A Theory of a Unilateral Parametric Amplifier Using Two Diodes

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This paper describes the theory of a unilateral parametric amplifier which contains two variable-capacitance diodes separated by a quarter wavelength at the signal frequency and a half wavelength at the idler frequency. It is shown that a broadband signal circuit is essential in order to obtain unilateral gain, and that matching conditions are obtainable even with a high gain. The optimum noise figure is slightly worse than that of a single-diode reflection-type amplifier; however, this amplifier has advantages if it is refrigerated at liquid helium temperature, because it does not require a circulator in front of its input port. The amplifier usually requires an isolator at its output port, since it does not have substantial loss in the reverse direction.

I. INTRODUCTION

The performance of a single-diode reflection-type parametric amplifier is often limited by the availability of a good circulator, which is essential to a practical amplifier. This becomes a more serious problem when the amplifier is to be refrigerated down to liquid helium temperature, since satisfactory circulator performance is then difficult to obtain.

A unilateral parametric amplifier with two diodes, originally proposed by Baldwin,[1] does not require any circulator or isolator in front of the amplifier if the signal source impedance is reasonably well matched. Therefore, this amplifier might avoid the difficult circulator problems.

This paper is prepared to show the theoretical characteristics of a unilateral parametric amplifier with two diodes separated by a quarter wavelength at the signal frequency. In Section II, an exact expression for the scattering matrix of a simplified model of the amplifier is obtained. In Sections III and IV, expressions for power gain, noise figure and bandwidth are calculated. In Sections V and VI, the optimum noise figure and some design considerations of the amplifier are described.

II. PRESENTATION OF THE SCATTERING MATRIX OF THE AMPLIFIER

Fig. 1 shows a basic configuration of the unilateral parametric amplifier. The black box shown in Fig. 1 is a symmetrical lossless reciprocal two-terminal-pair network whose image impedance Z and phase constant θ are assumed to be real quantities. Stationary susceptances of the diodes are considered as a part of the black box, and losses in the diodes are included in the external loads. C and C' in Fig. 1 represent the sinusoidally varying shunt capacitances which play an essential role in the mechanism of amplification. Parameters concerned with the right-side arm of the black box are indicated by primed letters, and those of signal and idler frequencies are indicated by suffixes 1 and 2 respectively.

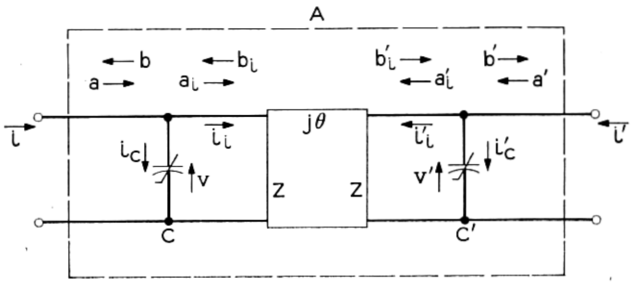

Fig. 1 — Basic configuration of the unilateral parametric amplifier.

Currents and voltages of the varying capacitance C are related by the following equations:

$$\begin{aligned} i_{c1} &= j\omega_1(c/2)v_2^* \\ i_{c2}^* &= -j\omega_2(c^*/2)v_1 \end{aligned} \tag{1}$$

where ω is angular frequency and the asterisk indicates a complex conjugate. The terminal currents and voltages of the capacitance C are expressed in terms of incident and reflected waves a, b, a_i, and b_i normalized by Z as follows:

$$\begin{aligned} v &= \sqrt{Z}(a + b) = \sqrt{Z}(a_i + b_i) \\ i &= \frac{1}{\sqrt{Z}}(a - b) \\ i_i &= \frac{1}{\sqrt{Z}}(a_i - b_i). \end{aligned} \tag{2}$$

The directions of flow of these waves are indicated by arrows in Fig. 1. The continuity equation of current at the terminal C is given by

$$i = i_c + i_i \, . \tag{3}$$

Combining (1), (2), and (3), we obtain the following equations:

$$\begin{aligned} a_{i1} &= a_1 - j\omega_1\dot{\kappa}a_2^* - j\omega_1\dot{\kappa}b_2^* \\ b_{i1} &= b_1 + j\omega_1\dot{\kappa}a_2^* + j\omega_1\dot{\kappa}b_2^* \\ a_{i2}^* &= a_2^* + j\omega_2\dot{\kappa}^*a_1 + j\omega_2\dot{\kappa}^*b_1 \\ b_{i2}^* &= b_2^* - j\omega_2\dot{\kappa}^*a_1 - j\omega_2\dot{\kappa}^*b_1 \end{aligned} \tag{4}$$

where $\dot{\kappa}$ is a complex number defined as:

$$\dot{\kappa} = \frac{c}{2} \cdot \frac{\sqrt{Z_1 Z_2}}{2} \, . \tag{5}$$

Equations similar to (4) are obtainable for the varying capacitance C' by replacing unprimed parameters by primed ones in (4).

Waves of the lossless reciprocal two-terminal-pair network are related by the following equations.

$$\begin{aligned} b_i &= e^{-j\theta}a_i' \\ b_i' &= e^{-j\theta}a_i \, . \end{aligned} \tag{6}$$

Substituting (4) and the similar equations for C' into (6), we obtain the following set of equations:

$$\begin{aligned} b_1 + j\omega_1\dot{\kappa}b_2^* + j\omega_1\dot{\kappa}'e^{-j\theta_1}b_2'^* &= e^{-j\theta_1}a_1' - j\omega_1\dot{\kappa}a_2^* - j\omega_1\dot{\kappa}'e^{-j\theta_1}a_2'^* \\ b_1' + j\omega_1\dot{\kappa}e^{-j\theta_1}b_2^* + j\omega_1\dot{\kappa}'b_2'^* &= e^{-j\theta_1}a_1 - j\omega_1\dot{\kappa}e^{-j\theta_1}a_2^* - j\omega_1\dot{\kappa}'a_2'^* \\ -j\omega_2\dot{\kappa}^*b_1 - j\omega_2\dot{\kappa}'^*e^{j\theta_2}b_1' + b_2^* &= j\omega_2\dot{\kappa}^*a_1 + j\omega_2\dot{\kappa}'^*e^{j\theta_2}a_1' + e^{j\theta_2}a_2'^* \\ -j\omega_2\dot{\kappa}^*e^{j\theta_2}b_1 - j\omega_2\dot{\kappa}'^*b_1' + b_2'^* &= j\omega_2\dot{\kappa}^*e^{j\theta_2}a_1 + j\omega_2\dot{\kappa}'^*a_1 + e^{j\theta_2}a_2^* \end{aligned} \tag{7}$$

