On the Response of Nonlinear Control Systems to Periodic Input Signals

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In this paper we study a broad class of nonlinear control systems containing a single memoryless nonlinear element. We present conditions under which there exists a unique periodic response, with a given period, to an arbitrary periodic input with the same period, and we derive an upper bound on the mean-square error incurred by applying the describing-function technique. The expression for the error reflects the intuitive engineering arguments that are often employed to justify the use of the describing-function method. Conditions are also presented under which subharmonic response components and self-sustained oscillations cannot occur.

I. INTRODUCTION

The describing-function technique is often used to determine the response of nonlinear control systems to sinusoidal input signals. In this approach,* which is applicable to systems of any order but which is ordinarily restricted to systems containing only one nonlinear element, it is assumed that the response is periodic, with only the component at the input frequency significant.

Although the describing-function technique is of considerable practical value and indeed is one of the most powerful analytical tools available to the control system synthesist, it appears that, except with regard to predicting the existence of self-sustained oscillations,[6] there has been no rigorous discussion of its validity.†

In this paper we study a broad class of nonlinear control systems containing a single memoryless nonlinear element. We present conditions under which there exists a unique periodic response, with a given period, to an arbitrary periodic input with the same period, and we

* The describing-function technique was discovered independently by engineers in at least five different countries.[1-5]

† However, some interesting relevant ideas have been presented by Johnson.[7]

derive an upper bound on the mean-square error incurred by applying the describing-function technique. The expression for the error reflects the intuitive engineering arguments that are often employed to justify the use of the describing-function method. Conditions are also presented under which subharmonic response components and self-sustained oscillations cannot occur.

Some mathematical preliminaries are considered in Section II. In Section III we describe the physical system to be studied, introduce some assumptions and notation, and discuss the describing-function technique. The remaining sections are concerned with mathematical results relating to the functional equation that governs the behavior of the physical system.

II. MATHEMATICAL PRELIMINARIES

Let $\mathcal{R} = [\Theta,\rho]$ be an arbitrary metric space.[8] A mapping $\mathbf{A}$ of the space $\mathcal{R}$ into itself is said to be a contraction if there exists a number $k < 1$ such that

$$\rho(\mathbf{A}x,\mathbf{A}y) \leqq k\rho(x,y)$$

for any two elements $x,y \in \Theta$. The contraction-mapping fixed-point theorem[8] is basic to much of the subsequent discussion. It states that every contraction-mapping defined in a complete metric space $\mathcal{R}$ has one and only one fixed point (i.e., there exists a unique element $z \in \Theta$ such that $\mathbf{A}z = z$). Furthermore $z = \lim_{n\to\infty} \mathbf{A}^n x_0$, where x_0 is an arbitrary element of Θ.

Let T be a real positive constant. The space of real-valued periodic functions of t with period T which are square-integrable over a period is denoted by $\mathcal{K}$. The norm of $g \in \mathcal{K}$ is denoted by $\| g \|$ and is defined by

$$\| g \|^2 = \frac{1}{T} \int_0^T g^2 dt$$

(i.e., $\| g \|$ is the rms value of g). With this norm $\mathcal{K}$ is a Banach space.

If $g \in \mathcal{K}$,

$$g = \underset{N\to\infty}{\text{l.i.m.}} \sum_{n=-N}^{N} g_n e^{in\omega_0 t}$$

where $\omega_0 = 2\pi/T$ and the Fourier coefficients g_n are given by

$$g_n = \frac{1}{T} \int_0^T g(t) e^{-in\omega_0 t} dt.$$

Parseval's identity reads:

$$\sum_{-\infty}^{\infty} | g_n |^2 = \| g \|^2.$$

Two elements of $\mathcal{K}$, g and h, are equivalent if $\| g - h \| = 0$.

In accordance with the usual notation, the norm of a linear operator $\mathbf{Q}$ defined on $\mathcal{K}$ is denoted by $\| \mathbf{Q} \|$.

The symbol $\mathfrak{L}_{1R}$ denotes the space of real-valued absolutely integrable functions defined on the real interval $(-\infty, \infty)$. We take as the definition of the Fourier transform of $f(t) \varepsilon \mathfrak{L}_{1R}$:

$$F(i\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt.$$

The symbol $\mathbf{I}$ is used throughout to denote the identity operator.

III. DESCRIPTION OF THE PHYSICAL SYSTEM, THE DESCRIBING-FUNCTION TECHNIQUE, AND THE PROJECTION OPERATOR P

We shall be concerned with the familiar nonlinear control system shown in Fig. 1.

