The Maser Rate Equations and Spiking

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The rate equations of Statz and De Mars giving the time development of the inversion and photon number in a maser or laser are discussed analytically with the aid of a mechanical analogy in which a particle moves in a potential well under the influence of a viscous damping force. The coordinate of this particle is analogous to the logarithm of the light output of the laser, and the amplitude, period, and damping of the motion can be directly related to the parameters of the rate equations. Simple analytic approximations are developed for all of the quantities of experimental interest in the spiking pattern of a laser. Four relationships are given, which do not contain any of the rate equation parameters, whereby a spike pattern can be tested to determine if it is consistent with the usual rate equations. Systematic procedures are described for extracting all of the information contained in spike patterns.

I. INTRODUCTION

The most fruitful approach for the discussion of maser and laser[1] behavior has been through rate equations describing the time rates of change of the atomic populations and the photon numbers of the electromagnetic field. Bloembergen[2] introduced rate equations for the populations in a paramagnetic maser and based his discussion on the steady-state solutions without explicitly considering the photon field. On the other hand, Shimoda, Takahasi, and Townes[3] have considered the photon rate equations and on this basis have given a theory of maser amplification without explicitly considering the atomic populations. Statz and De Mars[4] have shown that the transient behavior of masers depends upon coupled rate equations for both the populations and the photons. A number of authors have rederived these equations and discussed their applications to various maser systems. Considerable attention has been given to the question of whether these equations have periodic (undamped) solutions. It has been shown by Makhov[5] and by Sinnett[6] that the small-signal solutions are always damped, and it has been presumed and often confirmed by numerical computations that the same is true in the large-signal domain. Statz, et al.[7] have suggested that the damping of experimentally observed "spikes" in the output of lasers is probably sensitive to coherence and noise conditions, and not necessarily related to the damping predicted by the usual rate equations. They also suggest that the complicated spiking patterns frequently observed[8] are due to oscillation in many modes of the laser cavity. Despite these doubts which have been cast on the adequacy of the Statz-De Mars rate equations for describing laser behavior, they still remain the logical starting point for any discussion of the power output of lasers, whether it be transient or steady-state, and of the dependence of power on the quality of the cavity, the intensity of the pumping light, or the linewidth, relaxation time, and concentration of the active atoms.

Several ingenious suggestions have been made for modifying the rate equations so as to obtain periodic solutions. Statz and De Mars[4] and Makhov[5] propose that periodic solutions are obtained if terms are added to the rate equations representing cross relaxation in the inhomogeneously broadened maser transition. On the other hand, Shimoda[9] suggests that periodic spiking can result if the losses due to absorption in the cavity can be partially saturated by the buildup of laser oscillations. Although these and other modifications may ultimately prove to be justified and necessary in laser theory, we shall confine our attention in this paper to the original Statz-De Mars equations which relate the photons in a single cavity mode to a single quantity, the *inversion*, describing the atomic populations. We shall make it our task to understand as fully as possible the damped oscillatory solutions of these equations and how they may be applied to the study of the spiking phenomenon seen[8] in solid-state lasers.

Rate equations in the simple form have been successfully applied by McClung and Hellwarth[10] and by Vuylsteke[11] to the giant-pulse laser, which produces a single very short and very intense burst of radiation. Wagner and Lengyel[12] have shown that an exact analytic solution can be obtained to a simplified rate equation which neglects spontaneous emission and pumping during the pulse. By also neglecting the loss of photons in the cavity Dunsmuir[13] has obtained a still simpler analytic solution which is applicable to the rising portion of a spike or a giant pulse. We shall not consider further these exact solutions or the giant-pulse laser in this paper, but confine ourselves to those solutions representing repetitive pulsations, or spikes, in the ordinary laser. It is in this field that the greatest need now exists for an analytical discussion of the solutions of the rate equations.

A number of authors have endeavored to put the rate equations on a firmer theoretical basis. Following Anderson[14] and Clogston,[15] who first described masers in terms of the density matrix, treatments using the density matrix to derive rate equations have been given by Fain, et al.,[16] Kaplan and Zier,[17] and Pao.[18] By considering directly, without explicit use of the density matrix, the correlation functions of the electromagnetic field which are measured in simple maser experiments, McCumber[19] has shown that the maser medium acts like a dielectric of negative conductivity; he concludes that the field and the dielectric satisfy rate equations of the usual form, providing that the populations change by a small fractional amount during a coherence time (reciprocal linewidth) of the atomic system. Another formal theory, based on a successive approximation approach to the quantum mechanical equations of motion, has been applied by Haken and Sauermann[20] to the frequency shifts and interactions of cavity modes in the laser. An extensive survey with bibliography of the early formal work has been given by Lamb.[21] In the present paper we shall not go into the theoretical basis for the rate equations, but confine ourselves entirely to the problem of solving the equations in the large-signal domain.

