Methods of Orbit Refinement

By R. B. BLACKMAN

(Manuscript received January 20, 1964)

During the past six or seven years, several methods of orbit refinement were developed specifically for use with artificial satellites and spacecraft. This article describes these methods, and the classical method, in a uniform mathematical formalism in order to facilitate comparisons of their relative advantages and disadvantages for practical systems applications. However, such comparisons are made in this article only to the extent that motivated the development of the new methods.

I. INTRODUCTION

The accuracy to which the position of a satellite or spacecraft can be predicted depends upon the accuracy to which the "initial" position and velocity vector, or related orbital parameters, are known. Since these parameters can be determined only from observational data which inevitably contain observational errors, the accuracy to which they can be known depends upon the nature, the quantity, the accuracy, and the distribution (in space and time) of the observational data, and the way in which these data are processed. The accuracy of the orbital parameters, and of prediction, depend also upon the accuracy to which all of the forces acting on the satellite or spacecraft are known and taken into account. Clearly, the term "accuracy" must be taken largely in a statistical sense.

Orbit refinement is essentially data smoothing for the purpose of accurate prediction. Given the nature of the observational data, and the statistical properties of the observational errors, it is possible to formulate a method of data smoothing and prediction which is optimum in the sense of giving predictions with the greatest possible accuracy. However, such an optimum method will, in general, be useful only as a standard of comparative performance for more practical methods. The reason for this is that it has not been difficult to find simpler and therefore more practical methods which are nearly as accurate as the optimum method. (For example, see Ref. 1.)

It should be noted also that in the practical applications of data-

smoothing and prediction methods each application is usually characterized by a unique set of practical constraints. Thus, it has been a frequent experience that a method which was judged to be the most practical for one application was usually not judged to be the most practical for another application. In fact, it has frequently happened that a new method, perhaps new only in the sense that it is a composite of parts of older methods, was developed for a particular system.

The classical method of orbit refinement, the so-called "differential corrections" method (which is essentially the method of least squares, developed by K. F. Gauss in 1795) has served astronomers very well for over 150 years. However, this method becomes quite unwieldy for large quantities of observational data. Hence, the quantity of data to be processed was frequently reduced by the supplementary use of "normal places" and/or "smoothing," described briefly in Section 6.6.1, pp. 141–142 of Ref. 2, and later in this article. Thus, up to about 1955 no need was felt for another method of orbit refinement, although the basis for the development of alternative methods was implicit in a number of articles published in the field of general statistical analysis, such as Refs. 3–6.

With the development of artificial satellites and space probes, the need for alternative methods of orbit refinement began to be felt in some quarters. The first definite proposal of an alternative method, as far as the author is aware, was made by P. Swerling (Refs. 7–9). A somewhat different method, independently developed by the author (Ref. 10) was used in the Telstar I experimental satellite communications system (Refs. 11, 12). Some difficulties experienced with this method after about four weeks of successful operation led A. J. Claus (Ref. 13) to develop another method which is slightly different from Swerling's method. In addition to these methods, it is worthwhile to include a method of space navigation described by R. H. Battin (Ref. 14) because it involves a practical detail which, under favorable circumstances, may be profitably introduced into the other methods.

The essential details of these methods will be described here in a uniform mathematical formalism in order to reveal their basic similarities and differences, and in order to facilitate comparisons of their relative advantages and disadvantages for practical systems applications.

II. CLASSICAL DIFFERENTIAL CORRECTIONS METHOD. LEAST SQUARES

Let $\tilde{\varphi}$ be an n-rowed vector representation of the observational (angular) data. Every component of this vector is assumed to be labeled to identify it as either a declination angle or a right ascension angle, and to

specify the time at which it was observed. Let ϵ be the 6-rowed vector representation of a set of values of the orbital elements, and let $\varphi(\epsilon)$ be an n-rowed vector representation of the angles which would have been observed, assuming that the actual orbital elements are exactly represented by ϵ, and assuming that observations are made with ideal accuracy. If the observational errors are independently and normally distributed, with zero means and equal variances, the best estimate of the orbital elements is that value of ϵ which minimizes the quadratic form

$$Q = [\tilde{\varphi} - \varphi(\epsilon)]' \cdot [\tilde{\varphi} - \varphi(\epsilon)],$$

where the prime stands for transposition. This quadratic form is simply the sum of the squares of the components of the vector difference $\tilde{\varphi} - \varphi(\epsilon)$.

Let ϵ_0 be an initially assumed value for ϵ, close to the true value. Then, to the first-order term in $(\epsilon - \epsilon_0)$,

$$\varphi(\epsilon) = \varphi(\epsilon_0) + J \cdot (\epsilon - \epsilon_0), \tag{1}$$

where J is the n by 6 matrix symbolized by

$$J = \partial\varphi(\epsilon_0)/\partial\epsilon_0 . \tag{2}$$

Hence, to second-order terms,

$$Q = [r - J \cdot (\epsilon - \epsilon_0)]' \cdot [r - J \cdot (\epsilon - \epsilon_0)],$$

where r is the n-rowed vector residual

$$r = \tilde{\varphi} - \varphi(\epsilon_0). \tag{3}$$

Now, Q is a minimum with respect to ϵ if

$$J' \cdot [r - J \cdot (\epsilon - \epsilon_0)] = 0.$$

Written in the form

$$J' \cdot J \cdot (\epsilon - \epsilon_0) = J' \cdot r,$$

this corresponds to the set of 6 equations commonly called "normal equations." If $J' \cdot J$ is nonsingular, and if the value of ϵ which satisfies this equation is denoted by $\bar{\epsilon}$, then,

$$\bar{\epsilon} = \epsilon_0 + (J' \cdot J)^{-1} \cdot J' \cdot r. \tag{4}$$

This $\bar{\epsilon}$ is then substituted for ϵ_0 in (2), in (3), and in the right-hand member of (4), in order to obtain another $\bar{\epsilon}$. This substitution procedure is iterated until $\bar{\epsilon}$ has essentially converged. The final $\bar{\epsilon}$ is the least squares estimate of the orbital elements.

In a statistical sense, this estimate is unbiased to the first order in the observational errors, assuming that the errors are not biased. It is only asymptotically unbiased to the second order, but it is a "consistent" estimate in the sense that the probability is unity that it will be correct to the second order as $n \to \infty$. (See Section 4.3 for clarification.)

2.1 *Classical Method. Weighted Least Squares*

The method described in the preceding section is quite unwieldy for large values of n on account of the size of the J matrix. Some relief was obtained by resorting to various artifices. For example, if a number of observations were made at sufficiently short intervals of time, a straight average of these observations would be taken and treated as a single observation called a "normal place." (More elaborate methods of deriving normal places directly from the observational data are called "smoothing" by Baker and Makemson in Ref. 2.) Since normal places would be more accurate than single actual observations, in proportion to the number of actual observations which went into each of them, it was necessary to generalize the differential corrections method to some extent.

The quadratic form to be minimized is now

$$Q = [\tilde{\varphi} - \varphi(\epsilon)]' \cdot W \cdot [\tilde{\varphi} - \varphi(\epsilon)],$$

where W is an n by n diagonal matrix. In expanded form, it is

$$Q = \sum w_{ii} \cdot [\tilde{\varphi}_i - \varphi_i(\epsilon)]^2,$$

where the w_{ii} are the components of W. Thus, the quadratic form is simply a weighted sum of the squares of the components of the vector difference $\tilde{\varphi} - \varphi(\epsilon)$.