Assumption I: It is assumed throughout that $\mathbf{F}$ *(in Fig. 1) is a linear operator. Let* $\mathfrak{F} = \{ \cdots, F_{-2}, F_{-1}, F_0, F_1, F_2, \cdots \}$ *denote a countable set of complex constants such that* $\sup_n | F_n | < \infty$ *and* F_n *is equal to the complex conjugate of* F_{-n}. *Unless stated otherwise, it is assumed that the restriction of* $\mathbf{F}$ *to* $\mathcal{K}$ *is a bounded linear mapping of* $\mathcal{K}$ *into itself with the property that if* $g \varepsilon \mathcal{K}$ *and* $h = \mathbf{F}g$, *then* $h_n = F_n g_n$ *in which* g_n *and* h_n, *respectively, are the nth Fourier coefficients of* g *and* h. *(According to the Riesz-Fischer theorem,* $\mathbf{F}$ *is completely defined on* $\mathcal{K}$ *by* $\mathfrak{F}$.*)*

The class of operators consistent with Assumption I includes the important special case in which

$$\mathbf{F}g = \int_{-\infty}^{\infty} f(t - \tau)g(\tau)d\tau, \qquad g \varepsilon \mathcal{K}$$

where $f(t) \varepsilon \mathfrak{L}_{1R}$ (see Appendix A). Here $F_n = F(in\omega_0)$ where $F(i\omega)$ is the Fourier transform of $f(t)$.

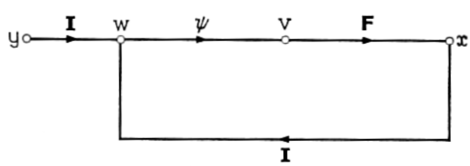

Fig. 1 — Nonlinear control system.

Assumption II: The nonlinear function ψ *in Fig. 1, which introduces the constraint* $v(t) = \psi[w(t)]$, *is assumed throughout to be real-valued, independent of t, and such that there exist two real constants* α *and* β $(\beta > 0)$ *with the properties that* $\frac{1}{2}(\alpha + \beta) = 1$ *and*

$$\alpha(\mu_1 - \mu_2) \leqq \psi(\mu_1) - \psi(\mu_2) \leqq \beta(\mu_1 - \mu_2)$$

for any real $\mu_1 \geqq \mu_2$.

The normalization $\frac{1}{2}(\alpha + \beta) = 1$ permits some simplification in the subsequent statements of results.

In Fig. 1 the input signal y is related to the output signal x by the functional equation

$$x = \mathbf{F}\psi[x + y].$$

3.1 *The Sinusoidal Response of the System in Fig. 1*

The response of the system in Fig. 1 to a sinusoidal input is frequently of engineering interest. Typically the system is of high order* so that the well-known special techniques applicable to second-order systems cannot be used. The describing-function approach simplifies the problem by assuming that the output is periodic and that the only significant frequency component of the output is that component at the input frequency. Hence it is assumed that the input to the nonlinear device is a sinusoid and that ψ is characterized by the ratio of the fundamental component of its output to the amplitude of the sinusoidal input (this ratio is called the describing function for ψ). Thus the nonlinear element is treated as an element with a gain that varies with input signal level, and to the extent that the describing-function approximation (sometimes called the "first harmonic approximation") is valid, the usual frequency response methods can be employed.

The first harmonic approximation is often "justified" on three grounds: first, no significant subharmonic components of $x(t)$ are ordinarily present; second, the harmonics of the output of ψ are ordinarily of smaller amplitude than the fundamental and, third, in most feedback systems **F** behaves as a low-pass filter with the result that the higher harmonics are significantly attentuated.

Aside from at least two computational difficulties[9] associated with the describing-function method, which can be remedied to a considerable extent with machine aids, "The third and most basic difficulty is related to the inaccuracy of the method and, in particular, to the

* That is, the nonlinear differential equation governing the system is typically of high order.

uncertainty throughout the analysis about the accuracy. There is [in the literature] no simple method for evaluating the accuracy of the describing-function analysis of a nonlinear system and no definite assurance that the results derived with the describing function are even approximately correct."[9] However, it should not be inferred that the accuracy is necessarily poor.[9,10] "Indeed the correlation between experimental and theoretical results is in many cases better than the accuracy of the design data."[9]

3.2 *The Role of the Projection Operator* **P**

A moment's reflection will show that the describing-function technique as applied to the system in Fig. 1 amounts to analyzing the approximating system that results by replacing the operator **F** with the operator $\tilde{\mathbf{F}}$ defined by

$$\frac{1}{T}\int_0^T [\tilde{\mathbf{F}}g]e^{-in\omega_0 t}dt = F_n g_n , \qquad n = \pm 1$$

$$= 0, \qquad n \neq \pm 1$$

where $g \in \mathcal{K}$, g_n is the nth Fourier coefficient of g and $T = 2\pi/\omega_0$ is the period of the input sinusoid.

