C *and* D

We assume throughout that

$$\mathbf{A}f = \int_{-\infty}^{\infty} a(t-\tau)f(\tau)d\tau, \qquad \mathbf{B}f = \int_{-\infty}^{\infty} b(t-\tau)f(\tau)d\tau$$

$$\mathbf{C}f = \int_{-\infty}^{\infty} c(t-\tau)f(\tau)d\tau, \qquad \mathbf{D}f = \int_{-\infty}^{\infty} d(t-\tau)f(\tau)d\tau$$

where each of the real symbolic functions $a(t)$, $b(t)$, $c(t)$, and $d(t)$ is most generally the sum of an element of $\mathcal{L}_{2R}$ and a delta function. It is assumed throughout that $|C(\omega)|$ and $|B(\omega)|$ are uniformly bounded for all ω and that $|A(\omega)|$ and $|D(\omega)|$ are uniformly bounded for all $\omega \in \Omega$. It follows that **C** and **B** are bounded mappings of $\mathcal{L}_{2R}$ into itself and that **A** and **D** are bounded mappings of $\mathcal{B}(\Omega)$ into itself.

3.2 *The Projection Operation and the Basic Flow Graph*

We shall suppose that s_2, the system's output signal, is the input to a device that projects signals in $\mathcal{L}_{2R}$ onto the subspace $\mathcal{B}(\Omega)$. This device may be thought of as representing an ideal transmission channel of the low-pass, bandpass, or multiband type. If the output of the device is denoted by s_3, then clearly

$$s_3 = \mathbf{P}s_2 = \mathbf{T}^{-1}P\mathbf{T}s_2 \tag{4}$$

where

$$\begin{aligned} P = P(\omega) &= 1, \qquad \omega \in \Omega \\ &= 0, \qquad \omega \notin \Omega \end{aligned}$$

and $\mathbf{T}s_2$ denotes S_2, the Fourier transform of s_2.

† This assumption is almost invariably satisfied in mathematical models of physical systems of interest.

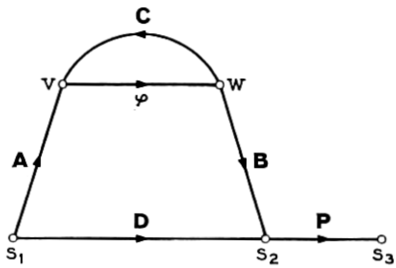

Fig. 1 — Signal-flow graph characterization of the relation between s_1, s_2, s_3, v, and w.

The equations we have introduced give rise to the signal-flow graph shown in Fig. 1 which summarizes the basic situation.

Our primary interest is in (*i*) obtaining conditions under which s_3 uniquely determines s_1, when s_1 is known to lie in the same subspace as s_3 [i.e., in $\mathcal{B}(\Omega)$], and (*ii*) obtaining a technique for recovering s_1.

3.3 *The Time-Variable Nonlinear Element*

We shall denote by $\psi(w,t)$ the inverse nonlinear characteristic; that is, $\psi(\varphi[v],t) = v$ for all v and t. It is assumed throughout that $\psi(0,t) = 0$ for all t, that $\psi\,[w(t)]$ is a measurable function of t whenever w is measurable, and that there exist two positive constants α and β with the properties that $\frac{1}{2}(\alpha + \beta) = 1$ and

$$\alpha(w_1 - w_2) \leqq \psi(w_1, t) - \psi(w_2, t) \leqq \beta(w_1 - w_2) \tag{5}$$

for all t and all $w_1 \geqq w_2$. Of course no loss of generality is introduced by the normalization $\frac{1}{2}(\alpha + \beta) = 1$, which happens to be convenient for our purposes. Observe that $0 < \alpha \leqq 1$.

It follows from (5) that

$$\beta^{-1}(v_1 - v_2) \leqq \varphi(v_1, t) - \varphi(v_2, t) \leqq \alpha^{-1}(v_1 - v_2)$$

for all t and all $v_1 \geqq v_2$. Observe that $w \in \mathfrak{L}_{2R}$ if and only if $v \in \mathfrak{L}_{2R}$.

3.4 *Assumptions Regarding* $|A(\omega)|$, $|B(\omega)|$, *and* $|D(\omega)|$

In addition to the uniform boundedness of $|A(\omega)|$, $|B(\omega)|$, $|C(\omega)|$, and $|D(\omega)|$ mentioned earlier, it is assumed, unless stated otherwise, that there exists a union of disjoint intervals Ω_D such that $\Omega_D \subseteq \Omega$,

$$\left.\begin{aligned} |D(\omega)| &= 0 \\ |B(\omega)| &\geqq k_1 \\ |A(\omega)| &\geqq k_2 \end{aligned}\right\} \omega \in \Omega_D ,$$

and

$$| D(\omega) | \geqq k_3 , \qquad \omega \, \varepsilon \, (\Omega - \Omega_D)$$

where k_1, k_2, and k_3 are positive constants. In most cases of engineering interest either $\Omega_D = \Omega$ or Ω_D is the null set.†

3.5 *Statement of Principal Results*

Our main result is

Theorem I: Let **A**, **B**, **C**, **D**, α, *and* ψ *be as defined in Sections* 3.1, 3.3, *and* 3.4. *Let*

$$\inf_{\omega \varepsilon (\Omega - \Omega_D)} | C - AD^{-1}B - 1 | > 1 - \alpha$$

$$\inf_{\omega \notin \Omega} | C - 1 | > 1 - \alpha.$$

Then to each $s_3 \, \varepsilon \, \mathcal{B}(\Omega)$ *there correspond unique functions* $s_1 \, \varepsilon \, \mathcal{B}(\Omega)$ *and* $w, v, s_2 \, \varepsilon \, \mathcal{L}_{2R}$ *such that*

$$s_3 = \mathbf{P}s_2$$

$$s_2 = \mathbf{D}s_1 + \mathbf{B}w$$

$$v = \mathbf{A}s_1 + \mathbf{C}w$$

$$v = \psi[w]$$

[*i.e., such that* (1), (2), (3), *and* (4) *are satisfied*]. *Furthermore if*

$$\bar{s}_3 = \mathbf{P}\bar{s}_2$$

$$\bar{s}_2 = \mathbf{D}\bar{s}_1 + \mathbf{B}\bar{w}$$

$$\bar{v} = \mathbf{A}\bar{s}_1 + \mathbf{C}\bar{w}$$

$$\bar{v} = \psi[\bar{w}]$$

where $\bar{w}, \bar{v}, \bar{s}_2 \, \varepsilon \, \mathcal{L}_{2R}$ *and* $\bar{s}_1$, $\bar{s}_3 \, \varepsilon \, \mathcal{B}(\Omega)$,

$$\| s_1 - \bar{s}_1 \| \leqq k_4 \| s_3 - \bar{s}_3 \|$$

where k_4 *is a positive constant that depends only on* **A**, **B**, **C**, **D** *and* ψ.

