THE BELL SYSTEM TECHNICAL JOURNAL

VOLUME XLIII JULY 1964 NUMBER 4, PART 2

Timing Errors in a Chain of Regenerative Repeaters, III

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(Manuscript received March 2, 1964)

We present here a general and rigorous theory of the jitter accumulation in a chain of regenerative repeaters. The sources of jitter are assumed to be the signal-dependent sources, as distinguished from purely random sources independent of the signal.

Our results show that while the absolute jitter and its dispersion grow without bound with the number of repeaters, the spacing and the alignment jitter remain bounded. In particular, the spacing jitter bounds are quite optimistic for most practical situations, viz., no greater than twice the absolute jitter injected at a single repeater. This result is of importance in that it ensures proper decoding of the binary signal. Its further importance is that it does ensure, in most cases, the validity of the basic model and thus the validity of other results obtained by that model. One such result shows that the alignment jitter is slowly-varying for repeaters farther along the chain. We also include some results which would be of use in computations, together with a simple example.

I. INTRODUCTION

1.1 *Purpose*

Pulse regeneration is an attractive feature of digital communication systems. A regenerator or a regenerative repeater must: (*i*) detect the presence or absence of a pulse at certain time instants which are, ideally, multiples of the basic pulse repetition period, (*ii*) regenerate the pulse shapes, and (*iii*) retime these pulses so they occur at appropriate instants of time in the outgoing signal. In practice, errors in detection due to noise, distortion, etc. in the system and errors in retiming the signal impose limitations on the operation of such systems. Except for noise which may lead to the detection of a pulse where no pulse exists in the original signal and vice versa, the imperfections in the system show up as a jittering of the pulse positions in the outgoing signal. In self-timing repeaters,[1] the jitter from all sources except random noise is also dependent on the signal itself.[2]

Several workers have investigated many different aspects of the timing jitter problem.[1-9] We study here the signal-dependent jitter due to repeated regeneration. This article leans quite heavily on our previous discussion, and in fact it supplements as well as complements the previous results referred to in this article as Parts I and II.[3]

1.2 *Background*

In our previous discussion, we showed how the timing jitter in a pulse train accumulates as the pulse train is repeatedly regenerated. The timing information was assumed to be extracted from the incoming signal itself, and the timing extractor was assumed to be a tuned circuit. Mistuning in the tuned circuit was assumed, as a convenience, to be the major source of jitter. However, the accumulation properties of timing jitter are not dependent on a particular source of jitter. We shall attempt to clarify this point here.

Our previous results showed that the displacements of the pulse positions from their original positions (or the "absolute jitter") increase indefinitely with the number of repeaters. The major component in the absolute jitter was found to be flat delay (i.e., the same displacement at every pulse position). A natural question follows: How does the absolute jitter behave if the flat delay is removed? This is the "dispersion" or the absolute jitter measured against the reference clock delayed by an appropriate amount. It was shown that the dispersion also increases without bound except when the pulse trains are severely constrained (e.g., periodic, finite, etc.). These results are valid even under the constraint that there exists at least one pulse in a predetermined number of the basic periods or "time slots."

It is worth noting that the absolute jitter and the dispersion have as counterparts the average and the variance of the random variable represented by the pulse displacement at any pulse position. We wish to emphasize, however, that our results are independent of any a priori statistics concerning the pulse train ensemble.

1.3 *Results*

In the present article, our most significant result concerns a more important parameter, viz., the "spacing jitter," which is the variation in the spacing between adjacent pulses. We show that the spacing jitter is bounded for an indefinitely long chain of repeaters, and the bound is directly related to the minimum pulse density. Such bounds may be precisely evaluated both for the infinite as well as the finite chain of repeaters. The importance of these results lies in the precise evaluation of the bounds, the means to control these bounds, and in our ability to relate these bounds either with resulting errors in the decoding of the signal,[2] or with distortion in the analog signal,[4] depending on whether the bounds are large or small. As will be seen later, these results also determine the validity or otherwise of all the other previous results on the timing problem, since the present results have a direct bearing on the validity of the model used by most people.

We also present a rather thorough discussion of the computational aspects of the problem in the Appendix. Special situations such as periodic patterns, truncations, pattern transitions, etc. are included in our discussion.

The case of nonidentical repeaters is examined briefly. It appears that the bounds on jitter are not appreciably different if the repeaters are not appreciably different. The "misalignment," or the jitter introduced by a single repeater in an already jittered pulse train, is also examined briefly. We show that the misalignment is slowly-varying for repeaters further along in the chain.

Let us emphasize, in conclusion, that our results are not dependent on any a priori statistics. Our analysis is quite rigorous once the basic model is derived. The basic model is essentially the same as that of other investigators, and the major assumption in the model asserts that a single repeater introduces only slowly-varying jitter in a jitter-free pulse train. Such an assumption is quite reasonable for any practical repeater.

1.4 *Organization*

We start with a mathematical statement of the problem. Our formulation shows that the input and the output jitter sequences are related to each other by a linear operator which maps the space of bounded sequences into itself. The dimensionality of the space is determined by the memory of the system, which is infinite for an infinitely long chain of repeaters. We are thus led to a discussion of the operator in a Banach space of infinite dimensions. The spectral properties of the operator determine the behavior of the absolute jitter and the dispersion. Next, we consider the spacing jitter obtained by a difference operation on the absolute jitter. The brief discussions on the misalignment, unequal repeaters, etc. follow. Finally, we present several results to facilitate computations.

II. STATEMENT OF PROBLEM

The basic model of the repeater is represented by a tuned circuit excited by a train of pulses. The natural frequency of the tuned circuit is assumed to be very close to the pulse repetition frequency. At upward (or downward) zero crossings of the response of the tuned circuit, timing pulses are generated which determine the instants of outgoing pulses; the presence or absence of a particular pulse in the output signal is determined by the presence or absence of a pulse in the input signal.