Assuming the same magnitude for the complex quantities $\dot{\kappa}$ and $\dot{\kappa}'$, and solving (7) with respect to b_1, b_1', b_2^*, and $b_2'^*$, we obtain an exact form of the scattering matrix for the circuit shown in Fig. 1. The matrix is as follows:

$$\begin{bmatrix} b_1 \\ b_1' \\ b_2^* \\ b_2'^* \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_1' \\ a_2^* \\ a_2'^* \end{bmatrix} \tag{8}$$

$$
\begin{aligned}
S_{11}\cdot\Delta = S_{22}\cdot\Delta &= 2\omega_1\omega_2\kappa^2 \exp(-j\theta_1)\,[\cos\theta_1 + \exp(j\theta_2)\cos\theta_p] \\
&\qquad - 4\omega_1^2\omega_2^2\kappa^4 \exp[j(\theta_2-\theta_1)]\sin\theta_1\sin\theta_2 \\
S_{33}\cdot\Delta = S_{44}\cdot\Delta &= 2\omega_1\omega_2\kappa^2 \exp(j\theta_2)\,[\cos\theta_2 + \exp(-j\theta_1)\cos\theta_p] \\
&\qquad - 4\omega_1^2\omega_2^2\kappa^4 \exp[j(\theta_2-\theta_1)]\sin\theta_1\sin\theta_2 \\
S_{12}\cdot\Delta &= \exp(-j\theta_1)\,\{1 + j2\omega_1\omega_2\kappa^2 \\
&\qquad \cdot\exp[j(\theta_2+\theta_p)]\sin\theta_1\} \\
S_{21}\cdot\Delta &= \exp(-j\theta_1)\,\{1 + j2\omega_1\omega_2\kappa^2 \\
&\qquad \cdot\exp[j(\theta_2-\theta_p)]\sin\theta_1\} \\
S_{34}\cdot\Delta &= \exp(j\theta_2)\,\{1 - j2\omega_1\omega_2\kappa^2 \\
&\qquad \cdot\exp[-j(\theta_1+\theta_p)]\sin\theta_2\} \\
S_{43}\cdot\Delta &= \exp(j\theta_2)\,\{1 - j2\omega_1\omega_2\kappa^2 \\
&\qquad \cdot\exp[-j(\theta_1-\theta_p)]\sin\theta_2\} \\
S_{13}\cdot\Delta = -(\omega_1/\omega_2)S_{42}\cdot\Delta &= -j\omega_1\kappa\exp(j\theta_p/2) \\
&\qquad \cdot\{1 + \exp[j(\theta_2-\theta_1-\theta_p)] - 4\omega_1\omega_2\kappa^2 \\
&\qquad \cdot\exp[j(\theta_2-\theta_1)]\sin\theta_1\sin\theta_2\} \\
S_{24}\cdot\Delta = -(\omega_1/\omega_2)S_{31}\cdot\Delta &= -j\omega_1\kappa\exp(-j\theta_p/2) \\
&\qquad \cdot\{1 + \exp[j(\theta_2-\theta_1+\theta_p)] - 4\omega_1\omega_2\kappa^2 \\
&\qquad \cdot\exp[j(\theta_2-\theta_1)]\sin\theta_1\sin\theta_2\} \\
S_{14}\cdot\Delta = -(\omega_1/\omega_2)S_{32}\cdot\Delta &= -j\omega_1\kappa\exp(j\theta_p/2)\,\{\exp(j\theta_2) \\
&\qquad + \exp[-j(\theta_1+\theta_p)]\} \\
S_{23}\cdot\Delta = -(\omega_1/\omega_2)S_{41}\cdot\Delta &= -j\omega_1\kappa\exp(-j\theta_p/2)\,\{\exp(j\theta_2) \\
&\qquad + \exp[-j(\theta_1-\theta_p)]\}
\end{aligned}
\tag{9}
$$

and

$$
\begin{aligned}
\Delta = 1 - 2\omega_1\omega_2\kappa^2(1 + \exp[j(\theta_2-\theta_1)]\cos\theta_p) \\
+ 4\omega_1^2\omega_2^2\kappa^4\exp[j(\theta_2-\theta_1)]\sin\theta_1\sin\theta_2
\end{aligned}
\tag{10}
$$

where κ and θ_p are real numbers, and represent the magnitudes of $\dot{\kappa}$ and $\dot{\kappa}'$, and a difference of pump phases at two diodes respectively as shown in the following equations:

$$\begin{aligned} \dot{\kappa} &= \kappa e^{j\theta_p/2} \\ \dot{\kappa}' &= \kappa e^{-j\theta_p/2}. \end{aligned} \tag{11}$$

Assuming the following phase relationships

$$\begin{aligned} \theta_1 &= (2m + 1)(\pi/2) \\ \theta_2 &= n\pi \\ \theta_p &= (2m + 2n + 1)(\pi/2), \end{aligned} \tag{12}$$

we obtain the following special relations for the waves:

$$\begin{aligned} b_1 &= (-)^{m+1} j a_1' \\ b_1' &= (-)^{m+1} j \frac{1 + 2\omega_1\omega_2\kappa^2}{1 - 2\omega_1\omega_2\kappa^2} a_1 + (-)^{n+1} j \frac{2\omega_1\kappa e^{-j\theta_p/2}}{1 - 2\omega_1\omega_2\kappa^2} a_2^* \\ &\quad - j \frac{2\omega_1\kappa e^{-j\theta_p/2}}{1 - 2\omega_1\omega_2\kappa^2} a_2'^* \\ b_2^* &= j \frac{2\omega_2\kappa e^{-j\theta_p/2}}{1 - 2\omega_1\omega_2\kappa^2} a_1 + \frac{2\omega_1\omega_2\kappa^2}{1 - 2\omega_1\omega_2\kappa^2} a_2^* + (-)^n \frac{1}{1 - 2\omega_1\omega_2\kappa^2} a_2'^* \\ b_2'^* &= (-)^n j \frac{2\omega_2\kappa e^{-j\theta_p/2}}{1 - 2\omega_1\omega_2\kappa^2} a_1 + (-)^n \frac{1}{1 - 2\omega_1\omega_2\kappa^2} a_2^* \\ &\quad + \frac{2\omega_1\omega_2\kappa^2}{1 - 2\omega_1\omega_2\kappa^2} a_2'^*. \end{aligned} \tag{13}$$

In (13) it is found that signal waves are completely matched in each direction, and that the signal incident wave a_1' propagates from the right to the left with no gain nor loss and has no interaction with the other three waves. When both idler ports are properly terminated, the transducer gain of this amplifier for the incident wave a_1 — that is, $|S_{21}|^2$ — equals the square of the ratio $(1 + 2\omega_1\omega_2\kappa^2)/(1 - 2\omega_1\omega_2\kappa^2)$. If the surge impedance of the circulator of the reflection-type amplifier at the signal frequency equals the image impedance Z_1 and the idler load impedance at the idler frequency equals the image impedance Z_2, the gain formula is identical to that of a single-diode reflection-type amplifier, $|(1 + 4\omega_1\omega_2\kappa^2)/(1 - 4\omega_1\omega_2\kappa^2)|^2$, except for a factor of $\frac{1}{2}$ in the pumping term $2\omega_1\omega_2\kappa^2$. Thus, a perfectly matched and unilateral parametric amplifier can be achieved by means of the phase synchronization shown in (12).

If all frequency characteristics of θ_1, θ_2, Z_1 and Z_2 are known, it is possible to calculate the frequency characteristic of the amplifier from

(9) and (10). Since this method is rather tedious, a simplified method will be developed in the following sections.

III. EFFECTS OF REFLECTIONS, BANDWIDTH

For the sake of analytical simplicity we first assume that phase constants are frequency independent and satisfy the condition given by (12), but we introduce the frequency-dependent reflection coefficients of the external loads. These reflection coefficients can be modified to include the effects of frequency-dependent phase constants.