Suppose that $\psi[w] = \mathbf{C}w + \mathbf{A}s_1$ {i.e., (2) with $v = \psi[w]$} possesses a unique solution $w \, \varepsilon \, \mathcal{L}_{2R}$ for any $s_1 \, \varepsilon \, \mathcal{B}(\Omega)$ and that if $\psi[\bar{w}] = \mathbf{C}\bar{w} + \mathbf{A}\bar{s}_1$

† The assumptions in this section facilitate a common treatment of these two important cases. Observe that, with the exception of these cases, it is assumed here that $| D(\omega) |$ is discontinuous on Ω. However, as indicated in the Appendix this is by no means a necessary condition for the recoverability of s_1.

in which $\bar{s}_1 \varepsilon \mathcal{B}(\Omega)$ and $\bar{w} \varepsilon \mathcal{L}_{2R}$, $\| w - \bar{w} \| \leqq k_5 \| s_1 - \bar{s}_1 \|$, where k_5 is a constant that does not depend on s_1 or $\bar{s}_1$. [A direct application of Theorem II (in Section IV) shows that this is the case if $\inf_{\omega} | C - 1 | > (1 - \alpha)$.] It follows directly from the properties of ψ and the assumptions regarding **A**, **B**, **C**, and **D** that if $s_1 \varepsilon \mathcal{B}(\Omega)$, there exist unique functions v, s_2, $s_3 \varepsilon \mathcal{L}_{2R}$ such that (1) (2), (3), and (4) are satisfied. Let $\mathbf{\Phi}$ denote the operator that associates with each $s_1 \varepsilon \mathcal{B}(\Omega)$ the corresponding s_3. The assumptions regarding $\psi[w] = \mathbf{C}w + \mathbf{A}s_1$ together with the boundedness of **B** and **D** imply that $\mathbf{\Phi}$ is a bounded mapping of $\mathcal{B}(\Omega)$ into itself. Under the conditions stated in Theorem I, $\mathbf{\Phi}$ possesses a bounded inverse.

The invertibility conditions are established in Section IV and the boundedness of $\mathbf{\Phi}^{-1}$ is considered in Section V.

The method used to establish the invertibility conditions is constructive. In particular, $\mathbf{\Phi}^{-1}s_3$ can be computed in accordance with a stable iteration procedure for which the successive approximations converge in the $\mathcal{L}_{2R}$ norm at a rate that is at least geometric. The approximations converge also in the supremum norm at a rate that is geometric or greater if $\mu(\Omega)$ is finite.

As indicated earlier, in most cases of engineering interest either $\Omega_D = \Omega$ (the single-loop feedback system case), or Ω_D is the null set (i.e., the magnitude of the "direct transmission" $D(\omega)$ is uniformly bounded away from zero on Ω). The invertibility conditions stated above are satisfied in many cases of practical interest.

The situation considered by Landau and Miranker,[1] and Zames[2] corresponds to one in which $\mathbf{A} = \mathbf{B} = \mathbf{I}$, $\mathbf{D} = \mathbf{C} = \mathbf{O}$, and $\Omega_D = \Omega$. The inequalities are obviously satisfied in this case. In fact they are satisfied when $\Omega_D = \Omega$ and $C(\omega) = 0$, $\omega \not\varepsilon \Omega$. More generally, observe that the inequalities are met if and only if $(C - AD^{-1}B)$, for all $\omega \varepsilon (\Omega - \Omega_D)$, and C, for all $\omega \not\varepsilon \Omega$, are bounded away from the disk centered in the complex plane at [1,0] and having radius $1 - \alpha$ where $0 < \alpha \leqq 1$.

IV. DERIVATION OF INVERTIBILITY CONDITIONS

In the following discussion we shall denote by $\mathbf{P}_D$ the operator that projects elements of $\mathcal{L}_{2R}$ onto $\mathcal{B}(\Omega_D)$. That is,

$$\mathbf{P}_D f = \mathbf{T}^{-1} P_D \mathbf{T} f, \qquad f \,\varepsilon\, \mathcal{L}_{2R} \tag{6}$$

where

$$P_D = P_D(\omega) = 1, \qquad \omega \,\varepsilon\, \Omega_D$$
$$= 0, \qquad \omega \not\varepsilon\, \Omega_D$$

and, as before, $\mathbf{T}f$ denotes the Fourier transform of f. Recall that $\mathbf{D}$ is an invertible mapping of $\mathcal{B}(\Omega - \Omega_D)$ into itself, that $\mathbf{A}$ and $\mathbf{B}$ are invertible mappings of $\mathcal{B}(\Omega_D)$ into itself, and that $\mathbf{D}$ annihilates $\mathcal{B}(\Omega_D)$. We shall denote by $\tilde{\mathbf{D}}^{-1}$ the inverse of the restriction of $\mathbf{D}$ to $\mathcal{B}(\Omega - \Omega_D)$, and by $\tilde{\mathbf{A}}^{-1}$ and $\tilde{\mathbf{B}}^{-1}$, respectively, the inverses of the restrictions of $\mathbf{A}$ and $\mathbf{B}$ to $\mathcal{B}(\Omega_D)$.

From (3) and (4)

$$s_3 = \mathbf{D}s_1 + \mathbf{PB}w, \qquad s_1 \,\varepsilon\, \mathcal{B}(\Omega) \tag{7}$$

and from (2) and $\psi[w] = v$

$$\psi[w] = \mathbf{C}w + \mathbf{A}s_1 . \tag{8}$$

Our objective is to determine w in order to find s_1 from (7) and (8). The corresponding functions s_2 and v can of course be computed from (3) and $v = \psi[w]$.

Since $\mathbf{D}$ annihilates $\mathcal{B}(\Omega_D)$, $\mathbf{P}_D s_3 = \mathbf{P}_D\mathbf{B}w$ and, since $\mathbf{P}_D$ and $\mathbf{B}$ commute,

$$\mathbf{P}_D w = \tilde{\mathbf{B}}^{-1}\mathbf{P}_D s_3 . \tag{9}$$

The problem therefore reduces to the determination of $(\mathbf{I} - \mathbf{P}_D)w$. Before proceeding it is convenient to set $w_a = \mathbf{P}_D w$ and $w_b = (\mathbf{I} - \mathbf{P}_D)w$, and to introduce

Definition I: Let

$$\begin{aligned} \eta(x) &= \beta - x, \qquad & x \leqq 1 \\ &= x - \alpha, \qquad & x \geqq 1. \end{aligned}$$

From (8),

$$(\mathbf{I} - \mathbf{P}_D)\psi[w_a + w_b] = \mathbf{C}w_b + \mathbf{A}(\mathbf{P} - \mathbf{P}_D)s_1 , \tag{10}$$

since $\mathbf{C}$ and $\mathbf{A}$ commute with $(\mathbf{I} - \mathbf{P}_D)$. From (7),

$$(\mathbf{P} - \mathbf{P}_D)s_3 = \mathbf{D}(\mathbf{P} - \mathbf{P}_D)s_1 + (\mathbf{P} - \mathbf{P}_D)\mathbf{B}w,$$

and

$$(\mathbf{P} - \mathbf{P}_D)s_1 = \tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3 - \tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B}w. \tag{11}$$

Thus,

$$(\mathbf{I} - \mathbf{P}_D)\psi[w_a + w_b] = \mathbf{C}w_b - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B}w_b + \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3$$

from which

$$(\mathbf{I} - \mathbf{P}_D)\{\psi[w_a + w_b] - \psi_0 w_b\}$$
$$= [\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} - \psi_0\mathbf{I}]w_b + \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3$$

where ψ_0 is a real constant to be chosen subsequently.