Despite the fact that the rate equations are generally accepted, and that the general nature of the solutions has been familiar from digital computer calculations for some time,[5,13,17,22] it is still quite inconvenient in any particular case to compare observed spiking patterns in solid-state lasers with predictions of the rate equations. It has been necessary either to use the small-signal solutions and hope that they are not too inaccurate in the large-signal domain, or to resort to machine calculations and try to fit three or more parameters to the data by trial and error. To be sure, much of the data on spiking is not amenable to analysis, consisting apparently of random spikes with widely varying amplitude, duration, and interval. Nevertheless, several laser systems are now known to give very regular pulsations of the type that might be consistent with the rate equations. Regular spiking patterns have been observed in $CaF_2:U^{+3}$ by Sorokin and Stevenson[23] and by O'Connor and Bostick,[24] and in $CaWO_4:N_d^{+3}$ by Johnson and Nassau.[25] Recently the effect has also been seen in a highly perfect ruby by Nelson and Remeika,[26] and in a confocal ruby by Johnson, et al.[27] More extensive studies of highly regular spiking have been reported by Gürs[28] and by Hercher.[29] Thus it is clear that good data on spiking patterns can be obtained, and it therefore becomes cogent to inquire into practical and convenient means for analyzing this data and obtaining information from it.

The information contained in spike patterns is of two distinct kinds, which we may call qualitative and quantitative. Qualitatively, we can determine quickly by means of relationships given here whether a spike pattern is *consistent* with the rate equations. If we determine that certain patterns are not consistent, we are spared the waste of time that would result from attempting to fit these patterns numerically by trial and error. It is not obvious at this writing that any of the regular patterns that have been reported are consistent, because the application of the consistency relationships requires a measurement of the ratio of the peaks to the valleys in the light output, and only the peaks are seen in pictures published so far. Thus we suggest that by extending spike studies to include the valleys new and interesting information can be obtained. It is to be expected that patterns which are regular but yet not consistent will turn out to be physically the most interesting of all, since they will point the way to new understanding of laser behavior. The quantitative information can only be obtained from patterns which are consistent, at least in some average sense, and consists in obtaining values for the physical constants which appear in the rate equations:

N = the *number* of active laser atoms in the optical cavity

t_r = the *relaxation* time of the upper laser level, usually due to spontaneous emission

t_p = the *photon* lifetime in the laser mode of the cavity

t_m = the *mode* time, the time for spontaneous emission into the laser mode

t_g = the *ground* state time, the time spent by an atom in the ground state before being excited by the pump

s = the *source* strength, the rate of production of laser photons by spontaneous emission, the pumping light, or any other noise source in the cavity, or any signal applied to the cavity. Although s may vary with time, it is convenient here to consider it with the constants.

In principle all of these quantities except s could be directly measured or calculated from independent measurements on the laser material, the cavity, and the pump. The spiking data would then serve as confirmatory evidence. In many cases, however, spike patterns may prove to be the most convenient method of measurement. The last quantity, s, is in some respects the most interesting. The first assumption would be that s is due entirely to spontaneous emission into the laser mode; if so, s could be calculated and comparison with the measured value would reveal the verity of the assumption. Experiments could be carried out with an external weak signal from a monochromator to test the response of the laser as observed in its spiking patterns. Thus we hope that the analysis given here will help to stimulate new experiments by making rate equation analysis more convenient for the experimentalist.

II. FORMULATION OF THE RATE EQUATIONS

The laser is a system consisting essentially of an *optical cavity* with very high Q in a few modes, low Q in all other modes, the *laser medium* containing the active atoms, and a *pump*, usually an intense light source, to excite the atoms into a broad band of excited states. It is a property of the laser medium that the atoms decay from these states in an extremely short time by nonradiative processes to one or more very sharp excited states, called the upper laser levels. The upper laser levels can decay radiatively to a sharp lower level, called the lower laser level, which may be the ground state. If the lower laser level is not the ground state, we shall assume that very rapid nonradiative decay processes return the atom to the ground state. Thus we may always neglect the population of the pumping band and of the lower laser level if the latter is not the ground state. The laser transition takes place from the lowest of the upper laser levels, but it may sometimes be necessary to take into account the populations of nearby levels in thermal equilibrium with this level. The statistical weights of the laser levels must also be taken into account.[30]

