Propagation of Light Rays through a Lens-Waveguide with Curved Axis

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The problem of the propagation of a light ray in a lens-waveguide with arbitrarily bent axis is solved. The solution can conveniently be expressed in form of an integral if the waveguide is bent sufficiently gradually.

In general, a ray which is incident on the axis of the straight lens-waveguide follows an undulating path after traversing the bend. The undulations can be kept arbitrarily small if the bend is tapered and if its curvature is sufficiently gentle. By properly dimensioning the bend, the undulations can be made to cancel out completely, so that a ray which follows the axis of the straight waveguide can be made to leave the bend on the axis of the outgoing straight waveguide.

The cases of circular and tapered bends as well as tilts and offsets are discussed.

I. INTRODUCTION

The invention of the laser has revived interest in light as a communications carrier. One of the many problems which have to be solved before a light communications system becomes feasible is the propagation of a light beam from transmitter to receiver. It is well known[1] that a sequence of converging lenses can guide a light beam and keep it from spreading. The losses of such a lens-waveguide can be calculated only by means of wave optics. However, even geometric optics can demonstrate the guiding properties of a lens-waveguide.[2] It can show that a light beam, once it is injected into a sequence of lenses, follows an undulating path without wandering away from the axis of the lens-waveguide.

The present paper is limited to describing the behavior of a light beam in a lens-waveguide whose axis is not straight everywhere, but which is allowed to follow bends of the transmission path or exhibit abrupt changes like tilts of its axis or an offset of one of its lenses from the axis on which all the other lenses are centered. The description is given in terms of geometric optics, so that no information about the losses of light transmission through a lens-waveguide with bent axis can be obtained. It is clear, however, that the loss of power from the light beam is caused partly by diffraction losses at the edge of the lenses. These losses are certainly minimized if the center of gravity of the light distribution in the beam, which is described as the light ray, remains as far from the edges of the lenses as possible. It is desirable, therefore, that the light ray follow the axis of the structure as much as possible. The mathematical description of the light ray presented in this paper shows that a ray which follows the axis of the straight lens-waveguide will be forced into an oscillatory trajectory whenever the axis of the lens-waveguide deviates from perfect straightness.

It is necessary, at times, to lead the lens-waveguide through a bend to circumvent obstacles which might lie in its path. If it is possible to design a bend such that a beam which is incident on the axis of the straight incoming waveguide will leave the bend on the axis of the outgoing straight waveguide, we will call such a bend one of optimum design. It is, of course, equally desirable also to keep the deviations of the beam from the waveguide axis on the bend as small as possible.

We show in this paper that one can inject a light beam into a circular section of the lens-waveguide in such a way that it travels through the bend at a constant distance from the waveguide axis. It can, furthermore, be readmitted into the outgoing straight section of the lens-waveguide so as to continue along its axis. An optimization of a circular bend is thus possible.

Another way of reducing the oscillations caused by a bend is to taper its radius of curvature gradually from its infinite value on the straight section to a minimum value and back to the infinite value of the outgoing straight section.

The theory of ray optics in a lens-waveguide with curved axis is a generalization of Pierce's theory.[2] The solution of the difference equation can be approximated by a convenient integral expression in the limit of a lens-waveguide whose axis changes direction only gradually.

II. RAY OPTICS OF THE CURVED LENS-WAVEGUIDE

The light ray is described by its distance r from the center of the lenses at the position of each lens (see Fig. 1). The lenses are spaced a distance L apart and have a focal length $f = 1/C$. The quantity C is known as the "lens power."