The analysis in this case will not be pursued beyond this point, since it is a special case of the analysis given in the next section. Suffice it to say that if the analysis were carried out for this special case, the results would be equivalent to the method used by astronomers when they deal with normal places, or with uncorrelated observations of different degrees of accuracy.

2.2 *General Form of the Classical Method.*

If the observational data are not all of the same nature (angles, ranges, and range-rates), and especially if some of the errors in the data are correlated, let Φ be the n by n covariance matrix of the n-rowed vector $\tilde{\varphi}$. For further generality, let ϵ be an m-rowed vector representation of a set

of values of parameters which, in addition to the 6 orbital elements, may include such quantities as the frequency of a satellite-borne Doppler source, and instrumental biases. Then the quadratic form to be minimized is

$$Q = [\tilde{\varphi} - \varphi(\epsilon)]' \cdot \Phi^{-1} \cdot [\tilde{\varphi} - \varphi(\epsilon)]. \tag{5}$$

Under the assumption that the components of $\tilde{\varphi}$ obey a joint n-dimensional normal (i.e., Gaussian) distribution with covariance matrix Φ, the value of ϵ which minimizes the quadratic form (5) is the maximum likelihood estimate of the true value of the parameters. Under any other symmetrical probability distribution of the errors in the observational data, this value of ϵ is simply the weighted least squares estimate of the parameters.

Substituting (1) into (5) we get

$$Q = [r - J \cdot (\epsilon - \epsilon_0)]' \cdot \Phi^{-1} \cdot [r - J \cdot (\epsilon - \epsilon_0)], \tag{6}$$

where J and r are defined by (2) and (3) except that J is now an n by m matrix, and ϵ_0 is an m-rowed vector. Now, Q is minimum with respect to ϵ if

$$J' \cdot \Phi^{-1} \cdot [r - J \cdot (\epsilon - \epsilon_0)] = 0.$$

Denoting the value of ϵ which satisfies this equation by $\bar{\epsilon}$, we have

$$\bar{\epsilon} = \epsilon_0 + C \cdot \rho, \tag{7}$$

where

$$C = (J' \cdot \Phi^{-1} \cdot J)^{-1}, \qquad \text{an } m \text{ by } m \text{ matrix}, \tag{8}$$

$$\rho = J' \cdot \Phi^{-1} \cdot r, \qquad \text{an } m\text{-rowed vector}. \tag{9}$$

Equation (7) is the generalization of (4) and, as in the case of (4), it is to be used iteratively until $\bar{\epsilon}$ has essentially converged.

After $\bar{\epsilon}$ has converged, C is its covariance matrix. This follows from the fact that (6) may be expressed in the form

$$Q = (\epsilon - \bar{\epsilon})' \cdot C^{-1} \cdot (\epsilon - \bar{\epsilon}) + \text{terms independent of } \epsilon. \tag{10}$$

In case the data are all of the same nature (all angles, or all ranges, or all range-rates), are all of the same accuracy, and the errors are not correlated, then, $C = \sigma^2 \cdot (J' \cdot J)^{-1}$.

The availability of the covariance matrix C of the estimate $\bar{\epsilon}$ offers the possibility of using the classical method in the intrapass stage of the pass-by-pass method described in Section IV, in order to reduce the amount

of observational data to be processed at any one time. However, it is not the only way to reduce the amount of observational data to be processed at any one time. There are other ways which, in particular, avoid the computation of the n by m matrix J, where n may be of the order of 400 for each pass. One such way is Swerling's method described in the next section. Another such way is cited as an example in Section 4.3. A modified form of the second way is described briefly in Section V.

III. SWERLING'S METHOD

Let ϵ_1 be a 6-rowed vector estimate of the osculating orbital elements at epoch t_1, and let C_1 be its covariance matrix. Let $\tilde{\varphi}$ be the n-rowed vector of new observational data more or less centered at epoch t_2, where $t_2 > t_1$, and let Φ be its covariance matrix. To obtain the least squares estimate of the osculating orbital elements at epoch t_2, we must first update (i.e., extrapolate) ϵ_1 and C_1. If $\hat{\epsilon}_1$ is the result of updating ϵ_1, the updated C_1 is

$$\hat{C}_1 = M \cdot C_1 \cdot M',$$

where M is the 6 by 6 matrix symbolized by

$$M = \partial \hat{\epsilon}_1 / \partial \epsilon_1 .$$

Then, assuming that the errors in the new data are not correlated with the errors in the old data, the quadratic form to be minimized is

$$Q = (\epsilon - \hat{\epsilon}_1)' \cdot \hat{C}_1^{-1} \cdot (\epsilon - \hat{\epsilon}_1) + [\tilde{\varphi} - \varphi(\epsilon)]' \cdot \Phi^{-1} \cdot [\tilde{\varphi} - \varphi(\epsilon)]. \quad (11)$$

This is essentially the sum of the right-hand members of (5) and (10). Now, to the first-order term in $(\epsilon - \hat{\epsilon}_1)$,

$$\varphi(\epsilon) = \varphi(\hat{\epsilon}_1) + J \cdot (\epsilon - \hat{\epsilon}_1), \quad (12)$$

where J is the n by 6 matrix symbolized by

$$J = \partial \varphi(\hat{\epsilon}_1) / \partial \hat{\epsilon}_1 . \quad (13)$$

Then, to second-order terms,

$$Q = (\epsilon - \hat{\epsilon}_1)' \cdot \hat{C}_1^{-1} \cdot (\epsilon - \hat{\epsilon}_1) + [r - J \cdot (\epsilon - \hat{\epsilon}_1)]' \cdot \Phi^{-1} \cdot [r - J \cdot (\epsilon - \hat{\epsilon}_1)],$$

where

$$r = \tilde{\varphi} - \varphi(\hat{\epsilon}_1). \quad (14)$$

Now, Q is minimum with respect to ϵ if

$$\hat{C}_1^{-1} \cdot (\epsilon - \hat{\epsilon}_1) - J' \cdot \Phi^{-1} \cdot [r - J \cdot (\epsilon - \hat{\epsilon}_1)] = 0.$$

Denoting the value of ϵ which satisfies this equation by $\bar{\epsilon}$, we have

$$\bar{\epsilon} = \hat{\epsilon}_1 + C \cdot \rho, \tag{15}$$

where

$$C = [\hat{C}_1^{-1} + J' \cdot \Phi^{-1} \cdot J]^{-1}, \tag{16}$$

$$\rho = J' \cdot \Phi^{-1} \cdot r. \tag{17}$$

Finally, since the quadratic form, to second-order terms, may be expressed in the form

$$Q = (\epsilon - \bar{\epsilon})' \cdot C^{-1} \cdot (\epsilon - \bar{\epsilon}) + \text{terms independent of } \epsilon, \tag{18}$$

it follows that C is the covariance matrix of $\bar{\epsilon}$. Equations (11), (15), (16), and (17) correspond respectively to equations (16), (25), (19), and (23) in Swerling's JAS paper (Ref. 8). Further, since ϵ_1 and C_1 may have been computed at the preceding stage in exactly the same way as $\bar{\epsilon}$ and C at the last stage, (11) corresponds also to equation (30) in Swerling's JAS paper.