The incident and reflected idler waves are related by the idler reflection coefficients Γ_2 and Γ_2' as follows:

$$\begin{aligned} a_2^* &= \Gamma_2^* b_2^* \\ a_2'^* &= \Gamma_2'^* b_2'^*. \end{aligned} \tag{14}$$

Substituting these relations into (13) and eliminating a_2^*, $a_2'^*$, b_2^* and $b_2'^*$ from the equation, we obtain the following equations:

$$\begin{aligned} b_1 &= S_{12}' a_1' \\ b_1' &= S_{21}' a_1 \end{aligned} \tag{15}$$

where

$$S_{12}' = (-)^{m+1} j$$

$$S_{21}' = (-)^{m+1} j \left[\frac{1 + 2\omega_1\omega_2\kappa^2}{1 - 2\omega_1\omega_2\kappa^2} + \frac{8\omega_1\omega_2\kappa^2 \left\{ 1 + \frac{1}{2}\left(\frac{1}{\Gamma_2^*} + \frac{1}{\Gamma_2'^*} \right) \right\}}{\left(2\omega_1\omega_2\kappa^2 - \frac{1 - 2\omega_1\omega_2\kappa^2}{\Gamma_2^*} \right)\left(2\omega_1\omega_2\kappa^2 - \frac{1 - 2\omega_1\omega_2\kappa^2}{\Gamma_2'^*} \right) - 1} \right]. \tag{16}$$

Equation (15) shows that any reflection in the idler circuits does not deteriorate the unilateral characteristic of the amplifier. (A change in idler phase, θ_2, however, causes deterioration in the unilateral characteristics of the amplifier: see Appendix.)

Next let us consider a special case in which the amplifier has symmetrical idler terminations, $\Gamma_2' = \Gamma_2$. Substituting this condition into the second equation of (16), we simplify the equation as follows:

$$S_{21}' = (-)^{m+1} j \frac{\dfrac{1 + 2\omega_1\omega_2\kappa^2}{1 - 2\omega_1\omega_2\kappa^2} - \Gamma_2^*}{1 - \Gamma_2^* \dfrac{1 + 2\omega_1\omega_2\kappa^2}{1 - 2\omega_1\omega_2\kappa^2}}. \tag{17}$$

The reflection coefficient Γ_2 is expressed in terms of the image impedance Z_2 and the idler load admittance Y_{T2} as follows:

$$\Gamma_2 = \frac{1 - Y_{T2}Z_2}{1 + Y_{T2}Z_2}, \tag{18}$$

where

$$Y_{T2} = G_{T2} + jB_{T2}. \tag{19}$$

This admittance may include parasitic elements of the diode. Substituting (5), (11) and (18) into (17), we obtain the following relation:

$$S_{21}' = (-)^{m+1} j \frac{Y_{T2}^* + \frac{1}{8}\omega_1\omega_2 \mid c \mid^2 Z_1}{Y_{T2}^* - \frac{1}{8}\omega_1\omega_2 \mid c \mid^2 Z_1}. \tag{17$'$}$$

This equation shows that the idler image impedance Z_2 does not affect the gain. Only the idler load admittance, Y_{T2}, is of primary importance for transmission gain of the amplifier. Therefore, we can assume hereafter that the idler image impedance equals $1/G_{T2}$ in order to avoid the reflection at the center frequency of amplification. The square root of the power gain at the center frequency, $\sqrt{PG}$, is given by the following relation:

$$\sqrt{PG} = g \equiv \frac{G_{T2} + \frac{1}{8}\omega_1\omega_2 \mid c \mid^2 Z_1}{G_{T2} - \frac{1}{8}\omega_1\omega_2 \mid c \mid^2 Z_1} \tag{20}$$

and a half-gain bandwidth, $(\Delta f)_{3\text{db}}$, is approximately determined from the frequencies where the susceptance component of the idler load equals the denominator of (20), i.e.,

$$\mid B_{T2} \mid = G_{T2} - \tfrac{1}{8}\omega_1\omega_2 \mid c \mid^2 Z_1 \tag{21}$$

where the right-hand side of (21) is a slowly varying function of frequency in comparison with B_{T2}.

Now let us consider effects of small reflections in the signal circuit. We assume that both the signal source and load admittances are represented by $(1/R_{01}) + jB_1$ as shown in Fig. 2, and waves a_{01}, b_{01}, a_{01}' and b_{01}' are normalized to R_{01} instead of Z_1. Expressing voltages and currents in terms of waves as was done in (2), we obtain the following equations from the equations of continuity of currents and voltages:

$$\begin{aligned} a_1 &= (k_1 - j\mu_1)a_{01} + (l_1 - j\mu_j)b_{01} \\ b_1 &= (l_1 + j\mu_1)a_{01} + (k_1 + j\mu_1)b_{01} \end{aligned} \tag{22}$$

where

$$\begin{aligned} k_1 &= \frac{1}{2}\left(\sqrt{\frac{R_{01}}{Z_1}} + \sqrt{\frac{Z_1}{R_{01}}}\right) \\ l_1 &= \frac{1}{2}\left(\sqrt{\frac{R_{01}}{Z_1}} - \sqrt{\frac{Z_1}{R_{01}}}\right) \\ \mu &= \frac{B_1}{2}\sqrt{R_{01}Z_1}\,. \end{aligned} \tag{23}$$

Substituting (23) into (15), we obtain the following relations:

$$\begin{aligned} b_{01} &= S_{11}^{(0)}a_{01} + S_{12}^{(0)}a_{01}' \\ b_{01}' &= S_{21}^{(0)}a_{01} + S_{22}^{(0)}a_{01}' \end{aligned} \tag{24}$$

$$\begin{aligned} S_{11}^{(0)} = S_{22}^{(0)} &= \frac{k_1 - j\mu_1}{k_1 + j\mu_1} \cdot \frac{-\Gamma_1^* + S_{12}'S_{21}'\Gamma_1}{1 - S_{12}'S_{21}'\Gamma_1^2} \\ S_{12}^{(0)} &= \frac{k_1 - j\mu_1}{k_1 + j\mu_1} \cdot \frac{S_{12}'(1 - |\Gamma_1|^2)}{1 - S_{12}'S_{21}'\Gamma_1^2} \\ S_{21}^{(0)} &= \frac{k_1 - j\mu_1}{k_1 + j\mu_1} \cdot \frac{S_{21}'(1 - |\Gamma_1|^2)}{1 - S_{12}'S_{21}'\Gamma_1^2} \end{aligned} \tag{25}$$

where Γ_1 is a reflection coefficient of the signal source and load normalized to Z_1, and is defined as follows

$$\Gamma_1 = \frac{l_1 - j\mu_1}{k_1 + j\mu_1} = \frac{\dfrac{1}{Z_1} - \left(\dfrac{1}{R_{01}} + jB_1\right)}{\dfrac{1}{Z_1} + \left(\dfrac{1}{R_{01}} + jB_1\right)}\,. \tag{26}$$

Equation (25) reveals that any reflections in signal circuits cause an

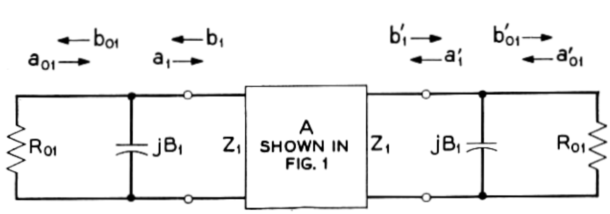

Fig. 2 — Forward and reverse waves in the signal circuits.