Let $\{\cdots, -2\tau, -\tau, 0, \tau, \cdots\}$ represent the instants of occurrence of pulses for the ideal pulse train. These would be the centers of the corresponding time slots. The occurrence or nonoccurrence of a pulse at $t = -n\tau$ is determined by the value of the random binary variable $\mathfrak{A}_n$. A pulse is present when $\mathfrak{A}_n = 1$ and no pulse is present when $\mathfrak{A}_n = 0$.

At this point it is convenient to assume that the pulse train consists of impulses located at the actual pulse positions defined above. The actual pulse shapes modify the zero crossings of the response of the tuned circuit, and a term representing such a correction can be added separately.

Finally, let ξ_k^l be the displacement of the kth pulse (originally located at $t = -k\tau$) at the output of the lth repeater in a chain of repeaters. At the input of the lth repeater the timing displacement is given by ξ_k^{l-1}, which is the jitter value at the output of the $(l - 1)$th repeater. The displacement (or jitter) is measured in radians, where 2π corresponds to the pulse interval τ.

In order to determine ξ_k^l, we merely find the appropriate zero crossing of the response of the tuned circuit excited by a train of pulses. The exciting pulse train is itself jittered and this fact is represented by the values of $\{\xi_k^{l-1}\}$. It turns out that the ξ_k^l is actually a nonlinear function of the set $\{\xi_{k+n}^{l-1}\}$ with $n = 0, 1, 2, \cdots$. Furthermore, it also depends on the original signal represented by the set $\{\mathfrak{A}_n\}$ and, of course, the Q of the resonant circuit. If, however, $\{\xi_{k+n}^{l-1}\}$ satisfy certain conditions, it is possible to represent the ξ_k^l as a linear function of the variables

$\{\xi_{k+n}{}^{l-1}\}$. The function is still dependent on $\{\mathfrak{a}_n\}$ and the Q of the circuit. This is the fundamental relation between the input and output jitter and, if there were no jitter introduced by the repeater, it would take the following form:[2]

$$\xi_k{}^l = \frac{\sum_{n=0}^{\infty} \mathfrak{a}_{n+k}\beta^n \xi_{n+k}{}^{l-1}}{\sum_{n=0}^{\infty} \mathfrak{a}_{n+k}\beta^n}, \qquad (k = 0, 1, 2, \cdots; l = 1, 2, \cdots), \quad (1)$$

where $\beta = \exp(-\pi/Q) \approx 1 - (\pi/Q)$ for large Q.

The conditions that must be satisfied by $\xi_k{}^{l-1}$ are the following:

$$|\xi_k{}^{l-1} - \xi_{k-1}{}^{l-1}| \ll \pi/Q \qquad \text{for all } k. \quad (2)$$

These conditions are unaltered when (1) is slightly modified to include the jitter introduced by each repeater [cf. (3)]. It is therefore very important to make sure that (2) holds in order for any of the results obtained on the basis of (1) to be valid. The quantity required to be small in (2) is the spacing jitter, whose behavior is important in that if it ever becomes too large there will occur errors and distortion in the decoded signal.[2,4] It is our intention here to investigate thoroughly the behavior of the spacing jitter. Equation (1) represents the way in which jitter propagates along a chain of repeaters. The jitter accumulation properties of a chain are, therefore, a consequence of this basic equation. Of course, at every repeater there is jitter injected in addition to the one propagated from previous repeaters. Since we are not discussing here effects of random noise, the sources of the injected jitter are signal-dependent, and they are identical if we assume identical repeaters. The jitter injected at every repeater by signal-dependent sources is thus identical. For such sources, this injection of jitter is simply additive either at the input end of the repeater[5] or at the output end.[2,3,6,7] For example, dispersion in the channel with a consequent imperfect detection of the pulse positions would be an additive source of jitter at the input of the repeater.[5] Certain nonlinear operations (such as limiting) on the response of the tuned circuit would inevitably alter the zero crossings, and this may be represented as an additive source at the output end of the repeater. If the mistuning of the resonant circuit is small, it can be shown that mistuning represents an additive source at the output end. Finite pulse widths also represent an additive source at the output.[2] In any case, such injection of jitter depends on the signal and repeater parameters. Since these are the same at every repeater, the injected jitter is the same at every repeater. Additivity of these sources is usually a consequence of the fact that the injected jitter is small. For convenience, we will refer all these sources to the output end and represent the injected jitter by $\{\xi_k^1\}$, the output of the first repeater, where the original pulse train is assumed to be jitter-free. It is somewhat inconvenient to refer all sources to the input, since it would involve inverting the functional relationship of (1). Our interest in this article is not to discuss the quantitative evaluation of $\{\xi_k^1\}$. For simple sources such as mistuning and finite pulse widths, it is relatively easy to evaluate $\{\xi_k^1\}$. We shall discuss elsewhere the question of determining $\{\xi_k^1\}$ when all sources are included. However, let us emphasize that we do not restrict the sources of jitter to short memory mechanisms.[5]

In this paper, our interest is to determine the behavior of $\{\xi_k^l\}$ for large l when the basic equation (1) is modified to include the injected jitter:

$$\xi_k^{\,l} = \frac{\sum_{n=0}^{\infty} \mathfrak{a}_{n+k}\beta^n \xi_{n+k}^{\,l-1}}{\sum_{n=0}^{\infty} \mathfrak{a}_{n+k}\beta^n} + \xi_k^{\,1}, \tag{3}$$

subject to condition (2), and where we assume that $\xi_k^0 = 0$. Condition (2) is satisfied by $\{\xi_k^0\}$ since they are all zero and by $\{\xi_k^1\}$ because for a practical repeater the jitter injected by a single repeater must be extremely small. We have made no assumptions on the nature of the signal. Thus, if it can be shown that condition (2) is valid for every l, then the results obtained by using (3) are valid. This is an important point which cannot be overemphasized. We will therefore pay particular attention to the behavior of the spacing jitter.