Thus, regarding $[\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} - \psi_0\mathbf{I}]$ as a mapping of the orthogonal complement of $\mathcal{B}(\Omega_D)$ into itself, and assuming that it possesses a bounded inverse $[\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} - \psi_0\mathbf{I}]^{-1}$,

$$\mathbf{R}w_b = w_b$$

where

$$\mathbf{R}w_b = [\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} - \psi_0\mathbf{I}]^{-1}(\mathbf{I} - \mathbf{P}_D)\{\psi[w_a + w_b] - \psi_0 w_b\}$$
$$- [\mathbf{C} - \mathbf{A}\mathbf{D}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} - \psi_0\mathbf{I}]^{-1}\mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3 .$$

The operator $\mathbf{R}$ is a mapping of a complete metric space into itself. We next establish a condition under which $\mathbf{R}$ is a contraction. Let $\mathbf{H} = [\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} - \psi_0\mathbf{I}]^{-1}$, and let f and g belong to the orthogonal complement of $\mathcal{B}(\Omega_D)$. Then

$$\| \mathbf{R}f - \mathbf{R}g \| \leqq \| \mathbf{H}(\mathbf{I} - \mathbf{P}_D) \| \cdot \| \psi[w_a + f] - \psi[w_a + g] - \psi_0(f - g) \|$$
$$\leqq \| \mathbf{H}(\mathbf{I} - \mathbf{P}_D) \| \, \eta(\psi_0) \, \| f - g \|,$$

since

$$\left| \frac{\psi[w_a + f] - \psi[w_a + g]}{f - g} - \psi_0 \right| \leqq \eta(\psi_0).$$

Thus $\mathbf{R}$ is a contraction for some ψ_0 if

$$r = \inf_{\psi_0} \| \mathbf{H}(\mathbf{I} - \mathbf{P}_D) \| \, \eta(\psi_0) < 1. \tag{12}$$

It turns out that the optimal choice of ψ_0 is unity, the median of α and β. Consequently we could have simply set $\psi_0 = 1$ at the outset. However, we prefer to establish the significance of this choice.

4.1 *Evaluation of* $\| \mathbf{H}(\mathbf{I} - \mathbf{P}_D) \|$

Let $H = [C - AD^{-1}(P - P_D)B - \psi_0]^{-1}$ with the understanding that $D^{-1}(P - P_D) = 0$, $\omega \not\in (\Omega - \Omega_D)$. Our result is†

Lemma I:

$$\| \mathbf{H}(\mathbf{I} - \mathbf{P}_D) \| = \operatorname*{ess\,sup}_{\omega \notin \Omega_D} | H(\omega) |.$$

† The notation $\operatorname*{ess\,sup}_{\omega} Q(\omega)$ denotes $\inf_{\mathfrak{N}} \sup_{\omega \notin \mathfrak{N}} Q(\omega)$ where $\mathfrak{N}$ is an arbitrary zero-measure subset of the real line.

Proof:

The norm of $\mathbf{H}(\mathbf{I} - \mathbf{P}_D)$ is $\sup\{\| z \|; \| f \| = 1\}$ where $z = \mathbf{H}(\mathbf{I} - \mathbf{P}_D)f$ and $f \varepsilon \mathfrak{L}_{2R}$. An application of the Plancherel identity yields, in terms of the frequency domain representation of $\mathbf{H}$,

$$\| z \|^2 = \frac{1}{2\pi} \int_{\omega \notin \Omega_D} | H(\omega) |^2 \cdot | F(\omega) |^2 \, d\omega.$$

Hence

$$\sup\{\| z \|; \quad \| f \| = 1\} \leqq \operatorname*{ess\,sup}_{\omega \notin \Omega_D} | H(\omega) |.$$

However if $\operatorname*{ess\,sup}_{\omega \notin \Omega_D} | H(\omega) | < \infty$, for any $\epsilon > 0$ there exists a set of nonzero measure $\mathcal{E}$ which is disjoint from Ω_D and such that $| H(\omega) | \geqq \operatorname*{ess\,sup}_{\omega \notin \Omega_D} | H(\omega) | - \epsilon$, $\omega \varepsilon \mathcal{E}$. Since $| F(\omega) |$ is permitted to be nonzero only on $\mathcal{E}$, it follows that

$$\sup\{\| z \|; \| f \| = 1\} \geqq \operatorname*{ess\,sup}_{\omega \notin \Omega_D} | H(\omega) | - \epsilon.$$

Thus if $\operatorname*{ess\,sup}_{\omega \notin \Omega_D} | H(\omega) | < \infty$,

$$\| \mathbf{H}(\mathbf{I} - \mathbf{P}_D) \| = \operatorname*{ess\,sup}_{\omega \notin \Omega_D} | H(\omega) |. \tag{13}$$

It is clear that (13) remains valid if $\operatorname*{ess\,sup}_{\omega \notin \Omega_D} | H(\omega) | = \infty$. This proves the lemma.

It follows from (12) and Lemma I that

$$r = \inf_{\psi_0} \operatorname*{ess\,sup}_{\omega \notin \Omega_D} | [C - AD^{-1}(P - P_D)B - \psi_0]^{-1} | \, \eta(\psi_0). \tag{14}$$

4.2 *Determination of ψ_0 and Statement of Theorem II*

The following lemma indicates that the optimal choice of ψ_0 is independent of $[C - AD^{-1}(P - P_D)B]$.

Lemma II: Let ξ be a complex number and suppose that

$$| \xi - \psi_0 |^{-1} \eta(\psi_0) < 1.$$

Then

$$| \xi - \psi_0 |^{-1} \eta(\psi_0) \geqq | \xi - 1 |^{-1} \eta(1).$$

Proof:

Suppose first that $\psi_0 \leqq 1$ and that

$$| \xi - \psi_0 | > k(\beta - \psi_0), \qquad k > 1.$$

Then, since $|\xi - \psi_0| \leqq |\xi - 1| + |1 - \psi_0|$,

$$|\xi - 1| + |1 - \psi_0| - k(1 - \psi_0) > k(\beta - 1),$$

and hence $|\xi - 1| > k(\beta - 1)$. Suppose now that $\psi_0 \geqq 1$ and that

$$|\xi - \psi_0| > k(\psi_0 - \alpha), \qquad k > 1.$$

Then,

$$|\xi - 1| + |\psi_0 - 1| - k(\psi_0 - 1) > k(1 - \alpha),$$

and hence $|\xi - 1| > k(1 - \alpha)$.