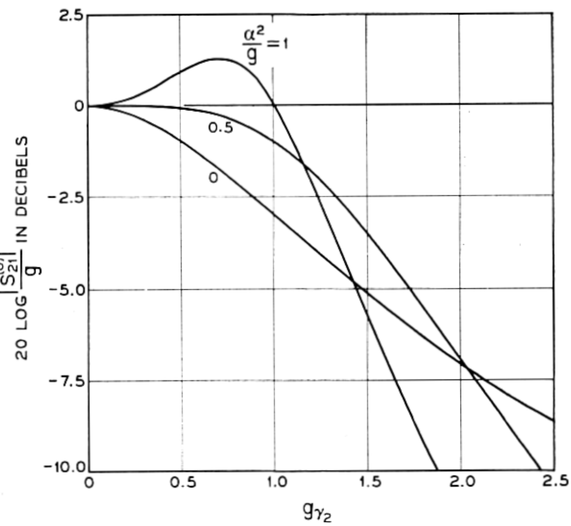

Fig. 3 — Normalized forward gain for various values of α^2/g vs normalized idler reflection coefficient $g\gamma_2$.

internal feedback and deteriorate the unilateral characteristic. To simplify the analysis, we assume that Γ_1 is proportional to Γ_2 and that both of them are small imaginary numbers, i.e., $\Gamma_2 = j\gamma_2$, $\Gamma_1 = j\alpha\gamma_2$, $|\Gamma_1|^2 \ll 1$, and $|\Gamma_2| \ll 1$. Substituting the first equation of (16) and (17) into (25), we obtain the following scattering matrix elements, which characterize the frequency dependence of the amplifier if $g \gg 1$:

reflection:

$$|S_{11}^{(0)}| = |S_{22}^{(0)}| \approx \frac{|\alpha|\,|g\gamma_2|}{\sqrt{\left\{1 - \frac{\alpha^2}{g}(g\gamma_2)^2\right\}^2 + (g\gamma_2)^2}}$$

reverse gain:

$$|S_{12}^{(0)}| \approx \frac{\sqrt{1 + (g\gamma_2)^2}}{\sqrt{\left\{1 - \frac{\alpha^2}{g}(g\gamma_2)^2\right\}^2 + (g\gamma_2)^2}} \tag{27}$$

forward gain:

$$|S_{21}^{(0)}| \approx \frac{g}{\sqrt{\left\{1 - \frac{\alpha^2}{g}(g\gamma_2)^2\right\}^2 + (g\gamma_2)^2}}.$$

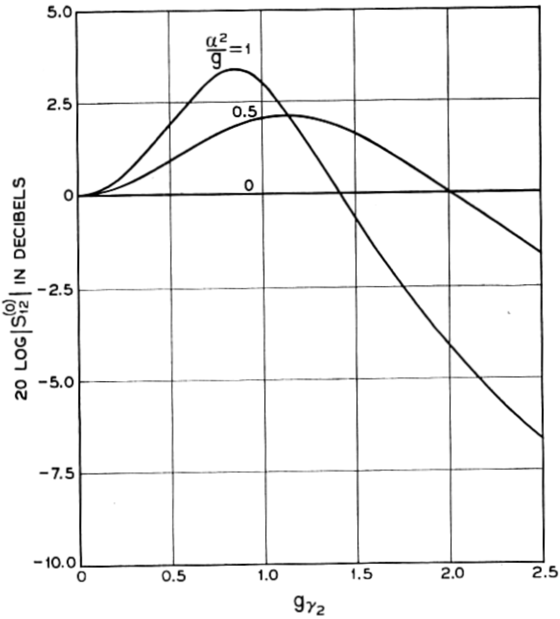

Fig. 4 — Reverse gain for various values of α^2/g vs normalized idler reflection coefficient $g\gamma_2$.

The characteristics of these elements with various values of the parameter α^2/g, as a function of $g\gamma_2$, are shown in Figs. 3, 4, and 5.

It is shown that the amplifier has a maximally flat gain characteristic when $\alpha^2/g = \frac{1}{2}$, as follows:

$$| S_{21}^{(0)} | \approx \frac{g}{\sqrt{1 + \frac{(g\gamma_2)^4}{4}}}. \tag{28}$$

A half-gain bandwidth is determined from the frequencies where the following relation is held:

$$| g\gamma_2 | \approx \sqrt{2}.$$

Similarly, for the amplifier with a matched signal source and load, the bandwidth is derived from the following equation:

$$| g\gamma_2 | \approx 1.$$

Therefore, by introducing a proper mismatch in the signal source and load and assuming that γ_2 is proportional to a frequency change, we can obtain about $\sqrt{2}$ times larger bandwidth than that of the matched amplifier. But this enlarged bandwidth is obtainable only at the expense of deterioration of unilateral characteristics, as shown in Fig. 4.

IV. EFFECTS OF LOSSES; NOISE FIGURE

In the previous discussions, losses in the diode have been considered a part of the external circuits. However, in order to study the noise performance of the amplifier the diode losses have to be separated from the external circuits.

In Fig. 6, G_d and G_d' represent the equivalent loss conductance of the diodes, and they are accompanied by the noise current generators i_{nd} and i_{nd}'. G_L and G_s represent the load and signal source conductance, respectively.

In order to eliminate feedback effect due to mismatches at the signal ports, we assume that the signal output port is perfectly matched to its image impedance Z_1. Thus, we can eliminate any feedback effect even if there is mismatch at the signal input port. The matching condition at the signal output port is expressed as follows:

$$(G_L + G_d')Z_1 = 1. \tag{29}$$

In order to feed the maximum power into the black box A shown in Fig. 6, the signal source impedance has to be matched to the input impedance of the amplifier, which includes the diode loss conductance G_d. This matching condition is expressed as follows:

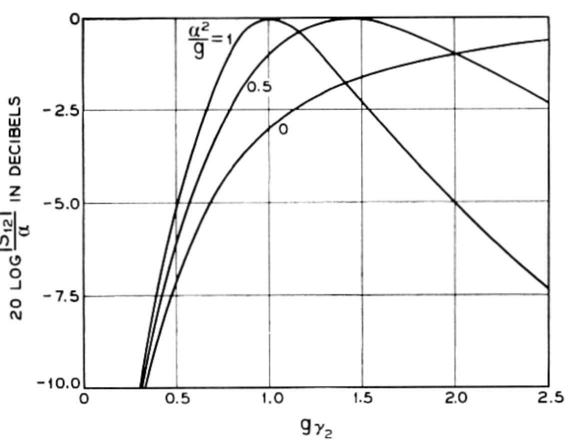

Fig. 5 — Normalized input reflection coefficient for various values of α^2/g vs normalized idler reflection coefficient $g\gamma_2$.

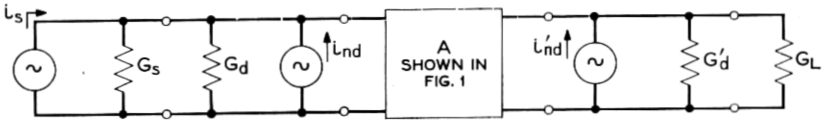

Fig. 6 — Noise sources of the unilateral parametric amplifier.

$$G_s Z_1 = 1 + G_d Z_1 . \tag{30}$$

Note that the condition in (30) does not necessarily mean that the signal source impedance is matched to Z_1.

The maximum power available into the black box A of Fig. 6 is given by

$$| a_1 |^2 = \frac{P_a}{G_s Z_1} = \frac{P_a}{1 + G_d Z_1}, \tag{31}$$

where P_a is the maximum available power from signal source. The power delivered to the load, denoted by P_0, is given by the following equation:

$$P_0 = G_L Z_1 \, | b_1' |^2. \tag{32}$$

Therefore, from the second equation of (13), and assuming no input signal at the idler frequency, we obtain the following gain expression:

$$PG = \frac{P_0}{P_a} = \frac{G_L Z_1}{1 + G_d Z_1} g^2, \tag{33}$$

where g is given by (20). [In (20) we assumed that the idler loads are not necessarily matched. However, as stated there, we can assume the matched idler loads without loss of generality.] This gain expression is very similar to that of a single-diode reflection-type amplifier, as mentioned in Section II.

Assuming noise sources are thermal and the temperatures of the amplifier, the signal source and the load are the same, denoted by $T°$ Kelvin, noise currents and noise wave are expressed in the following relation:

$$\frac{\overline{i_{nd}^2}}{4G_d} = \frac{\overline{i_{nd}'^2}}{4G_d'} = | a_1' |^2 = | a_2 |^2 = | a_2' |^2 = kT \, \Delta f \tag{34}$$

where $k = 1.38 \times 10^{-23}$ joule/°Kelvin is Boltzmann's constant and Δf is the infinitesimal bandwidth concerned.