At this point it is convenient to introduce

Definition I: Let $\mathfrak{N}$ denote a set of integers such that $-m \in \mathfrak{N}$ if $m \in \mathfrak{N}$. Let g be an arbitrary element of $\mathcal{K}$ with nth Fourier coefficient g_n. The projection operator **P** *is a linear mapping of $\mathcal{K}$ into itself defined by*

$$\frac{1}{T}\int_0^T [\mathbf{P}g]e^{-in\omega_0 t}dt = g_n , \qquad n \in \mathfrak{N}$$

$$= 0, \qquad n \notin \mathfrak{N}.$$

An obvious generalization of the describing-function technique is to take as the approximating system the system that is obtained by replacing **F** in Fig. 1 with **PF**. The results to be presented relate to this more general situation. Of course in the case of principal interest, $\mathfrak{N} = \{-1,1\}$ and $\mathbf{PF} = \tilde{\mathbf{F}}$.

IV. RESULTS RELATING TO THE FUNCTIONAL EQUATION $x = \mathbf{F}\psi[x + y]$

The proof of the following simple preliminary result is given in Appendix B.

Theorem I:

$$\| \mathbf{F} \| = \sup_n | F_n |.$$

If $\inf_n |1 - F_n| > 0$, *the operator* $(\mathbf{I} - \mathbf{F})$ *possesses a bounded inverse on* $\mathcal{K}$ *and*

$$\|(\mathbf{I} - \mathbf{F})^{-1}\mathbf{F}\| = \sup_n \left| \frac{F_n}{1 - F_n} \right|.$$

The principal result of this section is

Theorem II: Let $\mathbf{F}$, ψ, *and* β *be as defined in Section III. Let* $y \in \mathcal{K}$. *Suppose that*

$$r = \sup_n \left| \frac{F_n}{1 - F_n} \right| (\beta - 1) < 1.$$

Then there exists a unique $x \in \mathcal{K}$ *such that* $x = \mathbf{F}\psi[x + y]$. *In fact,* $x = \lim_{m\to\infty} x_m$ *where*

$$x_{m+1} = (\mathbf{I} - \mathbf{F})^{-1}\mathbf{F}\{\psi[x_m + y] - x_m\}$$

and x_0 *is an arbitrary element of* $\mathcal{K}$. *The mth approximation* x_m *satisfies*

$$\|x_m - x\| \leqq \frac{r^m}{1 - r} \|x_1 - x_0\|.$$

Proof:

Let $\psi[w] = \psi_0 w + \tilde{\psi}[w]$, where ψ_0 is a real constant such that $\inf_n |1 - \psi_0 F_n| > 0$ (since $r < 1$, there exists such a ψ_0). According to Theorem I, $(\mathbf{I} - \psi_0\mathbf{F})$ possesses a bounded inverse on $\mathcal{K}$. Hence the functional equation $x = \mathbf{F}\psi[x + y]$ can be written as $x = \mathbf{M}x$ where

$$\mathbf{M}x = (\mathbf{I} - \psi_0\mathbf{F})^{-1}\mathbf{F}\tilde{\psi}[x + y] + \psi_0(\mathbf{I} - \mathbf{F})^{-1}\mathbf{F}y.$$

In order to prove Theorem II it is sufficient to consider the case in which $\psi_0 = 1$. However, we prefer to bring out the fact that this choice of ψ_0, the median of α and β, is optimal in a significant sense.

It is evident that $\mathbf{M}$ is a mapping of $\mathcal{K}$ into itself. Let us consider the determination of a condition under which $\mathbf{M}$ is in fact a contraction-mapping of $\mathcal{K}$ into itself. Let $g,h \in \mathcal{K}$ and observe that

$$\begin{aligned}\|\mathbf{M}g - \mathbf{M}h\| &= \|(\mathbf{I} - \psi_0\mathbf{F})^{-1}\mathbf{F}\{\tilde{\psi}[g + y] - \tilde{\psi}[h + y]\}\| \\ &\leqq \|(\mathbf{I} - \psi_0\mathbf{F})^{-1}\mathbf{F}\|\|\tilde{\psi}[g + y] - \tilde{\psi}[h + y]\|.\end{aligned}$$

Since

$$\|\tilde{\psi}[g + y] - \tilde{\psi}[h + y]\| = \left\| \left(\frac{\psi[g + y] - \psi[h + y]}{g - h} - \psi_0 \right) (g - h) \right\|,$$

$$\|\tilde{\psi}[g + y] - \tilde{\psi}[h + y]\| \leqq \eta(\psi_0)\|g - h\|$$

where

$$\begin{aligned}\eta(\psi_0) &= \beta - \psi_0, \qquad \psi_0 \leqq 1\\ &= \psi_0 - \alpha, \qquad \psi_0 \geqq 1.\end{aligned}$$

Thus

$$\| \mathbf{M}g - \mathbf{M}h \| \leqq \|(\mathbf{I} - \psi_0\mathbf{F})^{-1}\mathbf{F} \| \, \eta(\psi_0) \| g - h \|,$$

and **M** is a contraction if

$$q(\psi_0) = \|(\mathbf{I} - \psi_0\mathbf{F})^{-1}\mathbf{F} \| \, \eta(\psi_0) < 1.$$

Using Theorem I,

$$q(\psi_0) = \sup_n \left| \frac{F_n}{1 - \psi_0 F_n} \right| \eta(\psi_0).$$

Assuming that there exists a ψ_0 such that $q(\psi_0) < 1$, the following result, which is proved in Ref. 11, implies that $\inf_{\psi_0} q(\psi_0) = q(1)$.