III. RECAPITULATION

In this section, we recapitulate the basic formulation of the problem developed in Parts I and II of this article. For details, the reader is referred to the original discussion. Define a vector

$$X_l = \{\xi_0^{\,l}, \xi_1^{\,l}, \xi_2^{\,l}, \cdots\}, \qquad (l = 1, 2, \cdots); \tag{4}$$

and $X_0 = 0$. Then, the basic equation (3) may be written as

$$X_l = T_0 X_{l-1} + X_1, \tag{5}$$

where

$$T_0 = \begin{bmatrix} \frac{\mathcal{A}_0}{s_0} & \frac{\mathcal{A}_1\beta}{s_0} & \frac{\mathcal{A}_2\beta^2}{s_0} & \cdots \\ 0 & \frac{\mathcal{A}_1}{s_1} & \frac{\mathcal{A}_2\beta}{s_1} & \cdots \\ 0 & 0 & \frac{\mathcal{A}_2}{s_2} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{6}$$

and

$$s_i = \sum_{n=0}^{\infty} \mathcal{A}_{n+i}\beta^n. \tag{7}$$

We have thus expressed our original problem in terms of an operator T_0 which maps the Banach space $\mathbf{1}_\infty$ (the space of bounded sequences) into itself. We are interested in the behavior of X_l as l approaches infinity. The operator T_0 is the most general one. However, there is some interest in the behavior of the jitter when the pulse trains are periodic and the operator T_0 under steady-state periodic condition (cf. Part I) becomes

$$A_0 = \begin{bmatrix} \frac{\mathcal{A}_0}{s_0'} & \frac{\mathcal{A}_1\beta}{s_0'} & \cdots & \frac{\mathcal{A}_{m-1}\beta^{m-1}}{s_0'} \\ \frac{\mathcal{A}_0\beta^{m-1}}{s_1'} & \frac{\mathcal{A}_1}{s_1'} & \cdots & \frac{\mathcal{A}_{m-1}\beta^{m-2}}{s_1'} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\mathcal{A}_0\beta}{s_{m-1}'} & \cdots & \cdots & \frac{\mathcal{A}_{m-1}}{s_{m-1}'} \end{bmatrix}, \tag{8}$$

where $s_n' = (1 - \beta^m)s_n$ for all n. Finally, it is more convenient to write the operator T_0 explicitly to indicate only those positions where the pulses are present. This leads us to the operator

$$T = \begin{bmatrix} \frac{1}{S_0} & \frac{\beta^{i_1}}{S_0} & \frac{\beta^{i_1+i_2}}{S_0} & \cdots \\ 0 & \frac{1}{S_1} & \frac{\beta^{i_2}}{S_1} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{9}$$

where

$$S_0 = s_0 \tag{10}$$

and

$$S_{n-1} = 1 + \beta^{i_n} S_n . \tag{11}$$

The vectors X_l are also assumed to be suitably modified. We will use the operator T of (9) to represent our quantities of interest.* The operator T_0 will give identical results. The special restriction to the periodic case will be of interest when we discuss computational aspects. The operator A_0 is also assumed suitably modified to A. Several parameters of interest may now be expressed in terms of the operator T and the injected jitter element X_1.

3.1 *Absolute Jitter*

From (5), we have

$$X_l = \left[\sum_{\nu=0}^{l-1} T^\nu \right] X_1 , \tag{12}$$

and in the limit

$$Y = \lim_{l \to \infty} \left[\sum_{\nu=0}^{l-1} T^\nu \right] X_1 . \tag{13}$$

3.2 *Dispersion*

This is obtained by subtracting the average delay from the absolute jitter. A delay element is represented to within a constant by $\{1,1,1, \cdots\}$. This element happens to be an eigenvector of T corresponding to an eigenvalue at $\lambda = 1$. Under the condition that $\lambda = 1$ is a pole (cf. Part II) of T, we can represent the dispersion by

$$X_l^D = (I - E_1) X_l = \left[I - E_1 \right] \left[\sum_{\nu=0}^{l-1} T^\nu \right] X_1 , \tag{14}$$

where E_1 is the projection operator E_1 ($\lambda = 1$; T) which takes on the value "one" in the neighborhood of $\lambda = 1$ and zero elsewhere.

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* It should be observed that the condition (2) is also modified. The right-hand side of (2) now becomes π/Q times the number of basic periods τ between two adjacent pulses.

3.3 *Spacing Jitter*

This is obtained by subtracting from the absolute jitter at one pulse position the absolute jitter at the adjacent pulse position (not necessarily the adjacent time slot).

$$X_l^s = \left[I - S \right] \left[\sum_{\nu=0}^{l-1} T^\nu \right] X_1, \tag{15}$$

where S is the shift operator given by

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{16}$$

3.4 *Alignment Jitter*

This quantity is given by the difference of X_{l+1} and X_l :

$$X_{l+1}{}^A = X_{l+1} - X_l = T^l X_1 . \tag{17}$$

Finally, we summarize here without proof the properties of the operator T which were determined in Part II of this article.

(a) The operator T is a bounded operator mapping $\mathbf{1}_p$ into itself for $1 \leqq p \leqq \infty$. In particular, the norm of T in $\mathbf{1}_\infty$ is equal to one (i.e., $| T | = 1$).

(b) The spectrum of T is a subset of the unit disk (i.e., $| \sigma(T) | \leqq 1$), and any pole λ of T with $| \lambda | = 1$ has order one.

(c) All points in the unit circle except $\lambda = 1$ are in $\rho(T)$, the resolvent of T. The point $\lambda = 1$ is an eigenvalue of T with the eigenvector $\{1, 1, \cdots\}$. The dimension of the eigenmanifold is one in this case.

(d) The point $\lambda = 1$ is the limit point of the point spectrum of T if the domain of T is unrestricted. It is a pole of T for very special cases, such as periodic pulse trains, truncated pulse trains, etc.