With regard to the updating of ϵ_1 and C_1 it should be noted that estimates of "time of perigee (or nodal) passage" should be labeled with the serial number of the passage. Thus, even in the hypothetical case of a purely Keplerian orbit, unless the serial number of the passage is intended to be the same in $\bar{\epsilon}$ as it is in ϵ_1, the vector $\hat{\epsilon}_1$ will differ from the vector ϵ_1. The component T_1 (time of perigee or nodal passage) of ϵ_1 will be increased to $\hat{T}_1$ in $\hat{\epsilon}_1$, where $\hat{T}_1$ is T_1 plus an integral multiple of the period estimate $2\pi a_1^{3/2}/\sqrt{\mu}$, where a_1 is the semimajor axis component of ϵ_1. The matrix M will therefore be a unity matrix except for an off-diagonal component $\partial \hat{T}_1/\partial a_1$ which is an integral multiple of $3\pi\sqrt{a_1/\mu}$. In this connection, A. J. Claus has found advantages in using the period instead of the semimajor axis as a component of ϵ_1, especially when perturbing forces are taken into account.

In the classical method n must be at least equal to the number of orbital elements, and it must include all of the available observational data — old observational data (which has been processed at least once before) as well as new. In Swerling's method n may be less than the number of orbital elements (possibly $n = 1$), and old observational data are represented by the 6 components of ϵ_1, the 21 distinct components of the symmetrical matrix C_1, and the epoch t_1. The chief objection to Swerling's method, as far at least as some applications are concerned, is its inability to omit any part of the old observational data without reprocessing the remainder of it. This objection is less serious if Swerling's

method is used only in the intrapass stage of the pass-by-pass method described in the next section.

IV. PASS-BY-PASS METHOD

This method consists essentially of two stages per pass. (These stages are not quite the same as Swerling's stages.) In the first (or *intrapass*) stage, a set of estimates of the orbital elements, and an associated covariance matrix, are computed from the observational data *for each pass*. In this stage, any method of computation which yields a covariance matrix for the estimates of the orbital elements may be used. In the second (or *interpass*) stage, the sets of single-pass (i.e., *intrapass*) estimates of the orbital elements are combined cumulatively (and possibly selectively), by a method in which the single-pass (i.e., *intrapass*) covariance matrices play important roles.

Let $\hat{\epsilon}_1$ and $\hat{C}_1$ have the same significance as in the description of Swerling's method, but let the new data be processed separately to obtain an independent vector estimate ϵ_2 of the osculating orbital elements at epoch t_2, with covariance matrix C_2. Then, the quadratic form to be minimized is

$$Q = (\epsilon - \hat{\epsilon}_1)' \cdot \hat{C}_1^{-1} \cdot (\epsilon - \hat{\epsilon}_1) + (\epsilon - \epsilon_2)' \cdot C_2^{-1} \cdot (\epsilon - \epsilon_2). \quad (19)$$

This is the sum of two terms similar to the right-hand member of (10). Now, Q is minimum with respect to ϵ if

$$\hat{C}_1^{-1} \cdot (\epsilon - \hat{\epsilon}_1) + C_2^{-1} \cdot (\epsilon - \epsilon_2) = 0.$$

Denoting the value of ϵ which satisfies this equation by $\bar{\epsilon}$, we have

$$\bar{\epsilon} = C \cdot (\hat{C}_1^{-1} \cdot \hat{\epsilon}_1 + C_2^{-1} \cdot \epsilon_2) \quad (20)$$

where

$$C = (\hat{C}_1^{-1} + C_2^{-1})^{-1}. \quad (21)$$

Finally, since (19) may be expressed in the form

$$Q = (\epsilon - \bar{\epsilon})' \cdot C^{-1} \cdot (\epsilon - \bar{\epsilon}) + \text{terms independent of } \epsilon, \quad (22)$$

it follows that C is the covariance matrix of $\bar{\epsilon}$. (See Appendix for a more illuminating derivation).

Note that a "fading memory" can be introduced into the pass-by-pass method (or any other method in which the old data are represented by ϵ_1, C_1, and t_1) simply by substituting kC_1 for C_1, where k is a scalar constant ($k > 1$) or

$$k = \exp [\gamma \cdot (t_2 - t_1)]$$

where γ is a constant ($\gamma > 0$). However, at the cost of storing a number of single-pass estimates of the orbital elements and the associated covariance matrices, old single-pass estimates of the orbital elements may be completely omitted at any time without having to reprocess any of the observational data on which the more recent single-pass estimates are based.

4.1 *An Example*

It is illuminating to see how the method of combination described in the preceding section works out in a simple problem. Consider an object traveling along the x axis at a known acceleration a. Let

$$\epsilon_1 = \begin{bmatrix} x_1 \\ v_1 \end{bmatrix}, \qquad \epsilon_2 = \begin{bmatrix} x_2 \\ v_2 \end{bmatrix},$$

be the estimates of position and velocity at epochs t_1 and t_2, and let

$$C_1 = C_2 = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_v^2 \end{bmatrix}.$$

If $t_2 - t_1 = \tau$, then

$$\hat{\epsilon}_1 = \begin{bmatrix} x_1 + v_1\tau + \frac{1}{2}a\tau^2 \\ v_1 + a\tau \end{bmatrix},$$

whence

$$M = \frac{\partial \hat{\epsilon}_1}{\partial \epsilon_1} = \begin{bmatrix} 1 & \tau \\ 0 & 1 \end{bmatrix}.$$

Then,

$$\hat{C}_1 = M \cdot C_1 \cdot M' = \begin{bmatrix} \sigma_x^2 + \tau^2\sigma_v^2 & \tau\sigma_v^2 \\ \tau\sigma_v^2 & \sigma_v^2 \end{bmatrix}.$$

Note that the correlation coefficient, which is

$$\frac{1}{\sqrt{1 + (\sigma_x/\tau\sigma_v)^2}},$$

is very close to unity for large values of τ. For this reason, or for other reasons, we may expect serious difficulties in numerical computations based on (20) and (21) as they stand. These difficulties are considerably reduced by the transformations described in the next section.