It follows from (14) and Lemma II that if $r < 1$,

$$\begin{aligned} r &= \operatorname*{ess\,sup}_{\omega \notin \Omega_D} |[C - AD^{-1}(P - P_D)B - 1]^{-1}| \eta(1) \\ &= \operatorname*{ess\,sup}_{\omega \notin \Omega_D} |[C - AD^{-1}(P - P_D)B - 1]^{-1}| (1 - \alpha). \end{aligned}$$

At this point we are in a position to state

Theorem II: Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, *and* $\mathbf{D}$ *be the bounded linear operators defined in Section 3.1. Let* $\mathbf{D}$, *but not necessarily* $\mathbf{A}$ *and* $\mathbf{B}$, *have the properties stated in Section 3.4. Let* $\tilde{\mathbf{D}}^{-1}$ *denote the inverse of the restriction of* $\mathbf{D}$ *to* $\mathcal{B}(\Omega_D)$, *and let* $\mathbf{P}_D$ *denote the operator that projects elements of* $\mathcal{L}_{2R}$ *onto* $\mathcal{B}(\Omega_D)$. *Suppose that*

$$r = max\,[r_1, r_2] < 1,$$

where

$$r_1 = \operatorname*{ess\,sup}_{\omega \varepsilon (\Omega - \Omega_D)} |[C - AD^{-1}B - 1]^{-1}| (1 - \alpha)$$

$$r_2 = \operatorname*{ess\,sup}_{\omega \notin \Omega} |[C - 1]^{-1}| (1 - \alpha).$$

Then for any w_a *and* g, *respectively elements of* $\mathcal{B}(\Omega_D)$ *and its orthogonal complement with respect to* $\mathcal{L}_{2R}$, *there exists a unique* w_b *in the orthogonal complement of* $\mathcal{B}(\Omega_D)$ *such that*

$$(\mathbf{I} - \mathbf{P}_D)\psi[w_a + w_b] = [\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B}]w_b + g.$$

In fact, $w_b = \lim_{i \to \infty} w_{bi}$ *where*

$$w_{b(i+1)} = [\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} - \mathbf{I}]^{-1}(\mathbf{I} - \mathbf{P}_D)\{\psi[w_a + w_{bi}] - w_{bi}\} - [\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} - \mathbf{I}]^{-1}g$$

and w_{b0} *is an arbitrary element in the orthogonal complement of* $\mathcal{B}(\Omega_D)$.

If $\bar{w}_b$ is a solution corresponding to $\bar{w}_a$ and $\bar{g}$,

$$\| w_b - \bar{w}_b \| \leqq \frac{r}{1-r} \| w_a - \bar{w}_a \| + \frac{r}{(1-r)(1-\alpha)} \| g - \bar{g} \|.$$

Proof:

With the exception of the last inequality, the proof follows from the fact that if $r < 1$, $\mathbf{R}$ (with $\psi_0 = 1$) is a contraction mapping of a complete metric space into itself.† The inequality is obtained as follows. Let $\mathbf{J} = [\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} - \mathbf{I}]^{-1}$ (i.e., let $\mathbf{J}$ be $\mathbf{H}$ with $\psi_0 = 1$). Then, since

$$w_b = \mathbf{J}(\mathbf{I} - \mathbf{P}_D)\{\psi[w_a + w_b] - w_b\} - \mathbf{J}g,$$

$$w_b - \bar{w}_b = \mathbf{J}(\mathbf{I} - \mathbf{P}_D)\{\psi[w_a + w_b] - \psi[\bar{w}_a + \bar{w}_b] - (w_a + w_b) + (\bar{w}_a + \bar{w}_b)\} - \mathbf{J}(g - \bar{g}).$$

Therefore

$$\| w_b - \bar{w}_b \| \leqq \| \mathbf{J}(\mathbf{I} - \mathbf{P}_D) \| \eta(1) \| w_a - \bar{w}_a + w_b - \bar{w}_b \| + \| \mathbf{J}(\mathbf{I} - \mathbf{P}_D) \| \cdot \| g - \bar{g} \|,$$

and since $r = \| \mathbf{J}(\mathbf{I} - \mathbf{P}_D) \| \eta(1)$, $\eta(1) = (1 - \alpha)$, and

$$\| w_a - \bar{w}_a + w_b - \bar{w}_b \| \leqq \| w_a - \bar{w}_a \| + \| w_b - \bar{w}_b \|,$$

$$\| w_b - \bar{w}_b \| \leqq \frac{r}{1-r} \| w_a - \bar{w}_a \| + \frac{r}{(1-r)(1-\alpha)} \| g - \bar{g} \|.$$

With regard to the "essential supremum" notation used in the statements of Lemma I and Theorem II, it is of course true that

$$\operatorname*{ess\,sup}_{\omega \notin \Omega_D} | H(\omega) | = \sup_{\omega \notin \Omega_D} | H(\omega) |$$

in at least almost all cases of engineering interest.

4.3 *The Complete Recovery Scheme*

Let us now consider our over-all objective, the recovery of s_1. From (8) and (11), using the definition of $\tilde{\mathbf{A}}^{-1}$,

$$(\mathbf{P} - \mathbf{P}_D)s_1 = \tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3 - \tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)Bw$$

$$\mathbf{P}_D s_1 = \tilde{\mathbf{A}}^{-1}\mathbf{P}_D\{\psi[w] - \mathbf{C}w\}.$$

† In particular, our assumption regarding the inverse of $[\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D) - \mathbf{B} - \mathbf{I}]$ is satisfied, since $| C - AD^{-1}(P - P_D) - 1 |$ is bounded away from zero for all ω in the complement of Ω_D.

Therefore,

$$s_1 = (\mathbf{P} - \mathbf{P}_D)s_1 + \mathbf{P}_D s_1 = [\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D) - \tilde{\mathbf{A}}^{-1}\mathbf{C}\tilde{\mathbf{B}}^{-1}\mathbf{P}_D]s_3 + \tilde{\mathbf{A}}^{-1}\mathbf{P}_D\{\psi[\tilde{\mathbf{B}}^{-1}\mathbf{P}_D s_3 + w_b]\} - \tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B}w_b \tag{15}$$

where we have used (9), the fact that $(\mathbf{P} - \mathbf{P}_D)\mathbf{B}w_a = 0$, and the identity $\tilde{\mathbf{A}}^{-1}\mathbf{P}_D\mathbf{C}\tilde{\mathbf{B}}^{-1}\mathbf{P}_D s_3 = \tilde{\mathbf{A}}^{-1}\mathbf{C}\tilde{\mathbf{B}}^{-1}\mathbf{P}_D s_3$. This proves the first part of Theorem I. The second part, which is concerned with the boundedness of $\mathbf{\Phi}^{-1}$, is considered in Section 5.1.

We define s_{1n}, the nth approximation to s_1, by

$$s_{1n} = [\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D) - \tilde{\mathbf{A}}^{-1}\mathbf{C}\tilde{\mathbf{B}}^{-1}\mathbf{P}_D]s_3 + \tilde{\mathbf{A}}^{-1}\mathbf{P}_D\{\psi[\tilde{\mathbf{B}}^{-1}\mathbf{P}_D s_3 + w_{bn}]\} - \tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B}w_{bn} \tag{16}$$

where w_{bn} is the nth approximation to w_b as defined in Theorem II. Observe that

$$s_{1n} - s_1 = \tilde{\mathbf{A}}^{-1}\mathbf{P}_D\{\psi[\tilde{\mathbf{B}}^{-1}\mathbf{P}_D s_3 + w_{bn}] - \psi[\tilde{\mathbf{B}}^{-1}\mathbf{P}_D s_3 + w_b]\} - \tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B}(w_{bn} - w_b),$$

from which, using the right inequality of (5) satisfied by ψ,

$$\| s_{1n} - s_1 \| \leqq \{ \| \tilde{\mathbf{A}}^{-1}\mathbf{P}_D \| \beta + \| \tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} \| \} \| w_{bn} - w_b \|. \tag{17}$$

An argument very similar to that used in the proof of Lemma I suffices to show that

$$\| \tilde{\mathbf{A}}^{-1}\mathbf{P}_D \| = \operatorname*{ess\,sup}_{\omega\varepsilon\Omega_D} | A^{-1} | \tag{18}$$

$$\| \tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} \| = \operatorname*{ess\,sup}_{\omega\varepsilon(\Omega-\Omega_D)} | D^{-1}B |. \tag{19}$$

Our assumptions regarding **A** and **B** imply that the right-hand side of (18) and the right-hand side of (19) are bounded. Therefore, since $w_b = \lim_{n\to\infty} w_{bn}$, (17) implies that $s_1 = \lim_{n\to\infty} s_{1n}$.