Though we have a mismatched signal source with respect to Z_1, the difference between the noise power from a matched signal source and that from the mismatched signal source is exactly the same as the noise power reflected at the terminal of this mismatched signal source. This reflected noise power is originally from the perfectly matched signal output network through the amplifier without gain nor loss in the reverse direction. Therefore, the resultant incoming noise wave into the amplifier is the same as mentioned in (34)

$$| a_1 |^2 = kT\Delta f. \tag{34$'$}$$

Substituting (34) and (34′) into the second equation of (13), we obtain

$$| b_1' |^2 = g^2 \left\{ 1 + \frac{\omega_1}{\omega_2} \left(1 - \frac{1}{g^2} \right) \right\} kT\,\Delta f.$$

Therefore, the noise output power due to $| b_1' |^2$ is given by

$$P_{01} = G_L Z_1 g^2 \left\{ 1 + \frac{\omega_1}{\omega_2} \left(1 - \frac{1}{g^2} \right) \right\} kT\,\Delta f.$$

The directly radiated noise power into the load from G_d' is also the other part of the noise output power. We obtain the following equation for this noise output power:

$$P_{02} = G_L Z_1 G_d' Z_1 kT\,\Delta f.$$

Therefore, we obtain the following expression for the noise figure of the amplifier:

$$\begin{aligned} F &= \frac{P_{01} + P_{02}}{(PG)kT\,\Delta f} \\ &= (1 + G_d Z_1) \left\{ 1 + \frac{G_d' Z_1}{g^2} + \frac{\omega_1}{\omega_2} \left(1 - \frac{1}{g^2} \right) \right\}. \end{aligned} \tag{35}$$

It is shown in (35) that $G_d Z_1$ and ω_1/ω_2 should be smaller in order to obtain a better noise figure. And the expression for the noise figure is very similar to that of a single-diode amplifier. Note that (35) was obtained on the assumption that the temperature of the resistive load or isolator after the amplifier is $T°$ Kelvin, and the noise figure is also normalized to $T°$ Kelvin. If the load temperature is 0° Kelvin, we have to subtract $G_d^2 Z_1^2 G_L Z_1/(1 + G_d Z_1)$ from the right-hand side of (35), which comes from the noise power from the load at $T°$ Kelvin.

If the load or isolator temperature, T_I, is higher than T, the noise power generated in the isolator travels toward the input side of the amplifier, reflects back at the input diode, and is amplified. Therefore, the higher the isolator temperature is, the smaller reflection must be kept at the input diode. If the isolator temperature is much higher than the amplifier temperature, a matched condition to Z_1 at the input diode, i.e.

$$G_sZ_1 + G_dZ_1 \approx 1 \tag{30'}$$

must be held in order to keep the effect of the hot isolator small. In this case, the power gain and the noise figure normalized to $T°$ Kelvin are approximately obtained in the following equations:

$$PG \approx G_{L1}Z_1G_sZ_1g^2 \tag{33'}$$

$$\begin{aligned} F \approx \frac{1}{1 - G_dZ_1} \bigg\{ 1 + \frac{G_d'Z_1}{g^2} + \frac{\omega_1}{\omega_2}\left(1 - \frac{1}{g^2}\right) \\ + \frac{T_I}{T} \cdot \frac{G_LZ_1}{4} (G_sZ_1 + G_dZ_1 - 1)^2 \bigg\} \quad \text{if} \quad \frac{T_I}{T} \gg 1. \end{aligned} \tag{35'}$$

The last term in parentheses represents the deterioration of noise figure due to input mismatching.

V. OPTIMUM NOISE FIGURE OF THE AMPLIFIER WITH LOSSY DIODES

To simplify the analysis, we assume that a variable-capacitance diode is represented by an equivalent circuit shown in Fig. 7(a). The junction capacitance $\tilde{C}_j$ is the only variable element and is given by the following equation:

$$\tilde{C}_j = C_j + c \cos \omega_p t \tag{36}$$

where ω_p denotes the pumping angular frequency and equals $\omega_1 + \omega_2$.

The characteristic quantities of the diode, ω_{cr}, ω_0 and Q_0 are defined as follows: the critical angular frequency

$$\omega_{cr} = \frac{1}{2} \frac{1}{C_jR_s} \frac{|c|}{C_j} \approx \omega\tilde{Q}, \tag{37}$$

where $\tilde{Q}$ is a dynamic quality factor of the diode; the self-resonant angular frequency

$$\omega_0 = \frac{1}{\sqrt{L_sC_j}}; \tag{38}$$

and the diode Q at the resonant frequency

$$Q_0 = \frac{\omega_0 L_s}{R_s}. \tag{39}$$

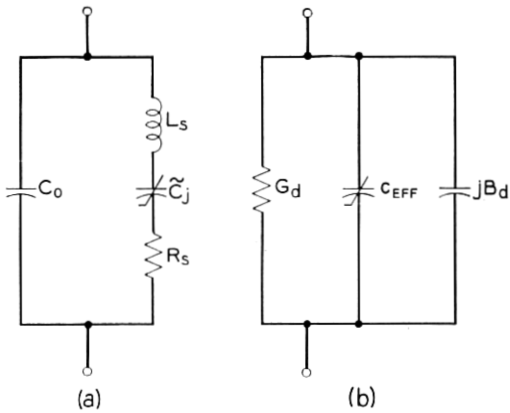

Fig. 7 — Equivalent circuits of a variable capacitance diode.

Transforming the circuit of Fig. 7(a) into the form shown in Fig. 7(b), where c_{eff} is an equivalent sinusoidally varying shunt capacitance, the new circuit parameters at frequencies away from the self-resonant frequency f_0 are given by the following relations:

$$B_d \approx \frac{1}{R_s Q_0}\left\{\frac{C_0}{C_j}\frac{\omega}{\omega_0} - \frac{1}{\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}}\right\}$$

$$G_d \approx \frac{1}{R_s Q_0^2}\frac{1}{\left(\dfrac{\omega}{\omega_0} - \dfrac{\omega_0}{\omega}\right)^2} \tag{40}$$

$$c_{\text{eff}} \approx \frac{\dfrac{\omega_0^2}{\omega_1\omega_2}}{\left(\dfrac{\omega_1}{\omega_0} - \dfrac{\omega_0}{\omega_1}\right)\left(\dfrac{\omega_2}{\omega_0} - \dfrac{\omega_0}{\omega_2}\right)} c.$$

Figs. 8 and 9 show the frequency characteristics of B_d and G_d respectively for various values of a parameter C_0/C_j.