Lemma I: Let ξ be a complex number and suppose that $| \xi - \psi_0 |^{-1}\eta(\psi_0) < 1$. Then

$$| \xi - \psi_0 |^{-1}\eta(\psi_0) \geqq | \xi - 1 |^{-1}\eta(1).$$

From this point on we assume that $\psi_0 = 1$ and we set $q(1) = r$. Thus the assumptions stated in Theorem II imply that **M** is a contraction. In view of the contraction-mapping fixed-point theorem, this establishes the existence and uniqueness of the function $x(t)$ and the fact that it can be determined in accordance with the stated iteration procedure.

The upper bound on $\| x_m - x \|$ follows directly from the fact that x can be written as

$$x = x_0 + \sum_{j=0}^{\infty} [x_{(j+1)} - x_j],$$

in which, for all $j \geqq 1$

$$\| x_{(j+1)} - x_j \| = \| \mathbf{M}x_j - \mathbf{M}x_{(j-1)} \| \leqq r \, \| x_j - x_{(j-1)} \|.$$

Remarks:

Observe that a nontrivial self-sustained periodic oscillation with period T cannot exist in the system of Fig. 1 if the hypotheses of Theorem II are satisfied and $\psi(0) = 0$ (since then $y = 0$ implies that $x = 0$).

When **F** is defined by

$$\mathbf{F}g = \int_{-\infty}^{\infty} f(t - \tau) g(\tau) d\tau$$

where $f(t) \varepsilon \mathcal{L}_{1R}$, Theorem II implies the following simple necessary condition for the occurrence of jump-resonance phenomena[9] in the system of Fig. 1:

$$\sup_{\omega} \left| \frac{F(i\omega)}{1 - F(i\omega)} \right| (\beta - 1) \geqq 1.$$

For example, let $F(i\omega) = -k[(i\omega + a)(i\omega + 1)]^{-1}$ where k and a are positive constants. Let

$$\begin{aligned} \psi[w] &= 2w, & |w| &\leqq c \\ &= 2c \operatorname{sign}(w), & |w| &> c \end{aligned}$$

where c is a positive constant. Then it is a routine matter to show that jump-resonance phenomena can occur only if $k > \frac{1}{2}(1 + a^2)$.

4.1 *Two Further Consequences of Theorem II*

Corollary I: Suppose that the hypotheses of Theorem II are satisfied. Then

$$\| x \| \leqq \frac{1}{1 - r} \| (\mathbf{I} - \mathbf{F})^{-1} \mathbf{F} \psi[y] \|.$$

Proof:

Set $m = 0$ and $x_0 = 0$ in the upper bound for $\| x_m - x \|$.

Corollary II: Suppose that the hypotheses of Theorem II are satisfied and that $\hat{x} \varepsilon \mathcal{K}$ satisfies $\hat{x} = \mathbf{PF}\psi[\hat{x} + y]$. *Then*

$$\| x - \hat{x} \| \leqq \frac{1}{1 - r} \| (\mathbf{I} - \mathbf{F})^{-1} \mathbf{F} (\mathbf{I} - \mathbf{P}) \psi[\hat{x} + y] \|.$$

Proof:

With $m = 0$ and $x_0 = \hat{x}$, the upper bound for $\| x_m - x \|$ yields

$$\| x - \hat{x} \| \leqq \frac{1}{1 - r} \| (\mathbf{I} - \mathbf{F})^{-1} \mathbf{F} \{ \psi[\hat{x} + y] - \hat{x} \} - (\mathbf{I} - \mathbf{PF})^{-1} \mathbf{PF} \{ \psi[\hat{x} + y] - \hat{x} \} \|.$$

Since $(\mathbf{I} - \mathbf{PF})^{-1} = (\mathbf{I} - \mathbf{P}) + (\mathbf{I} - \mathbf{F})^{-1}\mathbf{P}$ and $(\mathbf{I} - \mathbf{P})\hat{x} = 0$,

$$\| x - \hat{x} \| \leqq \frac{1}{1 - r} \| (\mathbf{I} - \mathbf{F})^{-1} \mathbf{F} (\mathbf{I} - \mathbf{P}) \psi[\hat{x} + y] \|.$$

Remarks:

Note that the hypotheses of Theorem II imply that there exists a unique $\hat{x} \varepsilon \mathcal{K}$ such that $\hat{x} = \mathbf{PF}\psi[\hat{x} + y]$.