The convergence of s_{1n} to s_1 established in the last paragraph is in the mean-square sense. If $\mu(\Omega) < \infty$, it is also true that s_{1n} converges to s_1 pointwise uniformly in t, that is

$$\lim_{n\to\infty} \sup_t | s_{1n} - s_1 | = 0.$$

This result follows from the inequality:†

† This inequality is proved in Ref. 1 for the case in which Ω is a single interval centered at the origin. The extension to arbitrary sets of finite measure is trivial.

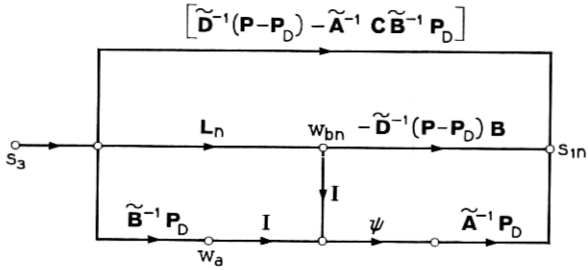

Fig. 2 — Idealized recovery scheme.

$$\sup | f(t) | \leqq \left(\frac{\mu(\Omega)}{2\pi} \right)^{\frac{1}{2}} \| f \|, \qquad f \varepsilon \mathcal{B}(\Omega)$$

and the fact that s_{1n} , $s_1 \varepsilon \mathcal{B}(\Omega)$.

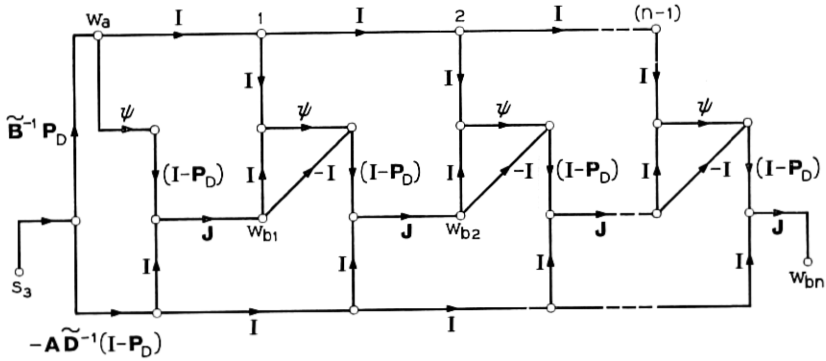

Fig. 3 — The iterative operation $\mathbf{L}_n$.

4.4 *Signal-Flow Graph for a Complete Recovery Scheme*

One complete idealized scheme for obtaining the nth approximation to s_1 , based on (16) and the solution for w_b given in Theorem II with $g = \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3$ and $w_{b0} = 0$, is summarized in Fig. 2. The iterative operation† $\mathbf{L}_n$ is shown in detail in Fig. 3 in which, as defined earlier,

† In the special case in which Ω_D is the null set and $C - AD^{-1}PB = 0$ identically in ω, $w = \varphi[\mathbf{A}\tilde{\mathbf{D}}^{-1}s_3]$ and hence the iteration stage is not required. The condition that $C - AD^{-1}PB$ vanish identically in ω, under which $\mathbf{\Phi}$ is by no means a trivial mapping of $\mathcal{B}(\Omega)$ into $\mathcal{B}(\Omega)$, is equivalent in engineering terms to requiring that the feedback transmission, for $\omega \notin \Omega$, and the null feedback transmission, for $\omega \varepsilon \Omega$, both vanish.

$\mathbf{J} = [\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} - \mathbf{I}]^{-1}$. Fig. 4 shows a flow-graph representation of $\mathbf{J}$ in terms of $[\mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B}]$ and elementary operations. The flow graphs in Figs. 2 and 3 simplify in obvious ways in the important special cases in which $\mathbf{D} = \mathbf{O}$ on $\mathcal{B}(\Omega)$ or $\mathbf{D}$ possesses a bounded inverse on $\mathcal{B}(\Omega)$.

The analog implementation of the scheme presented in Fig. 2 requires consideration of the time delay inherent in the approximation of the impulse response functions corresponding to the nonrealizable operators† $\mathbf{P}$ and $\mathbf{P}_D$, as well as the time delay that might be required in the approximation of $\mathbf{J}$. These considerations imply that time delay sections must be inserted at various points in the recovery system and that the time variation of the nonlinear elements must be staggered. Of course the output of the recovery system will be a delayed version of an approximation of $s_1(t)$.

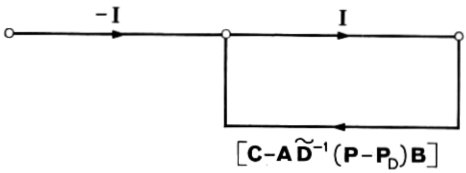

Fig. 4 — Flow-graph representation of the operator $\mathbf{J}$.

There are many variations possible in the implementation of the recovery system. For example, the iteration can be performed with a recording device and a *single* typical stage of the type used in Fig. 3.

V. RATE OF CONVERGENCE AND STABILITY OF THE RECOVERY SCHEME

The key element in the recovery scheme is of course the iteration procedure. We show first that the approximating functions w_{bi} converge to their limit w_b at a rate that is at least geometric. This type of convergence is a direct consequence of the fact that $w_{bi} = \mathbf{R}^i w_{b0}$ where $\mathbf{R}$ is a contraction mapping.