From (40), we obtain the following relation:

$$\frac{\omega_1\omega_2 \mid c_{\text{eff}} \mid^2}{4} = \frac{\omega_{cr}^2}{\omega_1\omega_2} G_{d1}G_{d2}. \tag{41}$$

For a high-gain amplifier, the following condition should be satisfied from (20):

$$G_{T2} = (1 + L_2)G_{d2} \approx \tfrac{1}{8}\omega_1\omega_2 \mid c_{\text{eff}} \mid^2 Z_1 \tag{42}$$

where L_2 is an external idler loading factor of the amplifier.* Substituting (41) into (42), the high-gain condition is expressed as follows:

$$\frac{1}{2}\frac{\omega_{cr}^2}{\omega_1\omega_2}\frac{G_{d1}Z_1}{1+L_2} \approx 1. \tag{42'}$$

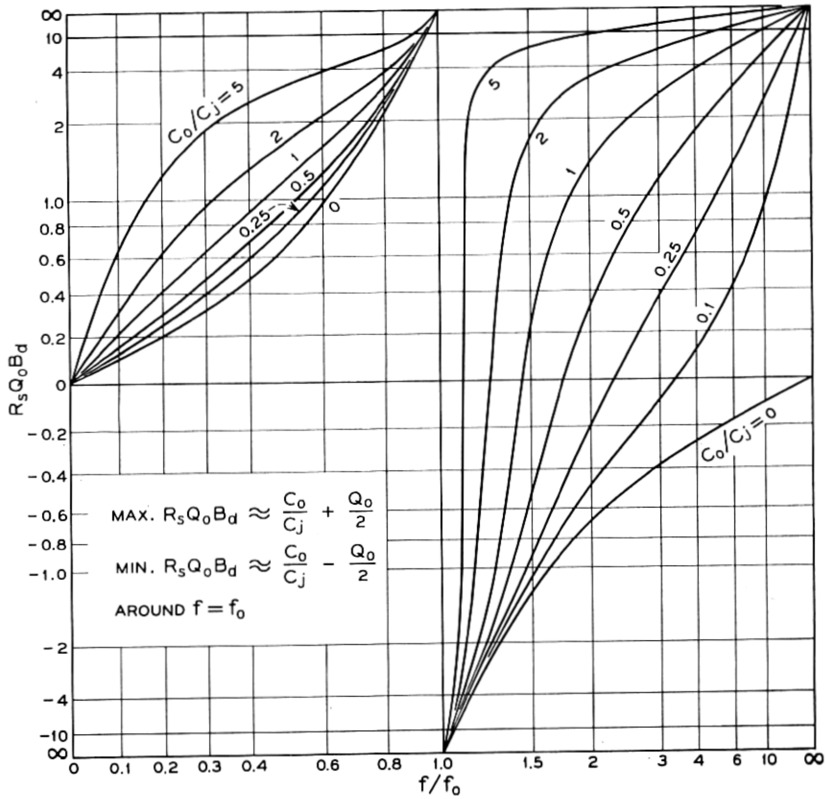

Fig. 8 — Frequency characteristics of an equivalent shunt susceptance of a diode.

From (35), a noise figure expression for a high-gain amplifier is approximately given by the following equation:

$$F \approx (1 + G_{d1}Z_1)\left(1 + \frac{\omega_1}{\omega_2}\right). \tag{35''}$$

Eliminating $G_{d1}Z_1$ from (35″) by using (42′), and differentiating (35″)

* Originally introduced by M. Uenohara.

with respect to ω_2/ω_1, we obtain the optimum ratio of idler-to-signal frequency for the optimum noise figure, and the optimum noise figure for a given signal frequency f_1, a diode critical frequency f_{cr} and an idler loading factor L_2

$$F_{\text{opt}} = \left\{1 + \frac{\omega_1}{\omega_{cr}} \sqrt{2(1 + L_2)}\right\}^2 \approx \left\{1 + \frac{1}{\tilde{Q}_1} \sqrt{2(1 + L_2)}\right\}^2 \tag{43}$$

$$\left(\frac{\omega_2}{\omega_1}\right)_{\text{opt}} = \frac{\dfrac{\omega_{cr}}{\omega_1}}{\sqrt{2(1 + L_2)}} \approx \frac{\tilde{Q}_1}{\sqrt{2(1 + L_2)}}. \tag{44}$$

Fig. 10 shows the numerical values of $(F)_{\text{opt}}$ and $(\omega_2/\omega_1)_{\text{opt}}$.

It is shown in (43) that the optimum noise figure is fairly good but slightly higher* than that of a single-diode parametric amplifier. The reason for the higher noise figure is attributed to the fact that, to obtain the same amount of gain with a given ratio of reactance swing, the quarter-wave coupled amplifier needs a lower idler frequency than that of a single-diode reflection-type amplifier. However, for reflection-type amplifiers it may not be possible to use the optimum idler frequency in order to obtain a wide bandwidth, and a small difference in thermal noise

* For a single-diode parametric amplifier, the optimum noise figure and the optimum ratio of idler-to-signal frequency are obtained in terms of a critical frequency and idler loading factor, as follows:

$$F_{\text{opt}} = \left\{\sqrt{1 + \left(\frac{\omega_1}{\omega_{cr}}\right)^2 (1 + L_2)} + \frac{\omega_1}{\omega_{cr}} \sqrt{1 + L_2}\right\}$$

$$\approx \left(\sqrt{1 + \frac{1 + L_2}{\tilde{Q}_1^2}} + \frac{1}{\tilde{Q}_1} \sqrt{1 + L_2}\right)^2$$

$$\left(\frac{\omega_2}{\omega_1}\right)_{\text{opt}} = \sqrt{\left(\frac{\omega_{cr}}{\omega_1}\right)^2 \frac{1}{1 + L_2} + 1} - 1$$

$$\approx \frac{\tilde{Q}_1}{\sqrt{1 + L_2}} \cdot \frac{1}{\sqrt{1 + \dfrac{1 + L_2}{\tilde{Q}_1^2}} + \dfrac{\sqrt{1 + L_2}}{\tilde{Q}_1}}$$

If $\tilde{Q}_1 \gg 1$, these equations yield

$$F_{\text{opt}} \approx \left(1 + \frac{1}{\tilde{Q}_1} \sqrt{1 + L_2}\right)^2$$

$$\left(\frac{\omega_2}{\omega_1}\right)_{\text{opt}} \approx \frac{\tilde{Q}_1}{\sqrt{1 + L_2}}.$$

These are the same shown in (43) and (44) if Q_1 is replaced by $Q_1/\sqrt{2}$.

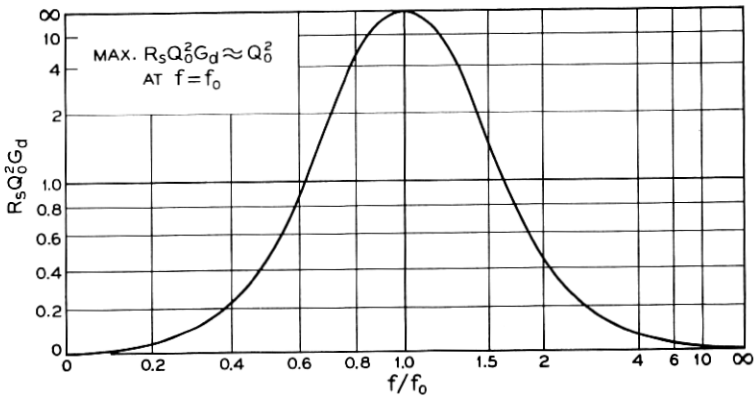

Fig. 9 — Frequency characteristic of an equivalent shunt conductance of a diode.

becomes negligible in comparison with the noise contribution from antenna circuit losses when the amplifier is refrigerated at a very low temperature; therefore, the disadvantage of slightly higher noise performance of the unilateral amplifier seems to be not always a serious problem in practical applications.

VI. DESIGN CONSIDERATION OF THE AMPLIFIER

In previous sections, we discussed the amplifier characteristics in general terms. Now we have to consider what kind of diode is necessary and what image impedance we have to choose in order to obtain a prescribed gain at a prescribed impedance of the system. And also we have to consider what characteristics the amplifier should have without pumping in order to have gain with pumping.