The bound on $\| x - \hat{x} \|$ can be expressed with the aid of Parseval's identity as

$$\| x - \hat{x} \| \leqq \frac{1}{1 - r} \left(\sum_{n \notin \mathfrak{N}} \left| \frac{F_n}{1 - F_n} p_n \right|^2 \right)^{\frac{1}{2}} \tag{1}$$

where p_n is the nth Fourier coefficient of $\psi[\hat{x} + y]$. Consider the usual describing-function case in which $\mathfrak{N} = \{-1,1\}$, and y is a sinusoid with period T. Assuming that ψ is an odd function so that $p_0 = 0$, (1) clearly shows that $\| x - \hat{x} \|$ is small when the amplitudes of the harmonics* of $\psi[\hat{x} + y]$ are sufficiently small or when the attenuation of the harmonics by $\mathbf{F}$ is sufficiently large. Thus subject to the key inequality

$$\sup_n \left| \frac{F_n}{1 - F_n} \right| (\beta - 1) < 1,$$

(1) makes precise the usual intuitive engineering arguments regarding the applicability of the describing-function method for determining the sinusoidal response of the system in Fig. 1.

It can be shown[12] that if $\alpha \geqq 0$, $\psi(0) = 0$, and $y = \mathbf{P}y$:

$$\| \mathbf{P}\psi[\hat{x} + y] \| \geqq \alpha \| \hat{x} + y \|.$$

Under these conditions

$$\begin{aligned} \|(\mathbf{I} - \mathbf{P})\psi[\hat{x} + y]\| &= (\| \psi[\hat{x} + y]\|^2 - \| \mathbf{P}\psi[\hat{x} + y]\|^2)^{\frac{1}{2}} \\ &\leqq (\beta^2 - \alpha^2)^{\frac{1}{2}} \| \hat{x} + y \|, \end{aligned}$$

and

$$\begin{aligned} \| x - \hat{x} \| &\leqq \frac{1}{1 - r} \|(\mathbf{I} - \mathbf{F})^{-1}\mathbf{F}(\mathbf{I} - \mathbf{P})\| (\beta^2 - \alpha^2)^{\frac{1}{2}} \| \hat{x} + y \| \\ &\leqq \frac{1}{1 - r} \sup_{n \notin \mathfrak{N}} \left| \frac{F_n}{1 - F_n} \right| (\beta^2 - \alpha^2)^{\frac{1}{2}} \| \hat{x} + y \|. \end{aligned}$$

Under the conditions stated in Theorem II, the response $x(t)$ can be determined in accordance with an iteration procedure for which the successive approximations converge in the mean-square sense at least a geometric rate. In particular, if we take $x_0 = \hat{x}$, the solution given by the describing-function method, the second approximation is

$$x_1 = (\mathbf{I} - \mathbf{F})^{-1}\mathbf{F}\{\psi[\hat{x} + y] - \hat{x}\},$$

* Of course here all even harmonic components vanish.

which can be evaluated in a relatively simple manner once $\hat{x}$ has been determined. Using Theorem II and the expression for $\| x_1 - x_0 \|$ obtained above,

$$\| x - x_1 \| \leqq \frac{r}{1 - r} \| (\mathbf{I} - \mathbf{F})^{-1}\mathbf{F}(\mathbf{I} - \mathbf{P})\psi[\hat{x} + y] \|.$$

4.2 *An Observation Relating to the Necessity of the Assumptions in Theorem II*

The basic assumption in Theorem II is:

$$\sup_n \left| \frac{F_n}{1 - F_n} \right| (\beta - 1) < 1.$$

This inequality is satisfied if and only if the numbers $(F_n)^{-1}$ are bounded away from the disk centered in the complex plane at (1,0) and having radius $(\beta - 1)$. It is of interest to note that there is a function ψ, in fact a linear ψ, that satisfies Assumption II and possesses the property that there is no function $x(t) \varepsilon \mathcal{K}$ such that $x = \mathbf{F}\psi[x + y]$ if, for some integer k:

(*i*) $F_k \neq 0$

(*ii*) F_k is a point on the real-axis diameter of the disk mentioned above, and

(*iii*) the kth Fourier coefficient of y does not vanish.

To prove this assertion observe that if the three conditions are satisfied, $\alpha \leqq (F_k)^{-1} \leqq \beta$, and $x = \mathbf{F}\psi[x + y]$ with $\psi[w] = (F_k)^{-1}w$, possesses no solution belonging to $\mathcal{K}$.

This observation suggests that the assumptions made in Theorem II are not too far from being necessary.