Since

$$w_{bi} = w_{b0} + [w_{b1} - w_{b0}] + [w_{b2} - w_{b1}] + \cdots + [w_{bi} - w_{b(i-1)}],$$

$$\| w_{bi} - w_b \| = \| [w_{b(i+1)} - w_{bi}] + [w_{b(i+2)} - w_{b(i+1)}] + \cdots \|$$

$$\leqq \| w_{b(i+1)} - w_{bi} \| + \| w_{b(i+2)} - w_{b(i+1)} \| + \cdots.$$

Repeated applications of the inequality:

† Of course we are ignoring the cases in which $\mathbf{P} = \mathbf{I}$ or $\mathbf{P}_D = \mathbf{O}$.

$$\| w_{bl} - w_{b(l-1)} \| = \| \mathbf{R} w_{b(l-1)} - \mathbf{R} w_{b(l-2)} \|$$
$$\leqq r \| w_{b(l-1)} - w_{b(l-2)} \|, \qquad l \geqq 2$$

lead to

$$\| w_{bi} - w_b \| \leqq \frac{r^i}{1-r} \| w_{b1} - w_{b0} \|. \tag{20}$$

If $w_{b0} = 0$, $w_{b1} = \mathbf{J}(\mathbf{I} - \mathbf{P}_D)\psi[\tilde{\mathbf{B}}^{-1}\mathbf{P}_D s_3] - \mathbf{J}\mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3$, and hence

$$\| w_{bi} - w_b \| \leqq \frac{r^i}{1-r} \| \mathbf{J}(\mathbf{I} - \mathbf{P}_D)\{\psi[\tilde{\mathbf{B}}^{-1}\mathbf{P}_D s_3] - \tilde{\mathbf{B}}^{-1}\mathbf{P}_D s_3$$
$$- \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3\} \|$$
$$\leqq \frac{r^i}{1-r} \| \mathbf{J}(\mathbf{I} - \mathbf{P}_D) \| \{\eta(1) \| \tilde{\mathbf{B}}^{-1}\mathbf{P}_D \|$$
$$+ \| \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D) \|\} \| s_3 \|$$
$$\leqq \frac{r^{i+1}}{1-r} \left\{ \| \tilde{\mathbf{B}}^{-1}\mathbf{P}_D \| + \frac{\| \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D) \|}{1-\alpha} \right\} \| s_3 \|$$

where, in accordance with the arguments used in the proof of Lemma I,

$$\| \tilde{\mathbf{B}}^{-1}\mathbf{P}_D \| = \operatorname*{ess\,sup}_{\omega \varepsilon \Omega_D} | B^{-1} |$$
$$\| \mathbf{A}\mathbf{D}^{-1}(\mathbf{P} - \mathbf{P}_D) \| = \operatorname*{ess\,sup}_{\omega \varepsilon (-\Omega_D)} | AD^{-1} |.$$

5.1 *Stability of the Recovery Scheme*

We consider here the degree of immunity of the recovery scheme to two important types of errors.

It is assumed first that the input to the recovery system, which we shall denote by $\bar{s}_3$, differs† from s_3. Let overbarred symbols denote signals due to the input $\bar{s}_3$. We have from (15)

$$\| s_1 - \bar{s}_1 \| = \| [\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D) - \tilde{\mathbf{A}}^{-1}\mathbf{C}\tilde{\mathbf{B}}^{-1}\mathbf{P}_D](s_3 - \bar{s}_3)$$
$$+ \tilde{\mathbf{A}}^{-1}\mathbf{P}_D\{\psi[\tilde{\mathbf{B}}^{-1}\mathbf{P}_D s_3 + w_b] - \psi[\tilde{\mathbf{B}}^{-1}\mathbf{P}_D \bar{s}_3 + \bar{w}_b]\}$$
$$- [\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B}](w_b - \bar{w}_b) \|$$
$$\leqq \| \tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D) - \tilde{\mathbf{A}}^{-1}\mathbf{C}\tilde{\mathbf{B}}^{-1}\mathbf{P}_D \| \cdot \| s_3 - \bar{s}_3 \| \tag{21}$$
$$+ \| \tilde{\mathbf{A}}^{-1}\mathbf{P}_D \| \beta\{ \| \tilde{\mathbf{B}}^{-1}\mathbf{P}_D \| \cdot \| s_3 - \bar{s}_3 \| + \| w_b - \bar{w}_b \| \}$$
$$+ \| \tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B} \| \cdot \| w_b - \bar{w}_b \|.$$

† The departure of $\bar{s}_3$ from s_3 might be due to the presence of noise in either the transmission channel or the initial stages of the receiver.

However, from Theorem II with $g = \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3$,

$$\| w_b - \bar{w}_b \| \leqq \frac{r}{1 - r} \| \tilde{\mathbf{B}}^{-1}\mathbf{P}_D \| \cdot \| s_3 - \bar{s}_3 \| + \frac{r}{(1 - r)(1 - \alpha)} \| \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D) \| \cdot \| s_3 - \bar{s}_3 \|. \tag{22}$$

In view of our earlier assumptions which imply the boundedness of all of the norms in (21) and (22), it is evident that there exists a positive constant k_4 such that

$$\| s_1 - \bar{s}_1 \| \leqq k_4 \| s_3 - \bar{s}_3 \| \tag{23}$$

for all $s_3, \bar{s}_3 \, \varepsilon \, \mathcal{B}(\Omega)$. In other words, our assumptions imply that $\mathbf{\Phi}^{-1}$ is bounded. This means that the error in the recovered signal is at most proportional to the error in the input to the recovery system. In particular, the recovered signal depends continuously on the input to the recovery system.

We show next that the recovery scheme is not critically dependent upon either an exact knowledge of the operator $\mathbf{J}$ or the projection property of $\mathbf{P}_D$. Specifically, we shall compare the functions w_b and $\hat{w}_b$ defined by

$$w_b = \mathbf{R}w_b, \quad \mathbf{R}w_b = \mathbf{J}(\mathbf{I} - \mathbf{P}_D)\{\psi[w_a + w_b] - w_b\} - \mathbf{J}\mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3 \tag{24}$$

$$\hat{w}_b = \hat{\mathbf{R}}\hat{w}_b, \quad \hat{\mathbf{R}}\hat{w}_b = \mathbf{Q}\{\psi[w_a + \hat{w}_b] - \hat{w}_b\} - \mathbf{S}\mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3 \tag{25}$$

where $\mathbf{Q}$ and $\mathbf{S}$ are bounded linear mappings of $\mathcal{L}_{2R}$ into itself. We assume that $r < 1$ and that

$$\hat{r} = \| \mathbf{Q} \| \, \eta(1) < 1. \tag{26}$$

Hence $\hat{\mathbf{R}}$ is assumed to be a contraction mapping of $\mathcal{L}_{2R}$ into itself. Note that inequality (26) is satisfied if $r = \| \mathbf{J}(\mathbf{I} - \mathbf{P}_D) \| \, \eta(1) < 1$ and $\| \mathbf{J}(\mathbf{I} - \mathbf{P}_D) - \mathbf{Q} \|$ is sufficiently small. A comparison of w_b and $\hat{w}_b$ yields an estimate of the error, due to the departure of $\mathbf{Q}$ from $\mathbf{J}(\mathbf{I} - \mathbf{P}_D)$ and to the departure of $\mathbf{S}$ from $\mathbf{J}$, in the limit function approached by the iteration procedure in the recovery system.