From the equivalent circuit of the diode shown in Fig. 7, the equivalent change in shunt susceptance ΔB due to a static change in the junction capacitance ΔC_j is given as follows:

$$\Delta B \approx \frac{\partial B}{\partial C_j} \Delta C_j = Q_0 G_d \frac{\omega_0}{\omega} \frac{\Delta C_j}{C_j} . \tag{45}$$

If the static change in the junction capacitance ΔC_j equals the amplitude of sinusoidal variation, $|c|$, (45) yields the resultant change in the equivalent susceptance as follows:

$$\Delta B = 2(\omega_{cr}/\omega) G_d . \tag{46}$$

Multiplying (46) by Z_1 and substituting it into (42′), we obtain the following equation:

$$Z_1 \Delta B_1 \approx 4 \frac{\omega_2}{\omega_{cr}} (1 + L_2). \tag{47}$$

It is seen in (47) that in order to obtain a high-gain amplifier, when the diode bias is varied over the entire range for making a passive test of the amplifier, the susceptance change normalized by the image impedance Z_1 at the signal frequency should be twice as much as the amount given in the right-hand side of (47). Also, the amplifier network should be a piece of matched transmission line if it is not pumped, and the two diodes in the amplifier should have the opposite direction of change in susceptance at the signal frequency when the bias voltage is changed in the same direction. If we have external idler ports for loading the amplifier, the two diodes should have the same direction of change in susceptance at the idler frequency. In Fig. 11 are shown required amounts of susceptance, which are twice the amount shown in (47), to fulfill the condition of unilateral amplification, as functions of idler to critical frequency ratio for various idler loading factors.

The bandwidth problem of this amplifier is not so straightforward as the noise figure problem. As mentioned in (21) of Section III, the ap-

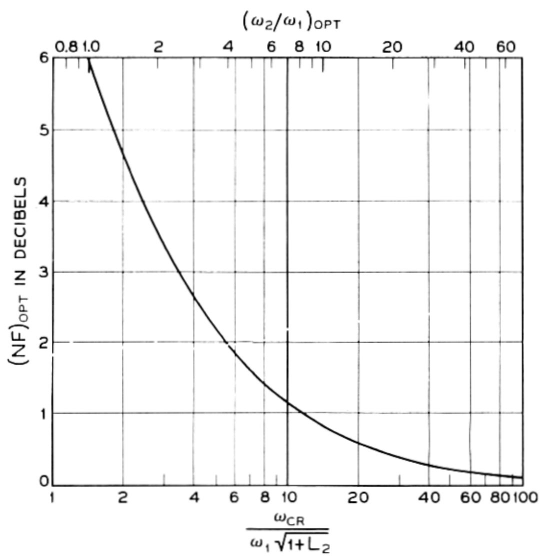

Fig. 10 — Optimum noise figure of the unilateral parametric amplifier.

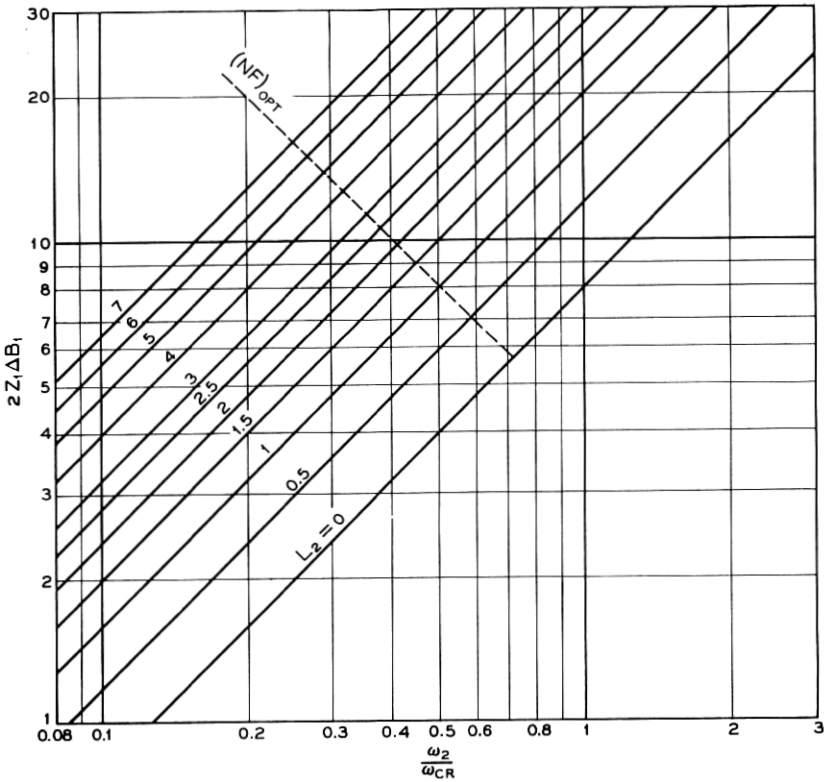

Fig. 11 — Required susceptance change at the signal frequency for high-gain amplification.

proximate bandwidth is determined by the idler circuit that can be improved by a multiple-tuning technique. However, in order to utilize a broadband idler circuit, the signal network should have a broad enough bandwidth to prevent any oscillations. The reason for this is as follows: in order to obtain a unilateral amplification a larger capacitance swing is needed than for a reflection-type amplification. Thus, wherever impedance conditions outside the signal frequency band become favorable for ordinary reflection-type amplification, the amplifier tends to change its mode of operation and breaks into oscillation. For this reason, not only must the image impedance of the network at the signal frequency be flat, but the phase constant also must be less frequency-sensitive in the frequencies where the amplifier has a gain. [For a closer investigation of the stability problem, we have to study (9).]

If both signal and idler frequencies are far from the self-resonant frequency of the diode, an unloaded Q of the diode at a given angular frequency ω, which is approximately given by

$$Q_{ud} \approx \frac{\omega}{2G_d} \frac{\partial B_d}{\partial \omega}, \tag{48}$$

is an important factor in determining the bandwidth of the amplifier. This factor is expressed by the following equation:

$$Q_{ud} \approx \frac{Q_0}{2} \left\{ \frac{\omega}{\omega_0} + \frac{\omega_0}{\omega} + \frac{C_0}{C_j} \frac{\omega}{\omega_0} \left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega} \right)^2 \right\}. \tag{48$'$}$$

Fig. 12 shows the frequency dependence of Q_{ud} for various values of C_0/C_j. In Fig. 12, it is shown that the idler frequency should be close to the self-resonant frequency in order to obtain a large bandwidth. If the idler circuit is a single-tuned circuit with a loaded Q of Q_{L2}, the gain-bandwidth product is roughly given by

$$\left(\frac{\Delta f}{f_1} \right)_{3\text{db}} \sqrt{PG} \approx \frac{f_2}{f_1} \frac{2}{Q_{L2}}. \tag{49}$$

6.1 *Numerical example*

Suppose the signal and idler frequencies and diode parameters are given as follows:

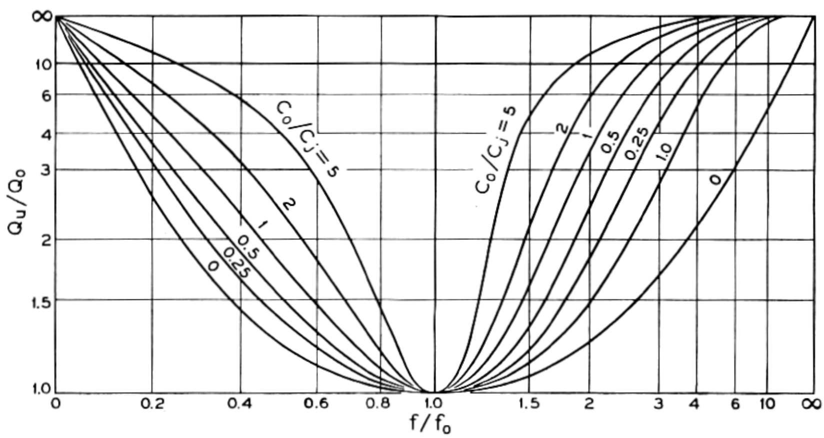

Fig. 12 — Frequency characteristic of a normalized unloaded Q of a diode.

$$f_1 = 4 \text{ gc}$$
$$f_2 = 8 \text{ gc}$$
$$C_j = 0.65 \text{ pf}$$
$$L_s = 0.6 \text{ nh}$$
$$C_0 = 0.15 \text{ pf}$$
$$f_{cr} = 30 \text{ gc}$$
$$\frac{|c|}{C_j} = 0.5.$$

From (38),

$$f_0 = \frac{1}{2\pi\sqrt{L_s C_j}} = 8.07 \text{ gc}.$$

Therefore f_0 is very close to f_2.