4.3 *On the Boundedness of the Solution*

Theorem III: Let $\mathbf{F}$ *be defined by*

$$\mathbf{F}g = \int_{-\infty}^{\infty} f(t - \tau)g(\tau)d\tau, \qquad g \varepsilon \mathcal{K}$$

where $f \varepsilon \mathcal{L}_{1R}$, *and let* ψ *satisfy Assumption II. Let* $x \varepsilon \mathcal{K}$ *satisfy* $x = \mathbf{F}\psi[x + y]$, $y \varepsilon \mathcal{K}$. *Suppose that* $(1 + |t|)f(t)$ *is square-integrable on* $(-\infty, \infty)$. *Then* $|x(t)|$ *is uniformly bounded on* $0 \leqq t \leqq T$.

Proof:

Let $h = \psi[x + y]$ and note that $h \varepsilon \mathcal{K}$. Using the Schwarz inequality,

$$| x(t)|^2 = \left| \int_{-\infty}^{\infty} (1 + | \tau |)f(\tau) \frac{h(t - \tau)}{(1 + | \tau |)} \, d\tau \right|^2$$
$$\leqq \int_{-\infty}^{\infty} [(1 + | \tau |)f(\tau)]^2 d\tau \int_{-\infty}^{\infty} \left| \frac{h(t - \tau)}{1 + | \tau |} \right|^2 d\tau.$$

The last integral can be bounded as follows:

$$\int_{-\infty}^{\infty} \left| \frac{h(t - \tau)}{1 + | \tau |} \right|^2 d\tau = \sum_{n=-\infty}^{\infty} \int_{nT}^{(n+1)T} \left| \frac{h(t - \tau)}{1 + | \tau |} \right|^2 d\tau$$
$$\leqq 2\left(1 + (T)^{-2} \sum_{n=1}^{\infty} n^{-2}\right) \int_0^T | h(t)|^2 d\tau.$$

Thus $| x(t)|$ is uniformly bounded on $0 \leqq t \leqq T$.

Remarks:

The hypothesis regarding $f(t)$ is almost always satisfied in cases of engineering interest.

If the hypotheses of Theorem III are satisfied and $| y(t)|$ is uniformly bounded on $[0,T]$, it follows that $x(t)$ is continuous on $[0,T]$, since $| h(t)|$ is uniformly bounded on $[0,T]$ and

$$\int_{-\infty}^{\infty} | f(t + \delta) - f(t)| \, dt \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

V. RESULTS RELATING TO AN IMPORTANT SPECIAL CLASS OF CONTROL SYSTEMS

One frequently encounters[9,10] discussions of systems of the type shown in Fig. 1 in which ψ is an odd function and the operator corresponding to $\mathbf{F}$ is characterized in the frequency domain by a transfer function (typically a real rational function in $i\omega$) with at least one pole at $\omega = 0$. The techniques presented earlier can be applied to situations of this type if $\mathbf{F}$ is replaced with $\mathbf{F}'$, the restriction of $\mathbf{F}$ to the subspace*

$$\mathcal{K}' = \left\{ g \mid g \, \varepsilon \, \mathcal{K}; \int_0^T g(t)e^{-in\omega_0 t} dt = 0, n \text{ even} \right\}.$$

The operator $\mathbf{F}'$ is completely defined on $\mathcal{K}'$ by the set of complex numbers $\mathfrak{F}' = \{\cdots, F_{-3}, F_{-1}, F_1, F_3, \cdots\}$. The result analogous to Theorem II is

Theorem IV: Let $\mathbf{F}'$ *and* $\mathcal{K}'$ *be as defined above. Let* ψ *and* β *be as defined*

* It is a simple matter to show that the linear manifold $\mathcal{K}'$ is in fact a subspace of $\mathcal{K}$. Consider any Cauchy sequence of elements of $\mathcal{K}'$. Since $\mathcal{K}$ is complete, the sequence converges to a function g belonging to $\mathcal{K}$. However, a direct application of Parseval's identity shows that this is impossible unless $g \, \varepsilon \, \mathcal{K}'$.

in Section III, with the further qualification that $\psi(-\mu) = -\psi(\mu)$ *for any real* μ. *Let* $y \; \varepsilon \; \mathcal{K}'$. *Suppose that*

$$q = \sup_{n \text{ odd}} \left| \frac{F_n}{1 - F_n} \right| (\beta - 1) < 1.$$

Then the conclusion of Theorem II remains valid if $\mathcal{K}$ *is replaced with* $\mathcal{K}'$, r *is replaced with* q, *and* $\mathbf{F}$ *is replaced with* $\mathbf{F}'$.

Proof:

The proof is essentially the same as for Theorem II. The assumption that ψ is odd is needed to verify that the operator corresponding to $\mathbf{M}$ (with $\psi_0 = 1$) is a mapping of the Banach space $\mathcal{K}'$ into itself.*

Of course Theorem IV implies results entirely analogous to Corollaries I and II.