From (24) and (25),

$$w_b - \hat{w}_b = (\mathbf{S} - \mathbf{J})\mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3 + \mathbf{J}(\mathbf{I} - \mathbf{P}_D)\{\psi[w_a + w_b] - w_b\}$$
$$- \mathbf{Q}\{\psi[w_a + w_b] - w_b\} + \mathbf{Q}\{\psi[w_a + w_b] - w_b\} - \mathbf{Q}\{\psi[w_a + \hat{w}_b] - \hat{w}_b\},$$

from which

$$\| w_b - \hat{w}_b \| \leqq \| (\mathbf{S} - \mathbf{J})\mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3 \| + \| [\mathbf{J}(\mathbf{I} - \mathbf{P}_D) - \mathbf{Q}] \{\psi[w] - w_b\} \| + \| \mathbf{Q} \| \, \eta(1) \, \| w_b - \hat{w}_b \|,$$

and

$$\| w_b - \hat{w}_b \| \leqq \frac{1}{1 - \hat{r}} \| (\mathbf{S} - \mathbf{J})\mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)s_3 \| + \frac{1}{1 - \hat{r}} \| [\mathbf{J}(\mathbf{I} - \mathbf{P}_D) - Q]\{\psi[w] - w_b\} \|.$$

Therefore, if the departure of $\mathbf{Q}$ from $\mathbf{J}(\mathbf{I} - \mathbf{P}_D)$ is not too large (i.e., if $\hat{r} < 1$), the error in the limit function approached by the iteration technique is, for fixed s_3 (and hence fixed w), at most a linear combination of two terms, one that approaches zero as $\| \mathbf{S} - \mathbf{J} \|$ approaches zero, and another that approaches zero as $\| \mathbf{J}(\mathbf{I} - \mathbf{P}_D) - \mathbf{Q} \|$ approaches zero.

VI. SOME NEGATIVE RESULTS

In this final section we consider some results that relate to the necessity of the conditions introduced earlier.

The equation $\psi[w] = \mathbf{C}w + \mathbf{A}s_1$, in which $s_1 \,\varepsilon\, \mathcal{B}(\Omega)$, plays a central role in defining the mapping $\mathbf{\Phi}$. As stated in Section 3.5, Theorem II implies that this equation possesses a unique solution $w \,\varepsilon\, \mathcal{L}_{2R}$ if

$$\inf_{\omega} | C - 1 | > 1 - \alpha. \tag{27}$$

It is of interest to note that there exists a function ψ such that the equation $\psi[w] = \mathbf{C}w + \mathbf{A}s_1$ possesses no solution $w \,\varepsilon\, \mathcal{L}_{2R}$ for any non-identically zero $\mathbf{A}s_1$ if (27) is not satisfied, Ω is a bounded set, and $\mathbf{C} = c\mathbf{I}$ where c is a real constant. This follows directly from the fact that if (27) is violated, $\alpha \leqq c \leqq (2 - \alpha) = \beta$. Specifically, throughout a neighborhood of the origin let ψ be independent of t and linear in w with slope c. Then clearly, $\psi[w] - cw = 0$ whenever $| w | < \epsilon$ where ϵ is some positive constant. Since $\mathbf{A}s_1$ is assumed to be nonzero almost everywhere, the validity of our assertion is evident.

Let $\mathbf{U}$ denote the mapping of the orthogonal complement of $\mathcal{B}(\Omega_D)$ into itself defined by $\mathbf{U}w_b = (\mathbf{I} - \mathbf{P}_D)\psi[w_a + w_b] - \mathbf{E}w_b$, where $w_a \,\varepsilon\, \mathcal{B}(\Omega_D)$ and $\mathbf{E} = \mathbf{C} - \mathbf{A}\tilde{\mathbf{D}}^{-1}(\mathbf{P} - \mathbf{P}_D)\mathbf{B}$. Theorem II asserts that $\mathbf{U}$ possesses a bounded inverse if $E(\omega) = C - AD^{-1}(P - P_D)B$, for all ω contained in the complement of Ω_D, is bounded away from the disk in the complex plane centered at [0,1] and having radius $(1 - \alpha)$.

Theorem III: Let γ be a real constant and let Ξ_1 denote an open interval contained in the complement of Ω_D such that $E(\omega)$ is continuous on Ξ_1 and

$$\inf_{\omega \varepsilon \Xi_1} | E(\omega) - \gamma | = 0.$$

Let ψ be independent of t and continuously differentiable with respect to x on an interval Ξ_2 where

$$\inf_{x \varepsilon \Xi_2} \left| \frac{d\psi(x)}{dx} - \gamma \right| = 0.$$

Then **U** *does not possess a bounded inverse.*

Remark: Note that the hypotheses regarding ψ are satisfied if ψ is independent of t, continuously differentiable with respect to x, and γ is any point on the real-axis diameter of the disk mentioned above. Of course we assume that

$$\inf_x \frac{d\psi(x)}{dx} = \alpha, \qquad \text{and} \qquad \sup_x \frac{d\psi(x)}{dx} = \beta.$$

Proof of Theorem III:

We need the following lemma.

Lemma III: Let Δ_1 denote the real interval $[-T,T]$, let ϵ_1 and ϵ_2 be real positive constants, and let $h(t)$ be a continuous real function defined on Δ_1. Then there exists a function $g(t)$ in the orthogonal complement of $\mathcal{B}(\Omega_D)$ (assuming that Ω_D is a proper subset of the real line) such that

$$| h(t) - g(t) | \leqq \epsilon_1 , \qquad t \, \varepsilon \, (\Delta_1 - \Delta_2)$$

where Δ_2 is a set of points contained in disjoint intervals of total measure not exceeding ϵ_2.

Proof:

If the complement of Ω_D contains an interval centered at the origin, the result is known and in fact is true with Δ_2 the null set. The following very direct argument makes use of the known result to treat the case in which the complement of Ω_D does not contain an interval centered at the origin.

Let ω_1 and ω_2 be real positive constants such that the interval $[\omega_1 - \omega_2, \omega_1 + \omega_2]$, where $\omega_1 > \omega_2$, is contained in the complement of Ω_D. Let Ω' be an interval of length $2\omega_2$ centered at the origin. Let Ω' be an interval of length $2\omega_2$ centered at the origin. Let $\{t_1, t_2, \cdots, t_n\} = \{t \mid t \, \varepsilon \, \Delta_1 ; \cos \omega_1 t = 0\}$. Let I_j denote an interval of length ϵ_2/n centered at t_j. For any $\epsilon_3 > 0$, there exists a function $l(t) \; \varepsilon \; \mathcal{B}(\Omega')$ such that

$$\left| l(t) - \frac{h(t)}{\cos \omega_1 t} \right| \leqq \epsilon_3 , \qquad t \; \varepsilon \; (\Delta_1 - \Delta_2)$$

where $\Delta_2 = \bigcup_{j=1}^{n} I_j$. Choose ϵ_3 such that $\epsilon_2 = \epsilon_3 \inf_{t\varepsilon(\Delta_1 - \Delta_2)} \cos \omega_2 t$. It is evident that $l(t) \cos \omega_1 t$ possesses the properties of $g(t)$ stated in the lemma.

To prove Theorem III it suffices to show that for any $\epsilon > 0$, there exist two functions w_{1b} and w_{2b}, belonging to the orthogonal complement of $\mathcal{B}(\Omega_D)$, such that $\| w_{1b} - w_{2b} \| = 1$ and $\| \psi[w_a + w_{1b}] - \psi[w_a + w_{2b}] - \mathbf{E}(w_{1b} - w_{2b}) \| < \epsilon$.