In the example under consideration there is of course no difficulty about inverting matrices analytically. It is found finally that

$$\bar{\epsilon} = \begin{bmatrix} \bar{x} \\ \bar{v} \end{bmatrix} \text{ at epoch } t_2 ,$$

where

$$\bar{x} = \frac{1}{4\sigma_x^2 + \tau^2\sigma_v^2} \{\sigma_x^2[2(x_1 + x_2) + \tau(v_1 + v_2)] + \sigma_v^2\tau^2 x_2\} ,$$

$$\bar{v} = \frac{1}{4\sigma_x^2 + \tau^2\sigma_v^2} \{2\sigma_x^2(v_1 + v_2 + a\tau) + \sigma_v^2\tau(x_2 - x_1 + \tfrac{1}{2}a\tau^2)\} ,$$

and

$$C = \frac{\sigma_x^2}{4\sigma_x^2 + \tau^2\sigma_v^2} \begin{bmatrix} 2\sigma_x^2 + \tau^2\sigma_v^2 & \tau\sigma_v^2 \\ \tau\sigma_v^2 & 2\sigma_v^2 \end{bmatrix} .$$

Under the assumptions implicit in C_1 and C_2, that x_1, v_1, x_2, and v_2 are not correlated, the equations for $\bar{x}$ and $\bar{v}$ may be verified by a more familiar method. We have in fact two independent estimates of the position at epoch t_2, viz.,

$$x_1 + \frac{v_1 + v_2}{2}\tau \quad \text{with variance } \sigma_x^2 + \frac{\tau^2\sigma_v^2}{2} ,$$

and

$$x_2 \quad \text{with variance } \sigma_x^2 .$$

Taking the weighted average of these two estimates, each estimate being weighted in inverse proportion to its variance, we get $\bar{x}$, as expressed above. The variance of $\bar{x}$ is the harmonic mean of the two variances, so that

$$\text{var } \{\bar{x}\} = \frac{\sigma_x^2(2\sigma_x^2 + \tau^2\sigma_v^2)}{4\sigma_x^2 + \tau^2\sigma_v^2} ,$$

in agreement with the equation for C.

Similarly, we have three independent estimates of the velocity at epoch t_2, viz.,

$$v_1 + a\tau \quad \text{with variance } \sigma_v^2 ,$$

$$v_2 \quad \text{with variance } \sigma_v^2 ,$$

and

$$\frac{x_2 - x_1}{\tau} + \frac{1}{2}a\tau \quad \text{with variance } \frac{2\sigma_x^2}{\tau^2} .$$

Taking the weighted average of these three estimates, each estimate being weighted in inverse proportion to its variance, we get $\bar{v}$, as expressed above. The variance of $\bar{v}$ is the harmonic mean of the three variances, so that

$$\text{var}\,\{\bar{v}\} = 2\sigma_x^2\sigma_v^2/(4\sigma_x^2 + \tau^2\sigma_v^2),$$

in agreement with the equation for C.

The covariance of $\bar{x}$ and $\bar{v}$ may also be verified, but the algebra is more involved.

4.2 *Transformation of Equations*

Omitting the explicit notational reference to updating (i.e., extrapolation) in (20) and (21), we have

$$\bar{\epsilon} = C\cdot(C_1^{-1}\cdot\epsilon_1 + C_2^{-1}\cdot\epsilon_2), \tag{23}$$

where

$$C = (C_1^{-1} + C_2^{-1})^{-1}. \tag{24}$$

These equations require the inversion of three matrices which, at least in the case of angles-only data, are usually extremely ill-conditioned. Hence, they have been transformed in order to reduce this difficulty. The transformations described here were developed by A. J. Claus and the author.

Introducing the identity

$$C_1^{-1}\cdot\epsilon_1 = (C_1^{-1} + C_2^{-1})\cdot\epsilon_1 - C_2^{-1}\cdot\epsilon_1$$

into (23), we get

$$\bar{\epsilon} = \epsilon_1 - C\cdot C_2^{-1}\cdot(\epsilon_1 - \epsilon_2).$$

Now,

$$\begin{aligned} C\cdot C_2^{-1} &= (C_1^{-1} + C_2^{-1})^{-1}\cdot C_2^{-1} = [C_2\cdot(C_1^{-1} + C_2^{-1})]^{-1} \\ &= (1 + C_2\cdot C_1^{-1})^{-1} = [(C_1 + C_2)\cdot C_1^{-1}]^{-1} \\ &= C_1\cdot(C_1 + C_2)^{-1}. \end{aligned}$$

Next, let

$$P_1 = S\cdot C_1\cdot S, \qquad P_2 = S\cdot C_2\cdot S, \tag{25}$$

where S is a diagonal matrix in which each diagonal term is the reciprocal of the square root of the sum of the corresponding diagonal terms of

C_1 and C_2. (Thus, every diagonal term of $P_1 + P_2$ is unity.) Then,

$$C_1 \cdot (C_1 + C_2)^{-1} = S^{-1} \cdot P_1 \cdot (P_1 + P_2)^{-1} \cdot S.$$

Hence,

$$\bar{\epsilon} = \epsilon_1 - S^{-1} \cdot P_1 \cdot (P_1 + P_2)^{-1} \cdot S \cdot (\epsilon_1 - \epsilon_2).$$

Similarly,

$$\bar{\epsilon} = \epsilon_2 + S^{-1} \cdot P_2 \cdot (P_1 + P_2)^{-1} \cdot S \cdot (\epsilon_1 - \epsilon_2).$$

Next, let W_1 and W_2 be arbitrary square matrices whose sum is a unity matrix. Then,

$$\bar{\epsilon} = W_1 \cdot \epsilon_1 + W_2 \cdot \epsilon_2 - R \cdot (\epsilon_1 - \epsilon_2),$$

where

$$R = (W_1 \cdot S^{-1} \cdot P_1 - W_2 \cdot S^{-1} \cdot P_2) \cdot (P_1 + P_2)^{-1} \cdot S.$$

Since $P_1 + P_2$ may yet be ill conditioned for inversion, we now use a well-known artifice, and write

$$R = [W_1 \cdot S^{-1} \cdot (P_1 \cdot G) - W_2 \cdot S^{-1} \cdot (P_2 \cdot G)] \cdot [(P_1 + P_2) \cdot G]^{-1} \cdot S,$$

where G may be regarded as another arbitrary square matrix although it merely represents a set of rules for combining rows and/or columns of $P_1 + P_2$, as well as of P_1 and P_2 individually. [The normalization of the matrix sum $C_1 + C_2$ to $P_1 + P_2$ and the preservation of its symmetry by the introduction of the matrix S, as in (25), simplifies the implementation of the matrix G as a set of operational rules.] Finally, W_1 and W_2 are restricted to diagonal matrices, so that

$$R = S^{-1} \cdot [W_1 \cdot (P_1 \cdot G) - W_2 \cdot (P_2 \cdot G)] \cdot [(P_1 + P_2) \cdot G]^{-1} \cdot S. \quad (27)$$

Noting that the right-hand member of (23) reduces to $2C$ if ϵ_1 is replaced by C_1 and ϵ_2 is replaced by C_2, it follows from (26) that

$$C = \tfrac{1}{2}[W_1 \cdot C_1 + W_2 \cdot C_2 - R \cdot (C_1 - C_2)]. \quad (28)$$

For further details, see Ref. 12.

4.3 *Debiasing Single-Pass Estimates*

Depending upon the method used to obtain single-pass estimates of the orbital elements in the first (or intrapass) stage of the pass-by-pass method of orbit refinement described in Section IV, the single-pass estimates may be biased on the average even if the errors in the observa-

tional data are not biased. Unless these biases are tolerable, they must of course be removed. In this section we will describe a method of removing these biases in the single-pass estimates of the orbital elements. However, biases in the single-pass estimates of the orbital elements due to biased errors in the observational data will not be removed by this method.

In the interest of simplicity, the description will be by analogy with a much simpler problem which involves only one observable coordinate, one parameter to be estimated, and does not involve time at all. We will consider two methods of estimation, of which the first is analogous to the classical method of orbit refinement, and the second is analogous to any method of obtaining single-pass estimates which are biased on the average even if the errors in the observational data are not biased.