VI. ON THE LACK OF SUBHARMONIC COMPONENTS IN THE RESPONSE

One of the key assumptions relating to the describing-function analysis of the system in Fig. 1 is that the response $x(t)$ (assuming it exists) is a periodic function with period T when $y(t)$ is a sinusoid with period T. In particular $x(t)$ is assumed not to contain subharmonic components. The techniques described earlier can be used to obtain explicit conditions under which this assumption is valid. The following theorem contains one such result. A preliminary fact[13] that is needed is: if $f(t) \; \varepsilon \; \mathcal{L}_{1R}$ and $F(i\omega) \neq 1$ for all real ω, then there exists a function $h(t) \; \varepsilon \; \mathcal{L}_{1R}$ with Fourier transform $F(i\omega)\,[1 - F(i\omega)]^{-1}$.

Theorem V: Let η *denote the space of bounded real-valued measurable functions defined on* $(-\infty, \infty)$. *Let* $\mathbf{F}$ *be defined by*

$$\mathbf{F}g = \int_{-\infty}^{\infty} f(t - \tau) g(\tau) d\tau, \qquad g \; \varepsilon \; \eta$$

where $f \; \varepsilon \; \mathcal{L}_{1R}$ *and let* ψ *and* β *be as defined in Section III. Suppose that* $F(i\omega) \neq 1$ *for all real* ω *and that*

$$(\beta - 1) \int_{-\infty}^{\infty} |\, h(t) \,|\, dt < 1,$$

where $h(t)$ *has Fourier transform* $F(i\omega)\,[1 - F(i\omega)]^{-1}$. *Let* $y \; \varepsilon \; \eta$. *Then there exists a unique* $x \; \varepsilon \; \eta$ *such that* $x = \mathbf{F}\psi[x + y]$. *Further,* $x(t)$ *is continuous and if* $y(t + T) = y(t)$, *then* $x(t + T) = x(t)$.

* If $g \; \varepsilon \; \mathcal{K}'$, $g(t) = -g(t + \tfrac{1}{2}T)$ for almost every t. Since ψ is odd, $\mathbf{M}$ preserves this property and hence $\mathbf{M}$ maps $\mathcal{K}'$ into itself.

Proof:

Arguments very similar to those presented in Ref. 13 can be used to show that under the conditions stated in the theorem, **F** is a bounded mapping of η into itself, that the operator $(\mathbf{I} - \mathbf{F})$ possesses a bounded inverse on η and that

$$(\mathbf{I} - \mathbf{F})^{-1}\mathbf{F}g = \int_{-\infty}^{\infty} h(t - \tau)g(\tau)d\tau, \qquad g \; \varepsilon \; \eta.$$

Thus, paralleling the proof of Theorem II, the functional equation $x = \mathbf{F}[x + y]$ can be written as $x = \mathbf{L}x$ where

$$\mathbf{L}x = (\mathbf{I} - \mathbf{F})^{-1}\mathbf{F}\{\psi[x + y] - x\}.$$

Let the norm of an element of η be defined by

$$\| g \|_{\infty} = \sup_{t} | g(t) |, \qquad g \; \varepsilon \; \eta.$$

With this norm η is a Banach space. It is a routine matter to verify that under the conditions stated in the theorem **L** is a contraction mapping of η into itself. This implies the existence of a unique solution $x(t) \; \varepsilon \; \eta$.

The continuity of $x(t)$ follows directly from the fact that $x = \mathbf{F}\psi[x + y]$ in which $\psi[x + y]$ is bounded and $f \; \varepsilon \; \mathfrak{L}_{1R}$.

If $y(t + T) = y(t)$, **L** is a contraction mapping of the following subspace of η into itself: $\{g \mid g \; \varepsilon \; \eta, g(t) = g(t + T)\}$ and hence there exists a unique solution belonging to this subspace.* This completes the proof of Theorem V.

VII. FINAL REMARKS

It seems likely to this writer that the contraction-mapping fixed-point theorem, and more generally the techniques of functional analysis, can be exploited with considerable profit by the control system synthesist. Indeed one objective of this paper is to stimulate engineering interest in these techniques.

The results in this paper can be extended to cover the analogous multiloop multi-nonlinear-element case (i.e., the case in which y, w, v, and x in Fig. 1 are N-vector valued functions of t, and ψ represents N nonlinear elements of the type considered earlier). In particular, the

* Alternatively, observe that $x(t + T)$ is a solution of the functional equation when $y(t + T) = y(t)$. Since the solution is unique, $x(t + T) = x(t)$.

corresponding extension of Theorem II is given in Theorem IV of Ref. 14.*

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APPENDIX A

Let $\mathbf{F}$ be defined by

$$\mathbf{F}g = \int_{-\infty}^{\infty} f(t - \tau)g(\tau)d\tau, \qquad g \in \mathcal{K}$$

where $f(t) \in \mathcal{L}_{1R}$.