IV. ABSOLUTE JITTER AND DISPERSION

The results for absolute jitter follow immediately from (13)

$$Y = \lim_{l \to \infty} \left[\sum_{\nu=0}^{l-1} T^\nu \right] X_1 .$$

Observe that $\lambda = 1$ is an eigenvalue of T, and since in general X_1 is any element of $\mathbf{1}_\infty$, the limit in the above equation approaches infinity as l

approaches infinity. In particular, the norm of the operator

$$K_l = \left[\sum_{\nu=0}^{l-1} T^\nu \right] \tag{18}$$

is l. We thus observe that, in general, the absolute jitter grows linearly with the number of repeaters.

If $\lambda = 1$ is a pole of T, the dispersion is given by

$$\begin{aligned} X_{l+1}^D &= \left[I - E_1 \right] \left[\sum_{\nu=0}^{l} T^\nu \right] X_1 , \\ &= \left[\sum_{\nu=0}^{l} T^\nu \right] \left[I - E_1 \right] X_1 \qquad \text{(since } E_1 T = T E_1\text{)}, \\ &= \left[\sum_{\nu=0}^{l} D^\nu \right] \left[E_D X_1 \right] \qquad \text{(cf. Part II)}; \end{aligned} \tag{19}$$

where $D = TE_D$ and $E_D = (I - E_1)$. Here $E_D X_1$ is the dispersion element due to the first repeater. The norm of the operator

$$K_{l+1}^D = \left[\sum_{\nu=0}^{l} D^\nu \right] \tag{20}$$

is bounded and converges to a finite value as l approaches infinity. This follows from the previous discussion (see Part II) where it was shown that

$$| D^m | \leqq M \alpha_0^m , \tag{21}$$

where M is a positive constant and $\alpha_0 < 1$. We therefore find that the dispersion is bounded provided that $\lambda = 1$ is a pole of T. This is true for certain highly constrained situations. In particular, this is true when the domain of T is restricted to a finite dimensional subspace of $\mathbf{1}_\infty$ which is invariant under T. Examples of such cases occur when the pulse trains are periodic, finite, etc.

On the other hand, when $\lambda = 1$ is not a pole of T the projection E_1 does not commute with T. In this case,

$$X_{l+i}^D = \left[I - E_1 \right] \left[\sum_{\nu=0}^{l} T^\nu \right] X_1 \tag{22}$$

where

$$E_1 T \neq T E_1 . \tag{23}$$

Furthermore, it can be shown that

$$| (I - E_1)T^m | = 1 \qquad \text{for all } m. \tag{24}$$

This is a consequence of the spectral properties of T summarized in the previous section, viz., the point $\lambda = 1$ is a limit point of the point spectrum of T. In fact, we show in Part II of this paper that there exist elements in $\mathbf{1}_\infty$ such that the norm of the operator

$$K_{l+1}{}^D = \left[I - E_1 \right] \left[\sum_{\nu=0}^{l} T^\nu \right] \tag{25}$$

is $(l + 1)$. It follows, therefore, that the dispersion grows without bound in the case of purely unconstrained pulse trains. The result is not altered even if some form of coding is provided to eliminate indefinitely long strings of zeros in the pulse trains.

V. SPACING JITTER

At the output of the lth repeater, the spacing jitter is given by (15),

$$X_l{}^s = \left[I - S \right] \left[\sum_{\nu=0}^{l-1} T^\nu \right] X_1 = [K_l{}^s] X_1 . \tag{26}$$

In particular, we are interested in knowing whether the quantity $X_l{}^s$ remains bounded as l approaches infinity. Secondly, if it remains bounded we wish to determine the least upper bound for each l. Since there are no restrictions on X_1 (other than the requirement of boundedness), we are interested in determining the norm in the limit of the operator $K_l{}^s$; or,

$$\lim_{l \to \infty} | K_l{}^s | = \lim_{l \to \infty} | (I - S) K_l | \tag{27}$$

when we know that

$$\lim_{l \to \infty} | K_l | \to \infty . \tag{28}$$

For physical systems we are also interested in $| K_l{}^s |$ for all l.

All of our results in this section depend upon an important lemma concerning the operator $K_l{}^s$. We assert that $K_l{}^s$ has a representation simpler than the one given in (26) and prove this assertion by verification. Define an operator

$$B = \text{diag} \cdot \{ (S_0 - 1)^{-1}, (S_1 - 1)^{-1}, (S_2 - 1)^{-1}, \cdots \}, \tag{29}$$

where $S_n \neq 1$ are defined in (11).* Then we assert the following
Lemma:

$$K_l^s = \left[I - S\right]\left[\sum_{\nu=0}^{l-1} T^\nu\right] = B[I - T^l]T^{-1}. \tag{30}$$

Proof: Observe that in (30)

$$B(I - T^l)T^{-1} = B(I - T)T^{-1}\left[\sum_{\nu=0}^{l-1} T^\nu\right].$$

So, we need merely show that

$$[I - S] = B(T^{-1} - I).$$

Or,

$$B^{-1} - B^{-1}S = T^{-1} - I,$$

where

$$B^{-1} = \text{diag}\cdot\{(S_0 - 1), (S_1 - 1), \cdots\}$$
$$= \text{diag}\cdot\{S_0, S_1, S_2, \cdots\} - I.$$

Thus we need to show that

$$\text{diag}\cdot\{S_0, S_1, S_2, \cdots\} - B^{-1}S = T^{-1}.$$

But from (11),

$$(S_{n-1} - 1) = \beta^{i_n}S_n,$$

so

$$B^{-1} = \text{diag}\cdot\{\beta^{i_1}S_1, \beta^{i_2}S_2, \cdots\}.$$

Therefore, the lemma is proven if

$$T^{-1} = \begin{bmatrix} S_0 & -\beta^{i_1}S_1 & 0 & 0 & 0 & \cdots \\ 0 & S_1 & -\beta^{i_2}S_2 & 0 & 0 & \cdots \\ \cdots & 0 & S_2 & -\beta^{i_3}S_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{31}$$

The truth of the above statement is verified directly by considering the products $T^{-1}T = TT^{-1} = I$. The validity of (30) is thus proven.