If we assume that all angles are so small that $\tan \alpha$ can be taken $= \alpha$, we obtain the following relations between the radii r_n and r_{n+1} of the

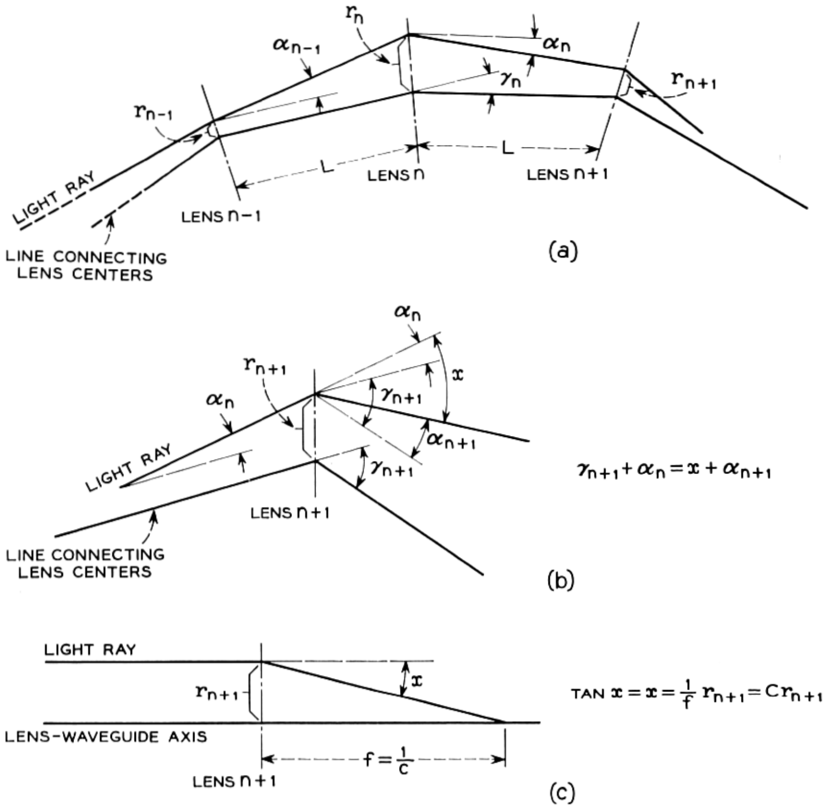

Fig. 1 — (a) Illustration of the bent lens-waveguide and the radii of the light beam at different lenses. (b) Relations among the various angles between the straight lines connecting successive lenses and the light beam and these straight lines. (c) Illustration of the relation between the angle x by which a thin lens deflects the light ray and the radius r_{n+1} of the ray.

ray at the positions of the nth and $(n + 1)$th lens

$$r_{n+1} = r_n + \alpha_n L \tag{1}$$

$$Cr_{n+1} = \gamma_{n+1} + \alpha_n - \alpha_{n+1} . \tag{2}$$

α_n is the angle between the ray and the straight line connecting successive lenses taken to the right of the nth lens. γ_n is the angle at the nth lens between the two straight lines connecting the lenses. The geometrical relations (1) and (2) can be read off Fig. 1(a) through (c).

Eliminating α_n and α_{n+1} from (1) and (2) results in

$$r_{n+2} - (2 - LC)r_{n+1} + r_n = L\gamma_{n+1} = y_{n+2} . \tag{3}$$

Equation (3) agrees with Pierce's equation for the straight lens-waveguide in the case $\gamma_{n+1} = 0$. In this case the equation has the solution

$$r_n = A \cos n\theta + B \sin n\theta \tag{4}$$

with

$$\cos \theta = 1 - (LC/2) \tag{5}$$

or equivalently

$$\sin \theta = \sqrt{LC} \sqrt{1 - \frac{LC}{4}}. \tag{6}$$

This solution shows that, in general, the ray oscillates around the axis of the lens-waveguide.

From Fig. 2 it is apparent that the quantity y_{n+2} occurring in (3) is given by

$$y_{n+2} = L^2/R_{n+1}. \tag{7}$$

Equation (3) is the mathematical description of the ray optics of the curved lens-waveguide. It allows the successive calculation of the