We show first that $\mathbf{F}$ is a bounded mapping of $\mathcal{K}$ into itself. Consider

$$h(t) = \int_{-\infty}^{\infty} f(t - \tau)g(\tau)d\tau.$$

Using Schwarz's inequality,

$$\begin{aligned} | h(t) | &\leqq \int_{-\infty}^{\infty} | g(t - \tau) | \cdot | f(\tau) |^{\frac{1}{2}} \cdot | f(\tau) |^{\frac{1}{2}} d\tau \\ &\leqq \left(\int_{-\infty}^{\infty} | g(t - \tau) |^2 \cdot | f(\tau) | \, d\tau \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} | f(\tau) | \, d\tau \right)^{\frac{1}{2}} \end{aligned}$$

from which

$$\int_0^T | h(t) |^2 dt \leqq \int_0^T \left[\int_{-\infty}^{\infty} | g(t - \tau) |^2 \cdot | f(\tau) | \, d\tau \right] dt \int_{-\infty}^{\infty} | f(\tau) | \, d\tau. \quad (2)$$

Since g has period T and $f \in \mathcal{L}_{1R}$,

$$\int_{-\infty}^{\infty} \left[\int_0^T | g(t - \tau) |^2 dt \right] | f(\tau) | \, d\tau = \int_0^T |g(t)|^2 \, dt \int_{-\infty}^{\infty} | f(\tau) | \, d\tau < \infty.$$

Hence Fubini's theorem implies that the order of integration in (2) can be interchanged. Thus

$$\int_0^T | h(t) |^2 dt \leqq \int_0^T | g(t) |^2 dt \left(\int_{-\infty}^{\infty} | f(\tau) | \, d\tau \right)^2$$

and since $h(t)$ is clearly real-valued and periodic in t with period T, $\mathbf{F}$ is a bounded mapping of $\mathcal{K}$ into itself.

Consider now the relation between the Fourier coefficients of h and g

* In Theorem IV of Ref. 14, the basic functional equation is written in terms of what corresponds here to w (since $x = w - y$, this is, of course, an unimportant difference), and the nonlinear functions are permitted to depend periodically on t with period T.

stated in Section III. We have

$$\int_0^T h(t)e^{-in\omega_0 t}\,dt = \int_0^T \left[\int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau\right] e^{-in\omega_0 t}dt.$$

By the same argument as before, the order of integration can be interchanged. Thus

$$\begin{aligned}\int_0^T h(t)e^{-in\omega_0 t}dt &= \int_{-\infty}^{\infty}\left[\int_0^T g(t-\tau)e^{-in\omega_0 t}dt\right] f(\tau)d\tau \\ &= \int_{-\infty}^{\infty}\left[\int_0^T g(t-\tau)e^{-in\omega_0 (t-\tau)}dt\right] e^{-in\omega_0\tau}f(\tau)d\tau \\ &= F(in\omega_0)\int_0^T g(t)e^{-in\omega_0 t}dt\end{aligned}$$

in which, clearly, $\sup_n |F(in\omega_0)| < \infty$.

APPENDIX B

Proof of Theorem I

The norm of $\mathbf{F}$ is: $\sup \{ \|\mathbf{F}g\|\,;\, g \,\varepsilon\, \mathcal{K},\, \|g\| = 1\}$. Using Parseval's identity, $\|\mathbf{F}g\|^2 = \sum_{-\infty}^{\infty} |F_n|^2 \cdot |g_n|^2$. Thus $\|\mathbf{F}\| \leqq \sup_n |F_n|$, and since it is clear that there exists a $g \,\varepsilon\, \mathcal{K}$ such that $\|g\| = 1$ and $\|\mathbf{F}g\| \geqq \sup_n |F_n| - \delta$, where δ is an arbitrary positive number, $\|\mathbf{F}\| = \sup_n |F_n|$.

Next, consider the invertibility of the operator $(\mathbf{I} - \mathbf{F})$ when $\inf_n |1 - F_n| > 0$. Let $w \,\varepsilon\, \mathcal{K}$. The hypothesis implies that

$$\sum_{-\infty}^{\infty} |1 - F_n|^{-2} |w_n|^2 < \infty,$$

where w_n is the nth Fourier coefficient of w. Thus, according to the Riesz-Fischer theorem, there exists a $z \,\varepsilon\, \mathcal{K}$ with Fourier coefficients $z_n = w_n(1 - F_n)^{-1}$. Parseval's identity implies that z satisfies the equation $(\mathbf{I} - \mathbf{F})z = w$ (in the sense that $\|(\mathbf{I} - \mathbf{F})z - w\| = 0$) and that

$$\|z\| \leqq \sup_n |1 - F_n|^{-1} \|w\|.$$

Thus $(\mathbf{I} - \mathbf{F})$ possesses a bounded inverse on $\mathcal{K}$.

The expression for $\|(\mathbf{I} - \mathbf{F})^{-1}\mathbf{F}\|$ given in Theorem I follows from arguments similar to those used to obtain the expression for $\|\mathbf{F}\|$.

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