* It is quite possible that the S_n approach unity. If any $S_n = 1$, it implies a finite pulse train, and the question is analyzed very simply in a finite dimensional space as was done for the periodic case. Such cases, however, do not give us information for the infinite pulse trains which are required for long chains of repeaters.

We can now state the following

Theorem: The operator K_l^s *is bounded for all* l *if* $\inf_n S_n \geqq \alpha > 1$.

Proof: From (30), we have

$$|K_l^s| = |B(I - T^l)T^{-1}|$$
$$\leqq |B| |I - T^l| |T^{-1}|.$$

We know that $|T^l| = 1$ for all l and $|T^{-1}|$ is finite from (31). Hence $|K_l^s|$ is bounded if B is bounded, which it is if $\inf_n S_n \geqq \alpha > 1$. The theorem is thus proven.

Corollary:

$$\lim_{l\to\infty} |K_l^s| < \infty.$$

This follows trivially since the boundedness of K_l^s was proven independently of l. Our assumption that $\inf_n S_n \geqq \alpha > 1$ is a simple assertion of the fact that indefinitely long strings of zeros are ruled out on any communication channel.

Next, we wish to determine what precisely is the norm of the element

$$X_l^s = K_l^s X_1$$

as it relates to the norm of X_1. Let us recall that X_1 is the jitter injected (at a single repeater) referred to the output of the repeater. The injected jitter referred to the input of the repeater is

$$X = T^{-1}X_1, \tag{32}$$

where T^{-1} is defined in (31). We find it more convenient to work with X in what follows. It is obvious, of course, that our entire discussion could have been carried out in terms of X from the very start. We chose not to do so in order to avoid a premature discussion of T^{-1}. The sequence represented by X naturally satisfies condition (2). In fact, the use of X allows a much simpler comparison of the spacing jitter at different repeaters along the chain. We are interested in the behavior of

$$X_l^s = [K_l^s T]X \tag{33}$$

for each l. This is obtained by a precise evaluation of the norm of

$$R_l^s = K_l^s T = B(I - T^l). \tag{34}$$

Let us first define $S_{\text{inf}} = \inf_n S_n$. Then, we state the following

Theorem: The norm of R_l^s is given by

$$| R_l^s | = \frac{2}{S_{\text{inf}}} \sum_{\nu=0}^{l-1} \frac{1}{S_{\text{inf}}^{\nu}}. \tag{35}$$

Proof:

$$R_l^s = B(I - T^l).$$

Let us consider the representation of $(I - T^l)$. The diagonal elements of $(I - T^l)$ are of the form $[1 - (1/S_n^l)]$, whereas the off-diagonal elements are all negative. Also, the sum of the elements in each row must be zero. If we multiply $(I - T^l)$ on the left by B, then the diagonal elements are of the form $[1/(S_n - 1)][1 - (1/S_n^l)]$. Again the off-diagonal elements are all negative and the sum of the elements in each row of $B(I - T^l)$ is zero. Hence, in $\mathbf{1}_\infty$

$$| B(I - T^l)| = 2\left(\frac{1}{S_{\text{inf}} - 1}\right)\left(1 - \frac{1}{S_{\text{inf}}^l}\right)$$

$$= \frac{2}{S_{\text{inf}}} \sum_{\nu=0}^{l-1} \frac{1}{S_{\text{inf}}^{\nu}}.$$

The theorem is thus proven. Observe that this is not just a bound but a precise norm. This value in the norm is taken by the spacing jitter X_l^s for some X whose norm is one. In other words, the value in (35) represents the maximum magnification of X that is possible to yield the value of the spacing jitter. It is interesting to note that this worst case occurs for each l for the same element X, viz., $\{\cdots, 1, -1, -1, \cdots\}$, where the $+1$ corresponds to the position of $1/S_{\text{inf}}$ in the matrix representing the operator T. Finally, to observe the maximum possible growth of the spacing jitter as it compares with the maximum possible spacing jitter at a single repeater, we compare the results for $l = 1$ and $l = \infty$.

$$R_1^s = \frac{2}{S_{\text{inf}}} \tag{36}$$

$$R_\infty^s = \frac{2}{S_{\text{inf}} - 1}. \tag{37}$$

The ratio of the quantities R_∞^s to R_1^s is less than two if S_{inf} is at least greater than two, which is to be expected in most physical situations. For example, if Q is of the order of 100 and there is at least one pulse present in fifteen time slots, then $S_{\inf}$ is at least two or greater. Thus, we observe that not only is there a bound on the growth of the spacing jitter but that the growth is monotonic and levels off rather fast. Of course, this is a rather general result which, if desired, can be more specifically stated in terms of the a priori probability distributions of the binary signal. Furthermore, the importance of this result lies in specifying the conditions for the validity of our model. Under most realistic situations it should be clear that conditions (2) are met at every repeater. Of course, there are situations when these conditions are not met and the results obtained by the use of our model [cf. (3)] can no more be relied upon. However, we show that under most situations the results are reliable and very optimistic, as shown by (36) and (37).

The above results are the most crucial ones in this paper. The rest of the paper is devoted to a varied miscellany that has some bearing on different aspects of the timing problem.

VI. ALIGNMENT JITTER

The alignment jitter in the $(l + 1)$th repeater is given by (17),

$$X_{l+1}^{A} = T^{l}X_{1}.$$

Obviously, for all l the alignment jitter is no greater in the norm than the absolute jitter X_1. This follows trivially since $| T^l |$ does not exceed one for any l.

For more detailed insight into the behavior of the alignment jitter, we need to discuss specific situations.

(a) If $\lambda = 1$ is a pole of T, then T^l converges to T^∞ and X_{l+1}^A settles down to a flat delay element for large l.

(b) If $\lambda = 1$ is not a pole of T, and $\inf_n S_n = 1$, T^l does not converge to T^∞. However, for large l the alignment jitter is slowly-varying.

(c) If $\lambda = 1$ is not a pole of T, and $\inf_n S_n = \alpha > 1$, the alignment jitter (for large l) varies even more slowly than it does in (b).