Let y be a function of the observable coordinate x, and let

$$x_i = x_0 + \epsilon_i \qquad (i = 1, 2, \cdots, n)$$

be the observed values of x, where the ϵ's are uncorrelated random errors with ave $\{\epsilon_i\} = 0$ and var $\{\epsilon_i\} = \sigma^2$ for every i. The most direct way of estimating $y_0 \equiv y(x_0)$ is obviously to compute first

$$\bar{x} = (1/n)\sum x_i ,$$

and then $y(\bar{x})$. The nature of the estimate obtained in this way is determined as follows. Since

$$\bar{x} = x_0 + \alpha,$$

where

$$\alpha = (1/n)\sum \epsilon_i ,$$

then, to second-order terms in the ϵ_i's,

$$y(\bar{x}) = y_0 + a_0\alpha + \tfrac{1}{2}b_0\alpha^2,$$

where $a_0 = dy_0/dx_0$, and $b_0 = d^2y_0/dx_0^2$. Now,

$$\text{ave } \{\alpha\} = 0, \qquad \text{ave } \{\alpha^2\} = \sigma^2/n.$$

Hence, to second-order terms in σ,

$$\text{ave } \{y(\bar{x})\} = y_0 + (b_0\sigma^2/2n),$$

and

$$\text{var } \{y(\bar{x})\} = \text{ave } \{[y(\bar{x})]^2\} - [\text{ave } \{y(\bar{x})\}]^2 = a_0^2\sigma^2/n.$$

If σ^2 is not known, an estimate of var $\{y(\bar{x})\}$ is given by

$$\text{var } \{y(\bar{x})\} \approx \frac{a_0^2}{n(n-1)} \sum (x_i - \bar{x})^2 .$$

Since

$$\lim_{n\to\infty} \text{ave } \{y(\bar{x})\} = y_0 ,$$

the estimate is asymptotically unbiased; and since

$$\lim_{n\to\infty} \text{var } \{y(\bar{x})\} = 0,$$

the estimate is "consistent" in the sense that the probability is unity that it will be correct to at least the second order as $n \to \infty$. These results are indicative of the nature of the estimates of orbital elements obtained by the classical method of orbit refinement, in which the estimates are such as to "predict" values of the observable coordinates which agree with the actual observations in the least squares sense.

Now, consider another way of estimating y_0. We compute $y(x)$ for each observed value of x, and define the estimate of y_0 as

$$\tilde{y}_0 = (1/n) \sum y_i \quad \text{where} \quad y_i = y(x_i).$$

The purpose of estimating y_0 in this way is to permit the estimation of var $\{\tilde{y}_0\}$ without using a_0. Thus,

$$\text{var } \{\tilde{y}_0\} \approx \frac{1}{n(n-1)} \sum R_i^2$$

where

$$R_i = y_i - \tilde{y}_0 .$$

The importance of this is that the analog of a_0 in the computation of orbital elements from, say, two complete radar fixes (each fix consisting of a range and two angles) is the inverse of a 6 by 6 matrix whose components are functions of the epochs of the two fixes. The computation of this matrix may be avoided by computing a set of orbital elements from each pair of radar fixes, averaging over the sets, and *estimating* the covariance matrix from the residuals.

The nature of the estimate $\tilde{y}_0$ is determined as follows. Since, to second-order terms,

$$y_i = y_0 + a_0\epsilon_i + \tfrac{1}{2}b_0\epsilon_i^2 ,$$

then

$$\tilde{y}_0 = y_0 + a_0\alpha + \tfrac{1}{2}b_0\beta,$$

where α is as previously defined, and

$$\beta = (1/n)\sum \epsilon_i^2.$$

Now,

$$\text{ave } \{\beta\} = \sigma^2.$$

Hence,

$$\text{ave } \{\tilde{y}_0\} = y_0 + (b_0\sigma^2/2),$$

and

$$\text{var } \{\tilde{y}_0\} = \text{ave } \{[\tilde{y}_0]^2\} - [\text{ave } \{\tilde{y}_0\}]^2 = a_0^2\sigma^2/n.$$

Thus, $\tilde{y}_0$ is a biased estimate of y_0. In particular, it should be noted that, while the variance of this estimate decreases with increasing n, the bias in the estimate is independent of n. Hence, the variance bears no relation to the accuracy of the estimate.

The bias may be removed by the following supplementary procedure.

1. Compute $\hat{x}_0$ such that

$$y(\hat{x}_0) = \tilde{y}_0.$$

This is analogous to computing artificial tracking data (at the same epochs as the actual tracking data) from the biased single-pass estimates of the orbital elements analogous to $\tilde{y}_0$. The method of computing tracking data from orbital elements must of course be numerically compatible with the method of computing orbital elements from tracking data in the absence of observational errors.

2. Compute

$$\hat{x}_i = 2\hat{x}_0 - x_i, \qquad i = 1, 2, \cdots, n.$$

This is analogous to combining the actual tracking data with the artificial tracking data computed in the preceding step. (The choice of a combination such that the random error in each $\hat{x}_i$ is equal in magnitude but opposite in sign to the random error in the corresponding x_i was suggested by D. R. Brillinger.)

3. Compute

$$\hat{y}_i = y(\hat{x}_i), \qquad i = 1, 2, \cdots, n.$$

This is exactly the same procedure used in computing $y_i = y(x_i)$.

4. Compute

$$\bar{y}_i = \tfrac{1}{2}(3y_i - \hat{y}_i), \qquad i = 1, 2, \cdots, n.$$

5. Compute

$$\bar{y}_0 = (1/n)\sum \bar{y}_i \quad \text{as the estimate of } y_0 .$$

The nature of the estimate $\bar{y}_0$ is determined as follows. To second-order terms,

$$\hat{x}_0 = x_0 + \alpha - \frac{b_0}{2a_0}(\alpha^2 - \beta) .$$

Hence,

$$\hat{x}_i = x_0 - (\epsilon_i - 2\alpha) - \frac{b_0}{a_0}(\alpha^2 - \beta), \qquad i = 1, 2, \cdots, n,$$

$$\hat{y}_i = y_0 - a_0(\epsilon_i - 2\alpha) + \frac{b_0}{2}(\epsilon_i^2 - 4\alpha\epsilon_i + 2\alpha^2 + 2\beta),$$

$$\bar{y}_i = y_0 + a_0(2\epsilon_i - \alpha) + \frac{b_0}{2}(\epsilon_i^2 + 2\alpha\epsilon_i - \alpha^2 - \beta),$$

$$\bar{y}_0 = y_0 + a_0\alpha + \tfrac{1}{2}b_0\alpha^2 .$$

Thus, to second-order terms in the ϵ_i's, $\bar{y}_0$ is the same as $y(\bar{x})$. It is asymptotically unbiased, and it is a consistent estimate of y_0. It may be absolutely debiased, to second-order terms in the ϵ_i's, by changing step 4 of the supplementary procedure to compute

$$\bar{y}_i = \frac{1}{2}\left[\left(3 + \frac{1}{n-1}\right)y_i - \left(1 + \frac{1}{n-1}\right)\hat{y}_i\right].$$