From (37),

$$30 \text{ gc} = \frac{1}{2\pi} \frac{1}{2} \frac{1}{C_j R_s} \frac{|c|}{C_j}$$
$$\therefore R_s = 2.05 \text{ ohms}.$$

From (39),

$$Q_0 = 14.8.$$

From the values of C_0 and C_j,

$$C_0/C_j = 0.231.$$

From the values of f_1, f_2 and f_0,

$$f_1/f_0 = 0.496$$
$$f_2/f_0 = 0.992$$
$$f_p/f_0 = (f_1 + f_2)/f_0 = 1.488.$$

From Figs. 8 and 9,

$$R_s Q_0 B_{d1} = 0.80 \quad \therefore B_{d1} = 26.3 \text{ m mhos}$$
$$R_s Q_0 B_{dp} = -0.85 \quad \therefore B_{dp} = -28.0 \text{ m mhos}$$
$$R_s Q_0^2 G_{d1} = 0.46 \quad \therefore G_{d1} = 1.02 \text{ m mhos}$$
$$R_s Q_0^2 G_{dp} = 1.44 \quad \therefore G_{dp} = 3.20 \text{ m mhos},$$

where the subscript p denotes a quantity for the pumping frequency. From (46),

$$\Delta B_1 = 15.3 \text{ m mhos.}$$

Assuming $L_2 = 0$, the right-hand side of (47) yields

$$4(\omega_2/\omega_{cr}) = 1.07.$$

Therefore, in order to have a good gain, the following condition should be held

$$Z_1 \Delta B_1 > 1.07$$

$$\therefore \; Z_1 > 70 \text{ ohms.}$$

Suppose we use 80 ohms for Z_1, which satisfies the above condition. The expected noise figure at room temperature is obtained from (35″) as follows:

$$F = 1.081 \times 1.49 = 1.61 = 2.1 \text{ db.}$$

If the amplifier and isolator are both refrigerated at 78° Kelvin, the noise temperature defined by

$$T_e = (F - 1) \times 290°\text{K}$$

becomes

$$T_e = 48°\text{K}$$

If the amplifier is further refrigerated at 42°K, the amplifier noise temperature is

$$T_e = 2.6°\text{K}.$$

If the input mismatching VSWR normalized to Z_1 is 1.2, the noise temperature of the amplifier increases by 1° Kelvin even with the isolator refrigerated at 78° Kelvin [cf. (35′)].

When we use a single-tuned idler circuit without external loading, $Q_{L2} \approx Q_0$, the percentage gain bandwidth product obtained from (49) is

$$(\Delta f/f_1)_{3\text{db}} \sqrt{PG} \approx 27 \text{ per cent.}$$

If we choose 16 db gain, the bandwidth is calculated as

$$(\Delta f/f_1)_{3\text{db}} \approx 4 \text{ per cent} \quad \text{or} \quad (\Delta f)_{3\text{db}} \approx 160 \text{ mc.}$$

With a double-tuning technique, the bandwidth may be improved up to 300 mc.

VII. CONCLUSION

Performance characteristics of a unilateral parametric amplifier with two diodes separated by a quarter wavelength at the signal frequency have been theoretically investigated on the basis of a scattering matrix representation. It is shown that a broadband signal circuit is essential in order to obtain a unilateral gain; otherwise, the amplifier may easily oscillate. The reason for this is that a unilateral amplifier requires a larger capacitance swing than an ordinary bilateral amplifier. It is shown that it is possible in principle to obtain any amount of low noise amplification without adjusting an input and output matching network of this amplifier, though the optimum noise figure is slightly higher than that of an ordinary reflection-type amplifier. This higher optimum noise figure does not always present a serious problem for practical applications, because the idler frequency is often determined by other factors such as broadbanding or pump availability. The bandwidth is primarily determined by the idler circuit. Broadbanding by introducing a mismatch in the signal circuit is not practical because it deteriorates the unilateral characteristic. A numerical example is given to show the potentiality of building an amplifier at 4 gc which has a 2.1 db noise figure and more than 300 mc bandwidth at 16 db gain. Since this amplifier does not have substantial reverse loss, it usually requires an isolator at its output port.

VIII. ACKNOWLEDGMENT

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APPENDIX

Substituting the following relations

$$\begin{aligned} \theta_1 &= (2m + 1)\frac{\pi}{2} + \delta_1 \\ \theta_2 &= n\pi + \delta_2 \\ \theta_p &= (2m + 2n + 1)\frac{\pi}{2} + \delta_p \end{aligned} \tag{50}$$

into (10), and assuming δ's are small quantities, each coefficient of the scattering matrix is modified as follows:

$$\Delta = 1 - 2\omega_1\omega_2\kappa^2 - j2\omega_1\omega_2\kappa^2(2\omega_1\omega_2\kappa^2\delta_2 + \delta_p) \tag{51}$$

$$S_{11}\cdot\Delta = S_{22}\cdot\Delta = j2\omega_1\omega_2\kappa^2(\delta_1 + 2\omega_1\omega_2\kappa^2\delta_2 + \delta_p)$$

$$S_{12}\cdot\Delta = (-)^{m+1}j[1 - 2\omega_1\omega_2\kappa^2 - j\{\delta_1(1 - 2\omega_1\omega_2\kappa^2)$$
$$+ 2\omega_1\omega_2\kappa^2(\delta_2 + \delta_p)\}]$$
$$S_{21}\cdot\Delta = (-)^{m+1}j[1 + 2\omega_1\omega_2\kappa^2 - j\{\delta_1(1 + 2\omega_1\omega_2\kappa^2)$$
$$- 2\omega_1\omega_2\kappa^2(\delta_2 - \delta_p)\}]$$
$$S_{13}\cdot\Delta = -(\omega_1/\omega_2)S_{42}\cdot\Delta = -\omega_1\kappa \exp(j\theta_{p0}/2)$$
$$\{\delta_1 + \delta_2(4\omega_1\omega_2\kappa^2 - 1) + \delta_p\}$$
$$S_{14}\cdot\Delta = -(\omega_1/\omega_2)S_{32}\cdot\Delta = (-)^n\omega_1\kappa \exp(j\theta_{p0}/2)\ (\delta_1 + \delta_2 + \delta_p) \quad (52)$$

etc., where

$$\theta_{p0} = (2m + 2n + 1)(\pi/2).$$

Equation (52) shows that the unilateral characteristic is deteriorated by the imperfect phase synchronization. And the amount of coupling between the signal wave in the reverse direction and the other three waves is determined by $\delta_1 + \delta_2 + \delta_p$ for a high-gain amplification where $2\omega_1\omega_2\kappa^2 \approx 1$. An imperfect pump synchronization represented by δ_p shifts the center of the amplification band as shown in (51). It is also shown in (51) that the bandwidth is affected mostly by a frequency-dependent idler phase constant if κ is kept constant.

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