All the above results follow from the properties of T. If $\lambda = 1$ is a pole of T, the result is obvious (cf. Part I). If it is not a pole of T, the results follow from the fine structure of the spectrum of T. For example, in situation (b), all points in the point spectrum except $\lambda = 1$ are poles of T, whereas this is not true in situation (c). The corresponding eigenvectors have different structures for the two cases (cf. Part II).

It follows that the situation of (a) is to be preferred over that of (c), which in turn is preferable to that of (b).

VII. NONIDENTICAL REPEATERS

This is a rather difficult matter to discuss with any great generality. What we hope to do here is to briefly indicate perhaps the most convenient formulation and to give some reasons for believing that the orders of magnitude of the jitter parameters are not changed for small differences in the repeaters.

There are essentially three possible ways in which the repeaters may differ: (*i*) injected jitter, (*ii*) Q of the repeater, and (*iii*) mistuning. These differences appear mathematically in terms of different operators, multiplicative coefficients in the power series, and so on. We examine each of these separately.

If we assume that the injected jitter differs slightly at each repeater, then we may write (5) as

$$X_l = TX_{l-1} + X_{\text{av}} + \Delta_l \,, \tag{38}$$

where X_{av} is the average injected jitter and Δ_l represents the deviation from this average in the lth repeater. The norm of X_l differs at most from the previous case of identical repeaters by

$$| \Delta_l + T\Delta_{l-1} + T^2\Delta_{l-2} + \cdots + T^{l-1}\Delta_1 | \leqq \sum_{i=1}^{l} | \Delta_i |. \tag{39}$$

If the $| \Delta_i |$ are quite small, it is clear that the results will not be appreciably different from the previous ones.

If the Q's are different, then our basic operator is different for each repeater. It would be almost impossible to analyze such a case in general. However, we can make certain observations if we put

$$T_l = T + K_l \,, \tag{40}$$

where T_l is the operator representing the lth repeater, which is assumed to be an operator T perturbed by an operator K_l. It is reasonable to expect $| K_l | \ll 1$. Then (5) becomes

$$X_l = TX_{l-1} + X_1 + K_lX_{l-1} \,. \tag{41}$$

If $| K_l | \leqq \epsilon \ll 1$, then the norm of X_l does not differ from the previous results by more than

$$\left(\frac{\epsilon}{1 - \epsilon}\right) | X_1 | \approx \epsilon | X_1 |. \tag{42}$$

Again, we see that the results are not appreciably different.

In the case of mistuning, (5) takes essentially the same form as (38),

$$X_l = TX_{l-1} + X_A + \epsilon_l X_B \,, \tag{43}$$

where X_A is the injected jitter due to sources other than mistuning, ϵ_l is the mistuning in the lth repeater, and $\epsilon_l X_B$ is the jitter due to mistuning (cf. Part I). Then

$$X_l = \left[\sum_{\nu=0}^{l-1} T^{\nu}\right] X_A + \left[\sum_{\nu=0}^{l-1} \epsilon_{l-\nu} T^{\nu}\right] X_B . \tag{44}$$

The contribution due to X_A is unaltered and the contribution due to X_B would depend on the specifications of ϵ_i. However, if we assume that the magnitudes of ϵ_i do not exceed one, then the contribution in the norm due to X_B cannot exceed l times the norm of X_B. Thus the worst case for the absolute jitter does indeed arise from the assumption of equal ϵ_i at their maximum values.

For spacing jitter we can make a slightly different statement for the contribution due to X_B. For one repeater, the worst case (in the norm sense) occurs for the maximum value of $\epsilon_1 = 1$, then the worst case for two repeaters is obtained by setting $\epsilon_2 = 1$. Setting the first two repeaters with $\epsilon_1 = \epsilon_2 = 1$, the worst case for a string of three repeaters is obtained by setting $\epsilon_3 = 1$ and so on. The statement is a simple consequence of the inequality

$$| (I - T^{l-1}) | < | (I - T^{l}) |. \tag{45}$$

It is believed that a similar statement can be made for the alignment jitter.

We thus observe that, when the differences in the repeaters are small, it is reasonable to expect that the results are not appreciably different from those obtained by assuming identical repeaters.

VIII. CONCLUSION

We have presented a general and rigorous theory of the jitter accumulation in a chain of regenerative repeaters. The sources of jitter are assumed to be the signal-dependent sources, as distinguished from purely random sources independent of the signal.

Our results show that while the absolute jitter and its dispersion grow without bound with the number of repeaters, the spacing and the alignment jitter remain bounded. In particular, the spacing jitter bounds are quite optimistic for most practical situations, viz., no greater than twice the absolute jitter injected at a single repeater. This result is of importance in that it ensures proper decoding of the binary signal. Its further importance lies in the fact that it does ensure, in most cases, the validity of the basic model and thus the validity of other results obtained by that model. One such result shows that the alignment jitter is slowly-varying for repeaters further along the chain. Finally, a brief discussion shows that the assumption of identical repeaters leads to results which are of the same order of magnitude as would be obtained if the repeaters differed by not too great an amount. Some results which would be of use in computations are to be found in the Appendix, together with an example.

In our discussion so far, we have investigated the accumulation properties of jitter due to repeated regeneration. We have made no attempt to determine the jitter introduced by a single repeater. Analytically, this problem is complicated not only by the nonlinearities involved, but also by a lack of complete knowledge as to the actual mechanisms involved. We propose instead, in a later paper, an experimental approach which allows these measurements to be carried out under steady-state conditions. This experiment also has some bearing on the question of simulation of long chains of repeaters.

IX. ACKNOWLEDGMENTS

I wish to take this opportunity to express my appreciation for the help I received in this entire endeavor from several of my colleagues. In particular, I wish to thank B. Liu and M. R. Aaron for many helpful discussions and criticisms. For their advice and encouragement, my thanks go to J. A. Young and M. Karnaugh.