The rate equations may always be written in the form (p = photon number, n = inversion)

$$\frac{dp}{dt} = -\frac{p}{t_p} + \frac{pn}{t_m} + s \tag{1}$$

$$\frac{dn}{dt} = \frac{n_0 - n}{t_0} - a\frac{pn}{t_m} \tag{2}$$

as long as we consider only a single mode of the cavity, the laser mode, and neglect all atomic populations except the upper laser level and the ground state. The time t_0 might be called the pumping relaxation time

$$\frac{1}{t_0} = \frac{1}{t_g} + \frac{1}{t_r}, \tag{3}$$

since it represents the characteristic time in the response of the population inversion n to the pump. The inversion n may always be written

$$n = N_u - (1/w)N_l, \tag{4}$$

where N_u, N_l are the populations of the upper and lower laser levels respectively and w is the statistical weight of the lower relative to that of the upper laser level. In view of our assumptions, w will come in only when the lower laser level is the ground state. Let us suppose that the upper laser level is in thermal equilibrium[26,30] with certain other states not directly involved in the laser transition such that there is a temperature-dependent probability P that an excited atom is in the upper laser level. Then we have for a *three-level system*, in which the lower laser level is the ground state,

$$\begin{aligned} a &= P + \frac{1}{w} \\ n_0 &= PN\, t_0 \left(\frac{1}{t_g} - \frac{1}{Pwt_r}\right); \end{aligned} \tag{5}$$

and for a *four-level system*, in which the lower laser level is not the ground state, we have

$$\begin{aligned} a &= P \\ n_0 &= PN\, t_0/t_g\,. \end{aligned} \tag{6}$$

If the relaxation of the upper laser level is predominantly by spontaneous emission to the lower laser level, there is a simple relation between t_r, t_m, and the linewidth $\Delta\nu$ (cps) of the laser transition

$$t_m = (8\pi\nu^2 n_{\text{ref}}^3/c^3)V\, t_r\Delta\nu, \tag{7}$$

where V is the volume of the cavity and n_{ref} is the refractive index. This relation should not be taken too literally, since it takes no account of the anisotropy of the laser medium or the polarization properties of the transition, and V would have to be replaced by a suitable effective "optical volume" if the laser material does not fill the cavity. Nevertheless, it points out the important fact that t_m varies with temperature in the same way as $\Delta\nu$ and therefore is subject to control in spiking studies. Another constant subject to convenient control is t_g, since $1/t_g$ is proportional to the pump intensity. Even when flash lamps are used for pumping it is still approximately valid to assume t_g is constant, since the time constant for the flash will usually be much longer than the interval between spikes. To assure that this is true, it would be advantageous to have the spike pattern commence when the flash is at its maximum intensity. It is not valid to assume without investigation that $1/t_g$ is proportional to the total energy dissipated in the flash. Spiking data should always include a record of the flash as a function of time and an indication of when the spike pattern occurred. Also subject to experimental control is t_p, the photon lifetime in the laser mode of the cavity. Presumably the losses in a good laser cavity can be estimated rather reliably, so that t_p can usually be directly calculated. The total number N of active laser atoms can ordinarily be determined in a given sample, but N does not appear to be a convenient parameter to vary in laser experiments. The signal or source strength s is best determined from an analysis of the spiking data as described in this paper. Although $s = s(t)$ will in general be a function of time, it will be shown that the spiking pattern depends only on the value of s at the instant when the net losses in the laser mode vanish due to the buildup of inversion. We shall denote this time by t_1 and the critical inversion by n_1, where

$$n_1 = t_m/t_p \tag{8}$$

is the celebrated Schawlow-Townes[1] criterion for the buildup of laser oscillation.

We now introduce dimensionless variables and parameters; in terms of the variables

$$\begin{aligned} \tau &= t/t_p \\ \eta &= n(t_p/t_m) = n/n_1 \\ \rho &= p(a\, t_0/t_m), \end{aligned} \tag{9}$$

and the parameters

$$\begin{aligned} \omega &= t_p/t_0 \\ \xi &= n_0(t_p/t_m) = n_0/n_1 \\ \sigma &= s(a\, t_p t_0/t_m), \end{aligned} \tag{10}$$

the rate equations (1), (2) become

$$\dot{\rho} = \frac{d\rho}{d\tau} = \sigma - \rho + \rho\eta \tag{11}$$

$$\dot{\eta} = \frac{d\eta}{d\tau} = \omega(\xi - \eta - \rho\eta). \tag{12}$$

Here ρ represents the *photons*, η the *inversion*, and τ the *time*, while ω represents the *pumping rate*, ξ the *limiting inversion* toward which the pump is tending to drive the system, and σ the *source*. We see that there are really only three parameters in the rate equations; it follows that three relationships among the six relevant physical parameters with which we started can be obtained from spiking studies. Although we shall assume in our analysis that ω and ξ are constant, our results will provide a useful adiabatic approximation for the case of slowly varying ω, ξ. Typical values of the parameters for a ruby laser will be given in the discussion of a numerical example. For the present we need only mention that σ is relevant only in the initial growth of ρ prior to the onset of laser gain.