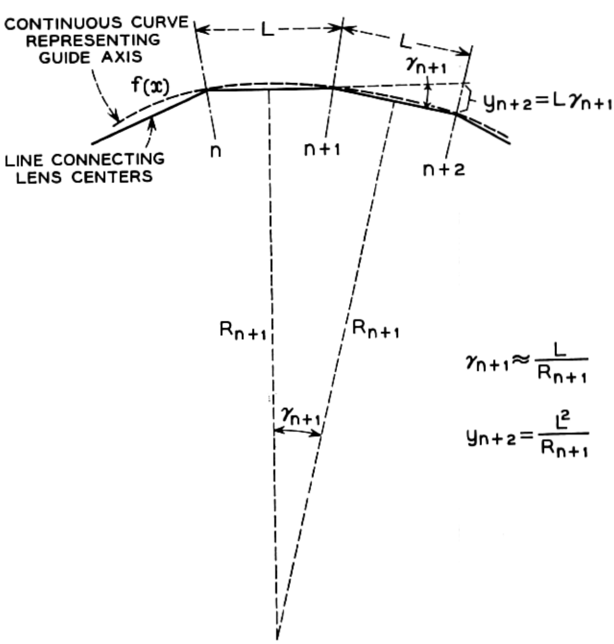

Fig. 2 — Definition of the quantities y_{n+2} and R_{n+1}.

distances r_n of the ray from the centers of the lenses for any distribution of lenses and suitable initial conditions.

III. SOLUTION OF THE RAY EQUATION

By standard methods for solving difference equations,[3] a rigorous solution of (3) was obtained:

$$r_n = A \cos n\theta + B \sin n\theta + \frac{L^2}{\sin\theta} \sum_{m=0}^{n-2} \frac{\sin\theta(n - m - 1)}{R_{m+1}}. \tag{8}$$

The sum in (8) is defined only for $n > 1$. The constants A and B are determined from the initial position of the ray at r_0 and r_1

$$A = r_0 \tag{9}$$

$$B = \frac{r_1 - r_0 \cos\theta}{\sin\theta}. \tag{10}$$

The validity of (8) can be checked by substitution into (3).

Equation (8) can be converted to a convenient integral expression. We use the identity

$$\sin\theta(n - m - 1) = \frac{\theta}{2\sin\theta} \int_m^{m+2} \sin\theta(n - x)\, dx \tag{11}$$

and obtain instead of (8)

$$r_n = A \cos n\theta + B \sin n\theta + \frac{\theta L^2}{2\sin^2\theta} \sum_{m=0}^{n-2} \frac{1}{R_{m+1}} \int_m^{m+2} \sin\theta(n - x)\, dx. \tag{12}$$

Now let us assume that R_m varies so slowly that we can write

$$R_m = R(x). \tag{13}$$

The value R_m defined in Fig. 2 has become the radius of curvature $R(x)$ of a curve $f(x)$ which smoothly connects the centers of the lenses. With the use of (13) the sum of (12) can be changed into an integral, and we obtain

$$r_n = A \cos n\theta + B \sin n\theta + \frac{L^2}{2\sin^2\theta} \left\{ \frac{\cos(n-1)\theta - \cos n\theta}{R_1} + \frac{1 - \cos\theta}{R_{n-1}} + 2\theta \int_1^{n-1} \frac{\sin\theta(n - x)}{R(x)} \right\} dx. \tag{14}$$

Equation (14) holds for arbitrary values of L and C as long as

$$0 < |1 - (LC/4)| < 1. \tag{15}$$

Its validity is restricted to the extent to which the approximation (13) holds. That is, (14) holds as long as the lens-waveguide is bent only gradually.

For completeness let us state the result in the limit $L \to 0$. It follows from (6) that $\theta \to \sqrt{LC}$ in that case. If we introduce the length $s = nL$ or $\sigma = xL$ as the new coordinate, we obtain from (14) in the limit of closely spaced lenses

$$r(s) = A \cos \kappa s + B \sin \kappa s + \frac{1}{\kappa} \int_0^s \frac{1}{R(\sigma)} \sin \kappa(s - \sigma)\, d\sigma \tag{16}$$

with

$$\kappa^2 = \lim_{L \to 0} (C/L). \tag{17}$$

The solution (16) could have been obtained by first converting the difference equation (3) into a differential equation. Equation (16) is the solution of this differential equation.