Then, in determining the nature of the estimate $\bar{y}_0$, we now have

$$\bar{y}_i = y_0 + \frac{a_0}{n-1}[(2n-1)\,\epsilon_i - n\alpha]$$

$$+ \frac{b_0}{2(n-1)}[(n-1)\,\epsilon_i^2 + 2n\alpha\epsilon_i - n\alpha^2 - n\beta],$$

$$\bar{y}_0 = y_0 + a_0\alpha + \frac{b_0}{2(n-1)}(n\alpha^2 - \beta).$$

Hence,

$$\text{ave}\,\{\bar{y}_0\} = y_0 .$$

Generally, if step 2 of the supplementary procedure is changed to compute

$$\hat{x}_i = \tfrac{1}{2}[(1 + w)\hat{x}_0 + (1 - w)x_i],$$

and step 4 is changed to compute

$$\bar{y}_i = \tfrac{1}{2}[(1 + W)y_i + (1 - W)\hat{y}_i],$$

then,

$$\bar{y}_0 = y_0 + a_0\alpha + \frac{(1 - W)b_0}{4}\left\{\frac{1 - w^2}{4}\alpha^2 + \left[\frac{3 + w^2}{4} + \frac{1 + W}{1 - W}\right]\beta\right\},$$

whence,

$$\text{ave } \{\bar{y}_0\} = y_0 + \frac{(1 - W)b_0\sigma^2}{4}\left[\frac{1 - w^2}{4n} + \frac{3 + w^2}{4} + \frac{1 + W}{1 - W}\right].$$

Hence, the estimate $\bar{y}_0$ is asymptotically unbiased if

$$W = \frac{w^2 + 7}{w^2 - 1},$$

and is absolutely unbiased if

$$W = \frac{w^2 + \dfrac{7n + 1}{n - 1}}{w^2 - 1}.$$

The choice of w should be made with some regard to the fact that

$$\text{var } \{\hat{x}_i\} = \sigma^2\left[1 - \frac{(n - 1)\ (1 + w)\ (3 - w)}{4n}\right].$$

The choice is $w = 3$ in step 2 of the supplementary procedure, and $W = 2$ in step 4, so that var $\{\hat{x}_i\} = \sigma^2 =$ var $\{x_i\}$.

V. CLAUS'S METHOD

The pass-by-pass method described in Section IV was used at the Andover, Maine, station of the Telstar I experimental satellite communications system. The intrapass estimates of the orbital elements were computed from angles-only data by a method described in some detail in Refs. 11 and 12. Suffice it here to say that a set of orbital elements is computed from each set of four sightlines (of which there may be as many as 200 in a single pass), the sets of orbital elements are averaged, and an *estimate* of the covariance matrix is computed from the residuals.

(Compare this outline with that of the example cited in Section 4.3, which uses complete radar fixes, including range. It should be noted that unless the number of sets of orbital elements is at least equal to the number of orbital elements the covariance matrix will be singular.) This method gave excellent results for about four weeks. After that period it began to give sporadically bad results — typical errors of 10^5 feet in single-pass estimates of the semimajor axis.

Extensive simulation studies by W. C. Ridgway III showed that most of the trouble was probably due to the sensitivity of single-pass estimates of the orbital elements to a bias error in the elevation angles, where these estimates are derived from single-tracker angles-only data. This result was subsequently confirmed by some formal analysis by the author. Ridgway's studies showed, in particular, that the sensitivity increases rapidly with decreasing length of pass, and with increasing maximum elevation angle, although sightlines at elevation angles over 82.5 degrees (or under 7.5 degrees) were not used. This is consistent with the fact that the sporadically bad results began to occur when the perigee of Telstar I had precessed sufficiently to make the passes at Andover substantially shorter than they were during the first week, and a substantially larger proportion of the passes had high maximum elevation angles. Ridgway's studies showed that, under these conditions, an error of 10^5 feet in the single-pass estimate of the semimajor axis could easily be due to a bias of 0.01 degree in the elevation angles.

In order to overcome the sensitivity of a single-pass single-tracker angles-only method to a bias error in the elevation angles, on short passes with high maximum elevation angles, A. J. Claus has developed a method of orbit refinement which permits the use of a few, perhaps only one or two, measurements of range in each pass. This method was intended to be used in the intrapass stage of the pass-by-pass method described in Section IV, but it may be used as a self-sufficient method, just as Swerling's method may be used either in the intrapass stage of the pass-by-pass method or as a self-sufficient method.

Although Claus developed his method with no foreknowledge of Swerling's method, it turns out that his method differs from Swerling's method only in the explicit introduction of an iterative routine which, as in the classical method, improves the estimates of the orbital elements.

Instead of (12) we now substitute (1) into (11) so that

$$Q = (\epsilon - \hat{\epsilon}_1)' \cdot \hat{C}_1^{-1} \cdot (\epsilon - \hat{\epsilon}_1) + [r - J \cdot (\epsilon - \epsilon_0)]' \cdot \Phi^{-1} \cdot [r - J \cdot (\epsilon - \epsilon_0)], \tag{29}$$

where J and r are defined by (2) and (3), and ϵ_0 has the same signifi-

cance as in the classical method. Now, Q is minimum with respect to ϵ if

$$\hat{C}_1^{-1} \cdot (\epsilon - \hat{\epsilon}_1) - J' \cdot \Phi^{-1} \cdot [r - J \cdot (\epsilon - \epsilon_0)] = 0.$$

Denoting the value of ϵ which satisfies this equation by $\bar{\epsilon}$, we have

$$\bar{\epsilon} = C \cdot [\hat{C}_1^{-1} \cdot \hat{\epsilon}_1 + J' \cdot \Phi^{-1} \cdot J \cdot \epsilon_0 + \rho], \tag{30}$$

where

$$C = [\hat{C}_1^{-1} + J' \cdot \Phi^{-1} \cdot J]^{-1}, \tag{31}$$

$$\rho = J' \cdot \Phi^{-1} \cdot r. \tag{32}$$

Equation (30) may be expressed in the form

$$\bar{\epsilon} = \hat{\epsilon}_1 + C \cdot [J' \cdot \Phi^{-1} \cdot J \cdot (\epsilon_0 - \hat{\epsilon}_1) + \rho], \tag{33}$$

where C, J, and ρ (the latter through r as well as J) depend implicitly on ϵ_0. Comparing this with (15), it will be seen that the iterative use of (30) or (33), substituting $\bar{\epsilon}$ for ϵ_0 at each iteration, until the difference between ϵ_0 and $\bar{\epsilon}$ is negligible, is precluded in Swerling's method by the constraint $\epsilon_0 \equiv \hat{\epsilon}_1$.

Finally, since (29) may be expressed in the form

$$Q = (\epsilon - \bar{\epsilon})' \cdot C^{-1} \cdot (\epsilon - \bar{\epsilon}) + \text{terms independent of } \epsilon, \tag{34}$$

it follows that C is the covariance matrix of $\bar{\epsilon}$.

VI. BATTIN'S METHOD

This method, as described in Ref. 14, is essentially a special case of Swerling's method. However, by introducing new data one at a time, Battin's method avoids the inversion of matrices. (Swerling's JAS article contains equations whereby matrix inversions are avoided if new data are introduced one at a time, but these equations appear at the end of the section entitled "Statistics of Propagated Errors," and their use for avoiding matrix inversions is not explicitly stated.)