Let ϵ_4, ϵ_5, and ϵ_6 be arbitrary positive constants. Since $\inf_{\omega\varepsilon\Xi_1} | E(\omega) - \gamma | = 0$ and $E(-\omega)$ is equal to the complex conjugate of $E(\omega)$, there exists an $\omega_3 \; \varepsilon \; \Xi_1$ such that $| E(\pm\omega_3) - \gamma | \leqq \frac{1}{2}\epsilon_4$. Let Π_1 and Π_2 denote two finite intervals of equal length $\mu(\Pi_1)$ contained in Ξ_1 and centered, respectively, at $-\omega_3$ and $+\omega_3$. Let $(w_{1b} - w_{2b}) \; \varepsilon \; \mathcal{B}(\Pi_1 \cup \Pi_2)$ with $\| w_{1b} - w_{2b} \| = 1$. Choose $\mu(\Pi_1)$ and T such that

$$\sup_{\omega\varepsilon\Pi_1} | E(\omega) - \gamma | \leqq \epsilon_4 , \qquad \| w_{1b} - w_{2b} \|_{\substack{|t|>T \\ t\varepsilon\Delta_3}} \leqq \epsilon_5$$

where Δ_3 is any subset of $\Delta_1 = [-T,T]$ with measure not exceeding k_6, a sufficiently small positive constant. The second inequality can always be satisfied since, in accordance with the inequality stated in Section 4.3, $\sup_t | w_{1b} - w_{2b} | \leqq [\pi^{-1}\mu(\Pi_1)]^{\frac{1}{2}}$.

Since $\inf_{x\varepsilon\Xi_2} | [d\psi(x)/dx] - \gamma | = 0$, there exists a real constant $x_0 \; \varepsilon \; \Xi_2$ such that

$$\left| \frac{\psi[w_a + w_{1b}] - \psi[w_a + w_{2b}]}{w_{1b} - w_{2b}} - \gamma \right| \leqq \epsilon_6 \tag{28}$$

whenever $| w_a + w_{1b} - x_0 |$ and $| w_{1b} - w_{2b} |$ are sufficiently small. We may assume that $\mu(\Pi_1)$ is so small that the condition on $| w_{1b} - w_{2b} |$ is satisfied. Choose w_{1b} in accordance with Lemma III so that (28) is satisfied on $(\Delta_1 - \Delta_2)$ where Δ_2 is a set of measure not exceeding k_6. Let $(\Delta_1 - \Delta_2)^*$ denote the complement of $(\Delta_1 - \Delta_2)$. Observe that

$$\begin{aligned} &\| \psi[w_a + w_{1b}] - \psi[w_a + w_{2b}] - \mathbf{E}(w_{1b} - w_{2b}) \| \\ &\quad \leqq \| \psi[w_a + w_{1b}] - \psi[w_a + w_{2b}] - \gamma(w_{1b} - w_{2b}) \| \\ &\qquad + \| (\mathbf{E} - \gamma\mathbf{I})(w_{1b} - w_{2b}) \| \end{aligned}$$

$$\leqq \epsilon_6 \| w_{1b} - w_{2b} \| + \| \psi[w_a + w_{1b}] - \psi[w_a + w_{2b}] - \gamma(w_{1b} - w_{2b}) \|_{(\Delta_1 - \Delta_2)^*}$$
$$+ \| (\mathbf{E} - \gamma \mathbf{I})(w_{1b} - w_{2b}) \|$$
$$\leqq \epsilon_6 + (\beta + |\gamma|)\epsilon_5 + \| (\mathbf{E} - \gamma \mathbf{I})(w_{1b} - w_{2b}) \|$$
$$\leqq \epsilon_6 + (\beta + |\gamma|)\epsilon_5 + \epsilon_4 .$$

This completes the proof.

APPENDIX

The purpose of this appendix is to briefly indicate an alternative technique for determining sufficient conditions for the recoverability of s_1.

Instead of the assumptions stated in Section 3.4 suppose that for some real constant ψ_0:

$$\inf_{\omega \varepsilon \Omega} | D - B(\psi_0 - C)^{-1} A | > 0$$

$$\| (\psi_0 \mathbf{I} - \mathbf{C})^{-1} \| \eta(\psi_0) = \operatorname{ess\,sup}_{\omega} | (\psi_0 - C)^{-1} | \eta(\psi_0) = q < 1.$$

These inequalities imply that $\{\mathbf{PD} + \mathbf{PB}(\psi_0\mathbf{I} - \mathbf{C})^{-1}\mathbf{A}\}$ possesses a bounded inverse on $\mathfrak{B}(\Omega)$ and that for any $g \varepsilon \mathfrak{L}_{2R}$ the equation $\psi[w] = \mathbf{C}w + g$ possesses a unique solution $w \varepsilon \mathfrak{L}_{2R}$.

From

$$\psi[w] = \mathbf{C}w + \mathbf{A}s_1 , \qquad s_3 = \mathbf{PB}w + \mathbf{D}s_1 , \tag{29}$$

and $\psi[w] = \psi_0 w + \tilde{\psi}[w]$ we have

$$s_3 = \{\mathbf{PD} + \mathbf{PB}(\psi_0\mathbf{I} - \mathbf{C})^{-1}\mathbf{A}\}s_1 - \mathbf{PB}(\psi_0\mathbf{I} - \mathbf{C})^{-1}\tilde{\psi}[w]. \tag{30}$$

Equation (30) can be written as

$$s_1 = \mathbf{M}s_1 + \{\mathbf{PD} + \mathbf{PB}(\psi_0\mathbf{I} - \mathbf{C})^{-1}\mathbf{A}\}^{-1}s_3$$

where

$$\mathbf{M}s_1 = \{\mathbf{PD} + \mathbf{PB}(\psi_0\mathbf{I} - \mathbf{C})^{-1}\mathbf{A}\}^{-1}\mathbf{PB}(\psi_0\mathbf{I} - \mathbf{C})^{-1}\tilde{\psi}[w].$$

Of course the dependence of the right-hand side on s_1 is through w.

Let $\bar{w}$ be the solution of $\psi[w] = \mathbf{C}w + \mathbf{A}s_1$ corresponding to $s_1 = \bar{s}_1$. Then by arguments similar to those leading to Theorem II,

$$\| w - \bar{w} \| \leqq \frac{1}{1 - q} \| (\psi_0 \mathbf{I} - \mathbf{C})^{-1} \mathbf{AP} \| \cdot \| s_1 - \bar{s}_1 \| .$$

Thus **M** is a contraction mapping of $\mathfrak{B}(\Omega)$ into itself if

$$p = \| \{\mathbf{PD} + \mathbf{PB}(\psi_0\mathbf{I} - \mathbf{C})^{-1}\mathbf{A}\}^{-1}\mathbf{PB}(\psi_0\mathbf{I} - \mathbf{C})^{-1} \| \eta(\psi_0)[1/(1 - q)] \| (\psi_0\mathbf{I} - \mathbf{C})^{-1}\mathbf{AP} \| < 1.$$

Hence if the received signal s_3 is known to be related to the transmitted signal $s_1 \varepsilon \mathfrak{B}(\Omega)$ by (29), s_1 can be recovered if our assumptions are satisfied and if $p < 1$. Using arguments similar to those leading to Lemma I,

$$p = \operatorname*{ess\,sup}_{\omega\varepsilon\Omega} \left| \frac{B}{D(\psi_0 - C) + BA} \right| \eta(\psi_0) \frac{1}{1 - q} \operatorname*{ess\,sup}_{\omega\varepsilon\Omega} \left| \frac{A}{\psi_0 - C} \right| .$$

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