IV. BENDS, TILTS AND OFFSETS

We study first the problem of transmitting a light ray along two sections of straight lens-waveguide which are connected by a circular bend. To make the problem more general we assume that the straight sections are not tangents of the circle at the point of contact and that the axis of the straight section of the guide does not go through the center of the first lens which is located on the circle. In other words, we assume that the circle is tilted by an angle α and offset by an amount "a" with respect to the incoming and outgoing straight lens-waveguides (see Fig. 3). For the solution of this problem we go back to the exact equation (8) and limit ourselves to a beam which enters the bend on the axis of the incoming straight guide ($A = 0$, $B = 0$). A study of the geometry shows that

$$R_1 = R_{N-1} = \frac{L^2}{a}; \qquad R_2 = R_{N-2} = \frac{2L^2R}{L^2 + 2\alpha LR - 2aR}; \qquad R_m = R$$

$$m \neq 1, 2, N - 1, N - 2.$$

Using the identity

$$\sum_{m=0}^{k} \sin \theta(n - m) = \frac{\sin\left(n - \frac{k}{2}\right)\theta \sin \frac{1}{2}(k + 1)\theta}{\sin \frac{1}{2}\theta}$$

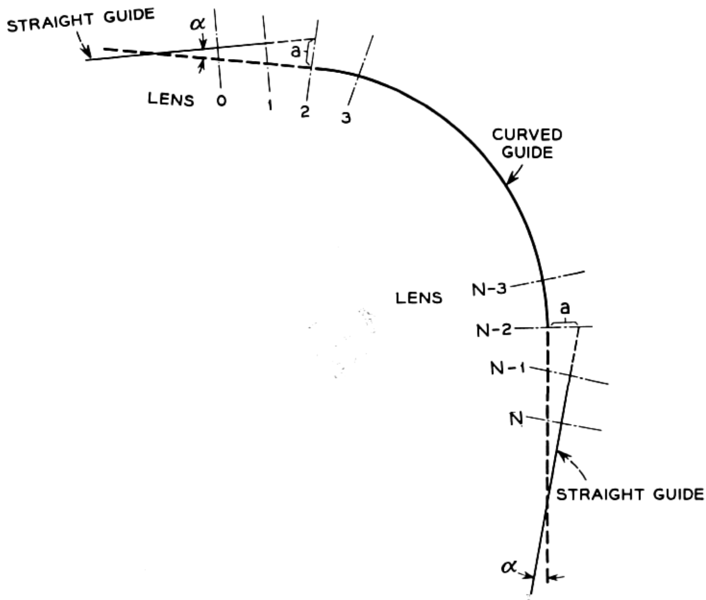

Fig. 3 — A circular bend to which the straight lens-waveguide is connected by an offset a and a tilt α.

we obtain from (8) on the circular bend

$$r_n = \frac{L^2}{\sin\theta}\left\{\left[\frac{a}{L^2}\sin\theta - \frac{\cot\frac{1}{2}\theta}{2R}\right]\cos\theta(n-2) + \frac{1}{L}\left[\alpha - \frac{a}{L}(1-\cos\theta)\right]\sin\theta(n-2) + \frac{\cot\frac{1}{2}\theta}{2R}\right\} \quad 2 \leqq n \leqq N-2 \tag{18}$$

and on the straight outgoing section

$$r_n = 2\frac{L^2}{\sin\theta}\sin\theta\left(n - \frac{1}{2}N\right)\left\{-\left[\frac{a}{L^2}\sin\theta - \frac{\cot\frac{1}{2}\theta}{2R}\right]\cdot\sin\theta\left(\frac{1}{2}N - 2\right) + \frac{1}{L}\left[\alpha - \frac{a}{L}(1-\cos\theta)\right]\cdot\cos\theta\left(\frac{1}{2}N-2\right)\right\} \quad N \leqq n < \infty. \tag{19}$$