In the notation of Section III, (15) may be written in the form

$$\bar{\epsilon} = \hat{\epsilon}_1 + [J' \cdot J + \sigma^2 \hat{C}_1^{-1}]^{-1} \cdot J' \cdot r, \tag{35}$$

where σ^2 is the variance of the scalar $\bar{\varphi}$, J is a 1 by 6 matrix (i.e., J' is a 6-rowed vector), and r is a scalar. Now, if

$$a = J \cdot \hat{C}_1 \cdot J' + \sigma^2 \quad \text{(a scalar)}, \tag{36}$$

then,

$$J' = J' \cdot \frac{J \cdot \hat{C}_1 \cdot J' + \sigma^2}{a},$$

$$= \frac{1}{a}[J' \cdot J \cdot \hat{C}_1 + \sigma^2] \cdot J',$$

$$= \frac{1}{a}\left[J' \cdot J + \sigma^2 \hat{C}_1^{-1}\right] \cdot \hat{C}_1 \cdot J'. \tag{37}$$

Substituting (37) for the last J' in (35), we get

$$\bar{\epsilon} = \hat{\epsilon}_1 + \frac{r}{a} \hat{C}_1 \cdot J'. \tag{38}$$

Further, since (16) may be written in the form

$$C = \sigma^2[J' \cdot J + \sigma^2 \hat{C}_1^{-1}]^{-1},$$

then,

$$C - \hat{C}_1 = \sigma^2[J' \cdot J + \sigma^2 \hat{C}_1^{-1}]^{-1} - \hat{C}_1,$$

and, therefore,

$$[J' \cdot J + \sigma^2 \hat{C}_1^{-1}] \cdot (C - \hat{C}_1) = -J' \cdot J \cdot \hat{C}_1.$$

Substituting (37) for J' in the right-hand member, we get

$$C = \hat{C}_1 - \frac{1}{a} \hat{C}_1 \cdot J' \cdot J \cdot \hat{C}_1. \tag{39}$$

Equations (36), (38), and (39) are equivalent to equations (30), (29), and (33) in Battin's paper (Ref. 14).

From this description of Battin's method, it is evident that the inversion of matrices may be avoided also in Claus's method if new data (perhaps only the range data) are introduced one at a time. Comparing (33), (31), and (32) with (15), (16), and (17), it is clear that (38) and (39), with (36), are valid in Claus's method if

$$r = \tilde{\varphi} - \varphi(\epsilon_0) + J \cdot (\epsilon_0 - \hat{\epsilon}_1), \tag{40}$$

and

$$J = \partial\varphi(\epsilon_0)/\partial\epsilon_0. \tag{41}$$

The initial value of ϵ_0 may be taken equal to $\hat{\epsilon}_1$, but thereafter $\bar{\epsilon}$ is repeatedly substituted for ϵ_0 until the difference between $\bar{\epsilon}$ and ϵ_0 is negligible.

VII. MOD-S AND MOD-C METHODS

Swerling's method and Claus's method can be modified to permit the introduction of new data n at a time, where $n < 6$, at the cost of having to invert a matrix of order n. This might be worthwhile for $n = 2$ or 3.

With regard to Swerling's method, let

$$A = J \cdot \hat{C}_1 \cdot J' + \Phi \qquad (\text{an } n \text{ by } n \text{ matrix}). \tag{42}$$

Then,

$$\begin{aligned} J' \cdot \Phi^{-1} &= J' \cdot \Phi^{-1} \cdot [J \cdot \hat{C}_1 \cdot J' + \Phi] \cdot A^{-1} \\ &= [J' \cdot \Phi^{-1} \cdot J + \hat{C}_1^{-1}] \cdot \hat{C}_1 \cdot J' \cdot A^{-1}. \end{aligned} \tag{43}$$

Substituting this into (17), substituting the resultant expression for ρ into (15), and taking account of (16), we get

$$\bar{\epsilon} = \hat{\epsilon}_1 + \hat{C}_1 \cdot J' \cdot A^{-1} \cdot r. \tag{44}$$

Equation (36) is a special case of (42), (38) is a special case of (44), and (39) is a special case of

$$C = \hat{C}_1 - \hat{C}_1 \cdot J' \cdot A^{-1} \cdot J \cdot \hat{C}_1 . \tag{45}$$

With regard to Claus's method, (42), (44), and (45) are valid if r and J are defined by (40) and (41), and the repeated substitution of $\bar{\epsilon}$ for ϵ_0 is carried out until the difference between ϵ_0 and $\bar{\epsilon}$ is negligible.

VIII. SUMMARY AND CLOSING REMARKS

8.1 *Summary*

The classical "differential corrections" method of orbit refinement, occasionally supplemented by the use of "normal places," is appropriate for astronomical bodies whose relative positions change very slowly, whose relative angular positions, viewed from the earth, can be measured with extreme optical precision, and which therefore require comparatively small quantities of observational data to establish their orbits with great accuracy. However, that method is very unwieldy for artificial earth satellites or short-range space probes, where the relative inaccuracy of the observational data must be offset by greater quantities of observational data.

For artificial earth satellites or short-range space probes, Swerling's method is more practical, chiefly because it does not require all of the observational data to be processed together. However, to omit any part

of the old observational data which has been processed by Swerling's method it is necessary to start all over again in the sense that all of the old observational data which are to be retained must be reprocessed in the same way, along with any new observational data which might be available.

The pass-by-pass method, on the other hand, operates on the principle of computing an independent set of estimates of the orbital elements for each pass, and combining the sets of single-pass estimates in an optimum way. Thus, entire blocks of observational data may be omitted without actually reprocessing any of the old observational data from which single-pass estimates of the orbital elements have already been computed. The method of obtaining the single-pass (intrapass) estimates is optional. In the Telstar I experiments, it consisted in dividing up the data (no range data) into mutually interlaced independent sets of four sightlines, computing a set of orbital elements from each set of four sightlines, averaging over the sets of orbital elements, and computing a covariance matrix for the average set, from the residuals. This particular intrapass method introduces biases (apart from the biases in the data) and a method of eliminating these computational biases was developed but was not used in the Telstar I experiments. After four weeks of successful operation, a more serious source of trouble arose, which was traced to the increasing sensitivity to bias (residual boresight error and sample bias) in the elevation angle data. This increasing sensitivity was associated with the precession of perigee to latitudes close to that of the tracker.

To overcome the sensitivity of the angles-only intrapass method to bias in the elevation angle data, Claus developed a method, intended to be used chiefly in the intrapass stage of the pass-by-pass method for Telstar II, which could accept occasional range data. This method is essentially Swerling's method supplemented by an iterative routine which improves its accuracy.

In Swerling's or Claus's method, the inversion of six-by-six or higher-order matrices can be avoided by borrowing a detail from Battin's method of spacecraft navigation, provided that the observational data are processed only one at a time, as in Battin's method. However, at the cost of inverting n-by-n matrices, where $n < 6$ (say, 2 or 3), the observational data may be processed n at a time, by modified forms of Swerling's or Claus's method.