APPENDIX

Eigenvalues and Eigenvectors of T

Practical systems call for the evaluation of jitter when the number of repeaters is finite. In such cases, we need not concern ourselves with infinite pulse trains. So long as the pulse trains are much longer than the effective memory of the system, the results obtained by considering finite pulse trains will be reliable. The results will also be reliable if the pulse trains are considered periodic with the period being greater than the memory of the system. Actually, as we shall see, the periodic pulse trains are much more difficult to work with than the finite ones. However, we shall discuss the periodic case in detail since much of the experimental work is carried out using periodic pulse trains. Finally, another case of interest is that in which there is a certain quiescent pattern which changes to a different one. This would include a periodic pattern changing to either a nonperiodic or a different periodic pattern. In each case, our interest is in determining the set of eigenvectors. Computations can then be carried out by expressing the injected jitter element in terms of the eigenvectors (cf. Part I). If the set of eigenvectors is not complete, we employ the standard procedure and use the so-called generalized eigenvectors. It should also be observed that the case of nonidentical repeaters is also handled using the same techniques. We present here certain simple algorithms for determining the eigenvalues and the eigenvectors for the several cases of interest.

A.1 *Truncated Pulse Trains*

This is the case where the pulse train is finite. The matrix T in (9) is now finite, upper triangular and the diagonal element in the last row is unity. Such a matrix also arises in the case of pattern transitions where originally the quiescent pattern is periodic with only one pulse present in each period. This is also the more realistic case because the tuned circuit is thus properly excited. We consider the truncated case, therefore, together with the pattern transition case.

A.2 *Pattern Transitions*

Here we have a steady-state periodic pattern which changes to a different pattern. The operator T can be represented as

$$T = \begin{bmatrix} P & C \\ 0 & A \end{bmatrix},$$

where P, representing the new pattern, is an upper triangular square matrix, A represents the quiescent periodic pattern, 0 is the null matrix, and C is the connecting matrix. The matrix A is either an arbitrary positive stochastic matrix for an arbitrary periodic pattern or it is a scalar (viz., unity) for the case of only one pulse present in each period. The latter case also occurs when the tuned circuit is excited by a reference pulse train such as 101010 $\cdots$.

In either case, the eigenvalues of T are given by the eigenvalues of P and those of A. The eigenvalues of the matrix A are discussed in the section dealing with the periodic case. The eigenvalues of P are given by the diagonal elements S_n^{-1}. If A is a scalar, the only other eigenvalue is unity. All the eigenvalues are thus determined.

Next, we observe that the eigenvalues of P are distinct. If not, for some n (say, $n = 0$),

$$S_0 = 1 + \beta^{i_1} + \beta^{i_1+i_2} + \cdots + [\beta^{i_1+i_2+\cdots+i_m}]S_0 ,$$

which implies a steady-state periodic pulse train, contradicting the transient nature of P. Thus the eigenvalues of T are distinct unless, of course, there is a periodic pattern involved. Let us reserve the periodic case for the next section. Then, the eigenvectors of T form a basis for the space of jitter vectors both for the truncated case and the pattern transition case when the quiescent pattern has only one pulse present in each period.

The eigenvectors are given by the algorithm

$$\xi_{n+1} = \frac{S_n - \frac{1}{\lambda}}{S_n - 1} \xi_n$$

(cf. Part II) for each eigenvalue λ.

The eigenvalues and the corresponding eigenvectors are thus very simply determined when either the reference pattern has only one pulse in each period or the pulse train is finite.

A.3 *Periodic Patterns*

Let us start by assuming that the period is m and there are n pulses in a period. Then

$$m = i_1 + i_2 + \cdots + i_n .$$

Let $\alpha_0 = (1 - \beta^m)$, and $D_i = S_i\alpha_0$. Then,

$$A = \begin{bmatrix} \frac{1}{D_0} & \frac{\beta^{i_1}}{D_0} & \frac{\beta^{i_1+i_2}}{D_0} & \cdots & \frac{\beta^{i_1+\cdots+i_{n-1}}}{D_0} \\ \frac{\beta^{i_2+\cdots+i_n}}{D_1} & \frac{1}{D_1} & \frac{\beta^{i_2}}{D_1} & \cdots & \frac{\beta^{i_2+\cdots+i_{n-1}}}{D_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\beta^{i_n}}{D_{n-1}} & \frac{\beta^{i_n+i_1}}{D_{n-1}} & \cdots & \cdots & \frac{1}{D_{n-1}} \end{bmatrix},$$

$$\Delta = \det(\lambda I - A)$$

$$= \frac{1}{D_0 D_1 \cdots D_{n-1}} \begin{vmatrix} \lambda D_0 - 1 & -\beta^{i_1} & \cdots & -\beta^{i_1+\cdots+i_{n-1}} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -\beta^{i_n} & -\beta^{i_n+i_1} & \cdots & \lambda D_{n-1} - 1 \end{vmatrix}.$$

$$\Delta = \frac{1}{D_0 D_1 \cdots D_{n-1}}$$

$$\cdot \begin{vmatrix} (\lambda D_0 - \alpha_0) & -\beta^{i_1} D_1 \lambda & 0 & 0 & \cdots & 0 \\ 0 & (\lambda D_1 - \alpha_0) & -\beta^{i_2} D_2 \lambda & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -\beta^{i_n} & -\beta^{i_n + i_1} & \cdots & & \cdots & (\lambda D_{n-1} - 1) \end{vmatrix}$$

$$= \Bigg[-\frac{1}{D_0} \beta^m \lambda^{n-1} + \left(\lambda - \frac{\alpha_0}{D_0}\right)\left(-\frac{1}{D_1} \beta^m \lambda^{n-2}\right) + \cdots$$

$$+ \left(\lambda - \frac{\alpha_0}{D_0}\right)\left(\lambda - \frac{\alpha_0}{D_1}\right) \cdots \left(\lambda - \frac{\alpha_0}{D_{n-3}}\right)\left(-\frac{1}{D_{n-2}} \beta^m \lambda\right)$$

$$+ \left(\lambda - \frac{\alpha_0}{D_0}\right) \cdots \left(\lambda - \frac{\alpha_0}{D_{n-2}}\right)\left(\lambda - \frac{1}{D_{n-1}}\right)\Bigg]$$

$$= P(\lambda) + Q(\lambda),$$

where

$$P(\lambda) = \sum_{i=0}^{n-1} \left(\lambda - \frac{\alpha_0}{D_i}\right)$$

and

$$Q(\lambda) = \sum_{\nu=0}^{n=1} -\frac{1}{D_\nu} \beta^m \lambda^{n-\nu-1} \prod_{k=1}^{\nu} \left(\lambda - \frac{\alpha_0}{D_{\nu-k}}\right).$$