By properly adjusting tilt and offset we can assure that the beam will continue to travel on the axis of the outgoing straight lens-waveguide without undulations. We adjust a and α so that both expressions in brackets in (19) vanish simultaneously. This leads to

$$a = \frac{L^2}{2R} \frac{\cot \frac{1}{2}\theta}{\sin \theta} = \frac{L}{RC} \tag{20}$$

and

$$\alpha = (a/L)(1 - \cos \theta) = L/2R. \tag{21}$$

These values for the tilt and offset, by which the circular bend is connected to the two sections of straight line, provide us with an optimum circular bend — that is, one which guides the light ray around a curve without causing it to undulate. It is apparent from (18) that these optimum values for tilt and offset assure simultaneously that the light beam traverses the circular section of the bend at a constant distance from the center of the lenses which is equal to the amount of offset (20). It is clear that the offset injects the beam into the circular bend just at the spacing at which the beam can travel around the circle without undulating.

There are other ways to design a bend to guide a light beam without introducing undulations of the outgoing beam. For example, if we connect the circular bend smoothly to the straight sections ($a = 0$ and $\alpha = 0$) we can still suppress undulations by choosing the length of the bend and the properties of the lenses such that $\sin \theta(\frac{1}{2}N - 2) = 0$. However, such a design is more complicated and depends on the length of the bend.

As a second example, we consider the case of a bend with tapered radius of curvature:

$$\frac{1}{R_t} = \begin{cases} \dfrac{4\delta}{D^2} Ly & 0 \leqq y \leqq \dfrac{N}{2} \\[2ex] \dfrac{4\delta}{D^2} (N - y)L & \dfrac{N}{2} \leqq y \leqq N . \end{cases} \tag{22}$$

Here, N is the number of periods L which fit on the bend (Fig. 4), δ is the angle through which the bend leads, and $D = NL$ is the total length of the bend.

The trajectory of the ray can be computed with the help of (14). We assume that the ray is incident on the axis of the incoming straight waveguide ($A = B = 0$).

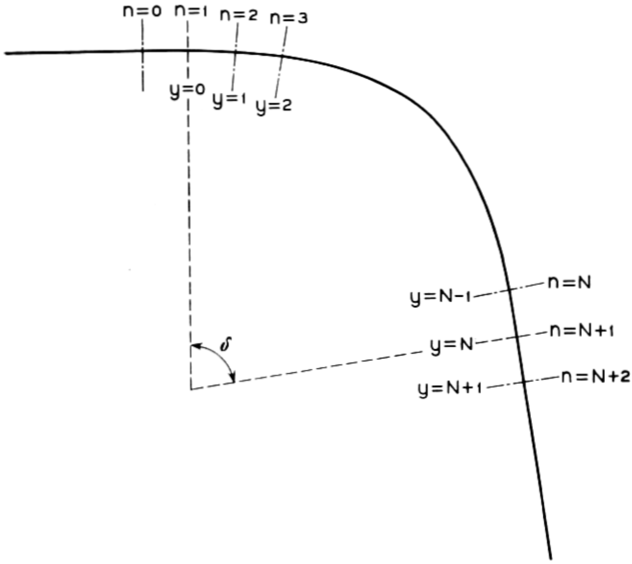

Fig. 4 — A tapered bend of the lens-waveguide.

For $2 \leqq n \leqq \frac{1}{2}N + 2$,

$$r_n = \frac{2\delta L^3}{D^2 \sin^2 \theta} \left\{ (n - 2)(1 + \cos \theta) + \frac{2}{\theta} [\sin \theta - \sin \theta(n - 1)] \right\}; \tag{23a}$$

for $\frac{1}{2} N + 2 \leqq n \leqq N + 2$,

$$r_n = \frac{2\delta L^3}{D^2 \sin^2 \theta} \left\{ (N - n + 2)(1 + \cos \theta) + \frac{2}{\theta} \left[2 \sin \theta \left(n - 1 - \frac{1}{2} N \right) - \sin \theta(n - 1) - \sin \theta \right] \right\}; \tag{23b}$$

for $N + 2 \leqq n < \infty$,

$$r_n = \frac{2\delta L^3}{D^2 \sin^2 \theta} \frac{2}{\theta} \left\{ 2 \sin \theta \left(n - 1 - \frac{1}{2} N \right) - \sin \theta(n - 1 - N) - \sin \theta(n - 1) \right\}. \tag{23c}$$