The comparisons of the methods described in this paper have been made only to the extent that motivated the development of the newer methods. The practical details of these methods are interchangeable to

a large extent, so that different and more appropriate combinations of these details may be made for other specific practical applications.

8.2 *Closing Remarks*

Claus has pointed out that, in principle, none of the alternative methods described in this paper, including the use of normal places in the classical method, can be as efficient, in a statistical average sense, as the classical method without normal places. The reason for this is simply that the classical method without normal places allows the maximum possible freedom in fitting the estimates of the orbital elements to the observational data. Hence, the choice of a method, or combination of methods, for a particular application, usually involves a small sacrifice in accuracy.

IX. ACKNOWLEDGMENTS

I wish to thank T. M. Burford, A. J. Claus, S. Darlington, F. T. Geyling, and F. W. Sinden for many comments on, and enlightening discussions of, the subject of this paper.

APPENDIX

Equations (20) and (21) may be derived in another way (essentially that used by Aitken in Ref. 3) which does not depend upon the minimization of a quadratic form and provides further insight into the significance of these equations.

Let x be the true value of an n-rowed vector (one-column matrix). Let $\tilde{x}$ be an unbiased estimate of x, with

$$\text{ave}\,\{(\tilde{x} - x)\cdot(\tilde{x} - x)'\} = \tilde{C},$$

where the prime stands for transposition and "ave" stands for ensemble average. The nth-order square matrix $\tilde{C}$ is the covariance matrix of $\tilde{x}$.

Let $\hat{x}$ be another unbiased estimate of x, with

$$\text{ave}\,\{(\hat{x} - x)\cdot(\hat{x} - x)'\} = \hat{C},$$

and let it be assumed that $\tilde{x}$ and $\hat{x}$ are independent, so that

$$\text{ave}\,\{(\tilde{x} - x)\cdot(\hat{x} - x)'\} = 0.$$

Now, consider the weighted linear average

$$\bar{x} = \tilde{W}\cdot\tilde{x} + \hat{W}\cdot\hat{x} \tag{46}$$

where $\tilde{W}$ and $\hat{W}$ are nth-order square matrices, with

$$\tilde{W} + \hat{W} = I \tag{47}$$

where I is the nth-order unit matrix. Since

$$\bar{x} - x = \tilde{W} \cdot (\tilde{x} - x) + \hat{W} \cdot (\hat{x} - x),$$

it readily follows that if

$$\bar{C} = \text{ave}\,\{(\bar{x} - x) \cdot (\bar{x} - x)'\}$$

then

$$\bar{C} = \tilde{W} \cdot \tilde{C} \cdot \tilde{W}' + \hat{W} \cdot \hat{C} \cdot \hat{W}'. \tag{48}$$

Each of the diagonal elements of $\bar{C}$, viz.,

$$\bar{C}_{ii} = \sum_{j,k} (\tilde{W}_{ij} \cdot \tilde{C}_{jk} \cdot \tilde{W}_{ik} + \hat{W}_{ij} \cdot \hat{C}_{jk} \cdot \hat{W}_{ik})$$

will be minimized under constraints on $\tilde{W}_{ik} + \hat{W}_{ik}$ by minimizing

$$\bar{C}_{ii} - 2 \sum_{k} \lambda_{ik} \cdot (\tilde{W}_{ik} + \hat{W}_{ik})$$

where the λ_{ik}'s are Lagrange multipliers. This requires that

$$\sum_{j} \tilde{W}_{ij} \cdot \tilde{C}_{jk} = \lambda_{ik}, \qquad \sum_{j} \hat{W}_{ij} \cdot \hat{C}_{jk} = \lambda_{ik},$$

for every i, k. In matrix notation,

$$\tilde{W} \cdot \tilde{C} = \lambda, \qquad \hat{W} \cdot \hat{C} = \lambda$$

where λ is the nth-order square matrix of the λ_{ik}'s. Hence,

$$\tilde{W} = \lambda \cdot \tilde{C}^{-1}, \qquad \hat{W} = \lambda \cdot \hat{C}^{-1}, \tag{49}$$

where, to satisfy (47),

$$\lambda = (\tilde{C}^{-1} + \hat{C}^{-1})^{-1}. \tag{50}$$

By (48) and (49)

$$\bar{C} = \lambda \cdot (\tilde{C}^{-1} + \hat{C}^{-1}) \cdot \lambda',$$

$$= \lambda' \text{ by (50)},$$

but, since λ is a symmetrical matrix,

$$\bar{C} = \lambda. \tag{51}$$

Finally, by (46), (49), and (51),

$$\bar{x} = \bar{C} \cdot (\tilde{C}^{-1} \cdot \tilde{x} + \hat{C}^{-1} \cdot \hat{x}), \tag{52}$$

where, by (50) and (51),

$$\bar{C} = (\tilde{C}^{-1} + \hat{C}^{-1})^{-1}. \tag{53}$$

This derivation shows that the diagonal elements of $\bar{C}$, which are the variances of the components of $\bar{x}$, have been minimized *independently of one another.*

REFERENCES

1. Blackman, R. B., Smoothing and Prediction of Time Series by Cascaded Simple Averages, 1960 I.R.E., Internatl. Conv. Record, **8**, Part 2, pp. 47–54, and I.R.E. Trans. Circuit Theory, **CT-7**, Special Supp., Aug., 1960, pp. 136–143.
2. Baker, R. M. L., Jr., and Makemson, M. W., *An Introduction to Astrodynamics*, Academic Press, New York, 1960.
3. Aitken, A. C., On Least Squares and Linear Combination of Observations, Proc. Roy. Soc. Edinb. A, **55**, 1934, pp. 42–47.
4. David, F. N., and Neyman, J., Extension of the Markoff Theorem on Least Squares, Statistical Research Memoir, London, **2**, 1938, pp. 105–116.
5. Plackett, R. L., A Historical Note on the Method of Least Squares, Biometrika, **36**, 1949, pp. 458–460.
6. Cohen, E. R., The Basis for the Criterion of Least Squares, Rev. Mod. Phys., **25**, 1953, pp. 709–713.
7. Swerling, P., A Proposed Stagewise Differential Correction Procedure for Satellite Tracking and Prediction, Rand Corporation Report P-1292, Jan. 8, 1958.
8. Swerling, P., First Order Error Propagation in a Stagewise Smoothing Procedure for Satellite Observations. J. Astronautical Sci., **6**, Autumn, 1959, pp. 46–52.
9. Swerling, P., Comment on A Statistical Optimizing Navigation Procedure for Space Flight, J. Amer. Inst. of Aeronautics and Astronautics, **1**, Aug. 1963, p. 1968.
10. Blackman, R. B., unpublished work.
11. Claus, A. J., Orbit Determination for Communication Satellites from Angular Data Only, paper delivered at the American Rocket Society 17th Annual Meeting, Nov. 13–18, 1962, Los Angeles.
12. Claus, A. J., Blackman, R. B., Halline, E. G., and Ridgway, W. C., III, Orbit Determination and Prediction, and Computer Programs, B.S.T.J., **42**, July, 1963, pp. 1357–1382.
13. Claus, A. J., private communication.
14. Battin, R. H., A Statistical Optimizing Navigation Procedure for Space Flight, Jour. ARS, **32**, Nov., 1962, pp. 1681–1696.