Finally, after some manipulation it can be shown that

$$\Delta = \left(\frac{\alpha_{0-1}}{\alpha_0}\right) \lambda^n + \frac{1}{\alpha_0} P(\lambda).$$

Thus the eigenvalues of A are given by the zeros of the polynomial

$$R(\lambda) = P(\lambda) - \beta^m \lambda^n,$$

when

$$P(\lambda) = \prod_{i=0}^{n-1} (\lambda - S_i^{-1}).$$

The zeros of $R(\lambda)$ can be obtained from those of $P(\lambda)$ by root-loci considerations or other numerical methods. Since β^m is usually small, this can be handled easily on a digital computer.

Next, let us discuss the eigenvectors corresponding to the eigenvalues given by the zeros of $R(\lambda)$. We show that for each distinct eigenvalue of A, the eigenvector is given by the algorithm

$$\xi_{k+1} = \frac{S_k - \dfrac{1}{\lambda}}{S_k - 1}\,\xi_k .$$

This would be true if $\xi_{k+n} = \xi_k$ for all k or, if

$$\frac{\left(S_{n-1} - \dfrac{1}{\lambda}\right)\left(S_{n-2} - \dfrac{1}{\lambda}\right)\cdots\left(S_0 - \dfrac{1}{\lambda}\right)}{(S_{n-1} - 1)(S_{n-2} - 2)\cdots(S_0 - 1)} = 1.$$

The above is true if

$$\begin{aligned} S_{n-1}S_{n-2}\cdots S_0\left(\frac{1}{\lambda^n}\right)P(\lambda) &= \prod_{k=1}^{n}(S_{k-1} - 1) \\ &= \prod_{k=1}^{n}(\beta^{i_k}S_k) \\ &= \beta^m S_1S_2\cdots S_n \\ &= \beta^m S_0S_1S_2\cdots S_{n-1} \qquad (\text{since } S_n = S_0), \end{aligned}$$

or if

$$P(\lambda) = \beta^m\lambda^n,$$

which is indeed true for every eigenvalue λ. For a repeated eigenvalue λ, if it exists, one must find a generalized eigenvector in the usual way.

We have thus given simple algorithms for determining the eigenvalues and the corresponding eigenvectors of A.

It should be mentioned that the algorithm for eigenvectors is the same as the one given above when A is a submatrix of T as in Section A.2.

A.4 *Example*

Let us consider a simple example to illustrate some of the points. Consider a system with repeaters having $Q = 100$ and signals having at least one pulse present in 10 time slots. Consider the case of a quiescent pattern (101010 $\cdots$ 10) suddenly changing to a new periodic pattern with one pulse in 10 time slots. We are interested in determining the behavior of the first pulse after the transition.

The operator T has the form

$$T = \begin{bmatrix} \frac{1}{1+\beta^{10}(1+\beta^2+\beta^4+\cdots)} & \frac{\beta^{10}}{1+\beta^{10}(1+\beta^2+\beta^4+\cdots)} & \cdots & \cdots \\ 0 & \frac{1}{(1+\beta^2+\beta^4+\cdots)} & \frac{\beta^2}{(\ )} & \cdots \\ 0 & 0 & \frac{1}{(1+\beta^2+\beta^4+\cdots)} & \cdots \\ \cdots & \cdots & \cdots & \end{bmatrix}.$$

Let the jitter vector $X_1 = \{\xi_0, \xi_1, \xi_1, \xi_1, \cdots\}$, where ξ_1 is the reference phase jitter for the quiescent pattern and $(\xi_0 - \xi_1)$ is the change due to the transition.

In our simple example it is clear that there are only two eigenvalues of T, viz., $\lambda_0 = (1 - \beta^2)/(1 - \beta^2 + \beta^{10})$ and $\lambda_1 = 1$. The eigenvectors corresponding to these eigenvalues are $e_0 = \{(\xi_0 - \xi_1), 0, 0, \cdots\}$ and $e_1 = \{\xi_1, \xi_1, \xi_1, \cdots\}$.

The absolute jitter obviously increases without bound with the number of repeaters. The dispersion remains bounded, since $\lambda_1 = 1$ is a pole of T in our simple case. In the limit, the dispersion is given by

$$X^D = \left(\frac{1}{1 - \lambda_0}\right) e_0 = \left(\frac{1 - \beta^2 + \beta^{10}}{\beta^{10}}\right) e_0 .$$

The spacing jitter in the limit is given by

$$\left(\frac{1 - \beta^2 + \beta^{10}}{\beta^{10}}\right) (\xi_0 - \xi_1)$$

in the zeroth position and zero elsewhere. Finally, the alignment jitter approaches e_1 in the limit.

The validity of the results is assured if

$$\left(\frac{1 - \beta^2 + \beta^{10}}{\beta^{10}}\right) (\xi_0 - \xi_1) \ll 10 \frac{\pi}{Q},$$

or if

$$(\xi_0 - \xi_1) \ll 10 \frac{\pi}{Q} \left(\frac{\beta^{10}}{1 - \beta^2 + \beta^{10}}\right) \approx \frac{\pi}{10}.$$

Thus, if the *jump* in the phase jitter, for a single repeater, due to the transition in pattern is much smaller than 18°, our results are valid. This requirement can be expected to be satisfied by most practical repeaters.[5]

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