Equation (23c) can be rewritten to show the amplitude of the undulation more clearly

$$r_n = \frac{16\delta L^3}{D^2\theta \sin^2 \theta} \sin^2 \theta \frac{1}{4} N \sin \theta \left(n - 1 - \frac{1}{2} N\right). \tag{24}$$

As in the case of the circular bend, the undulations can be made to cancel out if we design the lens-waveguide and the length of the bend such that $\sin \theta \frac{1}{4} N = 0$. It is also apparent that the amplitude of oscillation decreases rapidly if the length D of the bend is increased.

Finally, we state the equations for the case of a tilt and for an offset of one lens in the waveguide (see Figs. 5 and 6). Assuming, as always, that the incident ray follows the axis of the incoming straight waveguide, we obtain from (8) in case of a tilt which is spaced a distance from the lens,

$$r_n = \frac{\delta}{\sin \theta} \sqrt{(L - a)^2 + a^2 + 2a(L - a) \cdot \cos \theta} \sin [\theta(n - 1) - \vartheta] \tag{25}$$

with

$$\vartheta = \arctan \frac{a \cdot \sin \theta}{(L - a) + a \cos \theta}. \tag{26}$$

If one lens is offset by an amount Δ and if the incident ray follows the axis of the incoming straight waveguide

$$r_n = -\frac{4\Delta}{\sin \theta} \sin^2 \frac{1}{2} \theta \sin \theta(n - 2) \tag{27}$$

describes the ray on the outgoing straight waveguide.

V. DISCUSSION AND NUMERICAL RESULTS

In the preceding section we have found that a light ray which is incident along the axis of the incoming straight waveguide, in general leaves a bend, an offset or a tilt undulating around the axis of the straight outgoing waveguide. We have found that these undulations can be suppressed by properly designing the bends. In this last section we will discuss the maximum undulations of the outgoing light ray if no provision for canceling the undulations has been made.

If one does not intend to use an optimum design, one would not include a tilt or an offset in the bend but would connect the circular bend smoothly to the straight sections. If we take $a = \alpha = 0$ in (19) we see that the maximum amplitude which the undulations can reach is given by

$$r_{\max} = \frac{L^2}{R} \frac{\cot \frac{1}{2} \theta}{\sin \theta} = 2 \frac{L}{RC}. \tag{28}$$

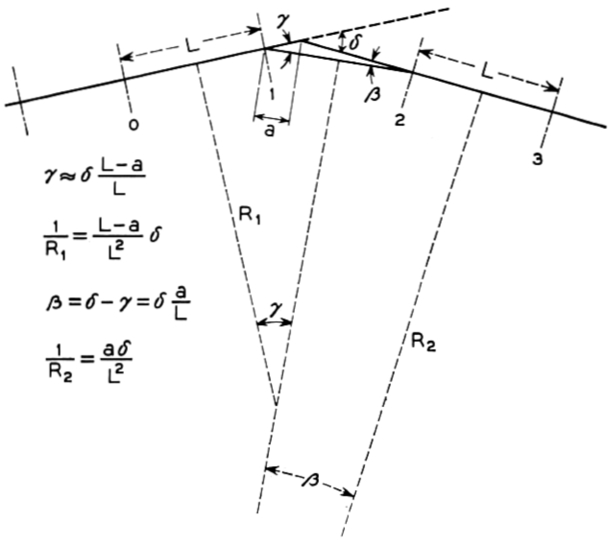

Fig. 5 — The lens-waveguide tilted by an angle δ a distance a from a lens.

The same equation describes the maximum deviation of the beam on the circular section of the bend.

The maximum deviation of the ray from the axis on the tapered bend can be obtained from (23). We assume that $N \gg 1$ and find that the maximum deviation on the tapered bend is

$$r_{\max} \approx \frac{\delta L^3 N}{D^2 \sin^2 \theta} (1 + \cos \theta) = \frac{2\delta L^2}{D(4 - LC)} \tag{29}$$

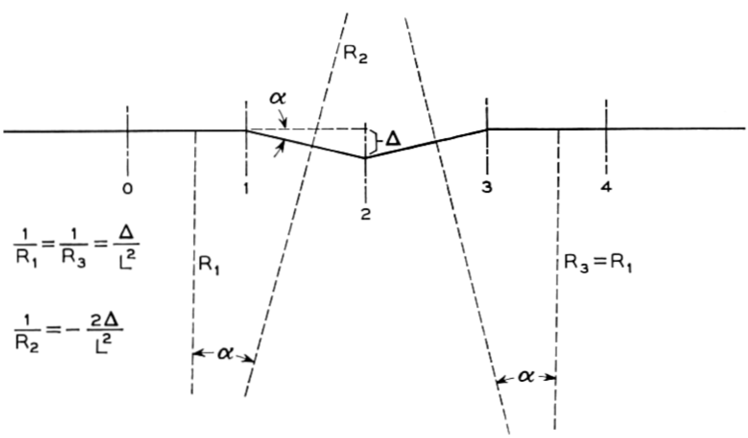

Fig. 6 — One lens of the waveguide is offset by an amount △.

while the maximum amplitude of the undulations on the straight outgoing section is given by (24)

$$r_{\max} = \frac{16\delta L^3}{D^2\theta \sin^2 \theta} = \frac{64\delta L^2}{D^2\left[\arccos\left(1 - \frac{LC}{2}\right)\right] C(4 - LC)}. \tag{30}$$

A comparison of (29) and (30) shows that the maximum deviation on the tapered bend is considerably larger than the maximum amplitudes of the undulating beam after it has left the bend. The maximum deviation on the tapered bend is larger than the maximum amplitude of the undulating beam leaving the bend by a factor of roughly $\frac{1}{32}DC$. The taper has the effect of canceling some of the oscillations which build up in the bend.

Equations (29) and (30) show that the amplitudes become infinitely large if $LC = 4$. This failure to guide the beam around a bend is not apparent from (28) for the circular bend. In fact, a beam which enters the circular bend on the axis of the incoming waveguide tangentially to the circle does not experience such a catastrophe. However, we see from (18) or (19) that this geometry is rather unique, since in general both (18) and (19) become infinite for $\sin \theta = \sqrt{LC(1 - LC/4)} = 0$. Only in the case considered ($a = \alpha = 0$) can the beam be confined even if $LC = 4$.

Table I lists the values of $r_{\max}$ for the circular and the tapered bend for the case $LC = 2$ (confocal geometry). It is further assumed that both bends lead the waveguide through an angle of $\delta = 90°$.

The maximum amplitude of undulation of a ray which has traversed a tilt is, according to (25),

$$r_{\max} = 2\delta \frac{\sqrt{L^2 - aLC(L - a)}}{\sqrt{LC}\sqrt{4 - LC}} \tag{31}$$

and the amplitude after an offset is, from (27)

$$r_{\max} = 2 \frac{\Delta\sqrt{LC}}{\sqrt{4 - LC}} \tag{32}$$

Table I — Values of $r_{\max}$ for Confocal Geometry

$r_{\max}/L$		D/L			
		10	100	1000	10,000
Circular bend		$1.57\ 10^{-1}$	$1.57\ 10^{-2}$	$1.57\ 10^{-3}$	$1.57\ 10^{-4}$
Tapered bend	on bend	$1.57\ 10^{-1}$	$1.57\ 10^{-2}$	$1.57\ 10^{-3}$	$1.57\ 10^{-4}$
	after bend	$1.60\ 10^{-1}$	$1.60\ 10^{-3}$	$1.60\ 10^{-5}$	$1.60\ 10^{-7}$

Both of these amplitudes become infinite for $LC = 4$. However, if the tilt is placed exactly halfway in between two lenses ($2a = L$), (31) becomes $r_{\max} = \delta L/2$ if $LC = 4$. In this way even a lens-waveguide with concentric geometry can be tilted without loss of the beam.

VII. ACKNOWLEDGMENT

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