If we regard σ as a constant, the steady-state solution of (11) and (12) is

$$\begin{aligned} \rho_\infty &= \tfrac{1}{2}\{(\xi + \sigma - 1) + [(\xi + \sigma - 1)^2 + 4\sigma]^{\frac{1}{2}}\} \\ \eta_\infty &= \tfrac{1}{2}\{(\xi + \sigma + 1) - [(\xi + \sigma - 1)^2 + 4\sigma]^{\frac{1}{2}}\}. \end{aligned} \tag{13}$$

The sign of the radical is determined by the requirement that $\rho \geqq 0$. In the limit $\sigma \to 0$ we obtain two cases, depending on whether $\xi < 1$ or $\xi > 1$. For $\xi < 1$

$$\begin{aligned} \rho_\infty &\underset{\sigma \to 0}{\longrightarrow} \sigma/(1 - \xi) \\ \eta_\infty &\longrightarrow \xi(1 - \rho_\infty) \end{aligned} \qquad \xi < 1; \tag{14}$$

this is the case in which the limiting inversion n_0 is less than n_1 given by (8), and laser oscillation does not occur. For $\xi > 1$

$$\begin{aligned} \rho_\infty &\underset{\sigma \to 0}{\longrightarrow} \xi - 1 \\ \eta_\infty &\longrightarrow 1 \end{aligned} \qquad \xi > 1; \tag{15}$$

this describes the steady state of laser oscillation, which is usually approached through a series of sharp pulses in $\rho(\tau)$ called *relaxation oscillations*, or *spikes*.

III. THE NATURE OF THE SPIKING SOLUTIONS TO THE RATE EQUATIONS

In this section we shall consider the nature of the solutions to (11) and (12) when $\xi > 1$ and the initial conditions on ρ and η correspond to very few photons and a small inversion $n \ll n_1$. This may be contrasted with the situation in the giant-pulse laser in which immediately after switching $n \gg n_1$. We shall also assume the *spiking condition*

$$\omega\xi \ll 1, \tag{16}$$

which will be satisfied whenever spiking can be observed. It will be apparent later that when (16) is not satisfied there will be no spikes, but ρ will smoothly approach ρ_∞. This is the case in gas lasers, where the pumping rate $1/t_g$ has to be very high to overcome the high relaxation rate $1/t_r \sim 10^8 \text{ sec}^{-1}$.

In view of (16) and (12), $\eta(\tau)$ will increase slowly with time and $\dot{\rho}$ in (11) can be neglected; thus we have

$$\rho(\tau) \approx \bar{\rho}(\tau) = \sigma(\tau)/(1 - \eta(\tau)) \tag{17}$$

until τ approaches τ_1 and η approaches unity. The behavior of $\rho(\tau)$ is shown in Fig. 1 for the initial condition $\rho(0) = 0$. Initially ρ rises with slope $\sigma(0)$ and asymptotically approaches the adiabatic solution $\bar{\rho}(\tau)$ which it follows for a relatively long time until $\tau \to \tau_1$. It follows that the initial condition on $\rho(\tau)$ is unimportant. As τ passes through τ_1, where

$$\eta(\tau_1) = 1, \tag{18}$$

$\rho(\tau)$ remains finite and is no longer given by (17); we then leave the *adiabatic phase* and enter the *spiking phase* of the time development of $\rho(\tau)$. According to (11) the slope of $\rho(\tau)$ at τ_1 is $\sigma(\tau_1) \equiv \sigma_1$. Thus we construct an approximate solution by smoothly joining (17) to a line of slope σ_1 as shown in Fig. 1. It follows that

$$\rho(\tau_1) = \rho_1 = 2\sigma_1/(\omega\beta)^{\frac{1}{2}}, \tag{19}$$

where

$$\beta = \xi - 1. \tag{20}$$

This is our first important result. It shows that the source occurs in spiking theory only as a parameter, the single value σ_1. Thus we may drop the subscript on σ and let $\sigma_1 = \sigma$, a constant.

Once we enter the spiking phase, $\eta(\tau)$ remains close to unity, fluctuat-

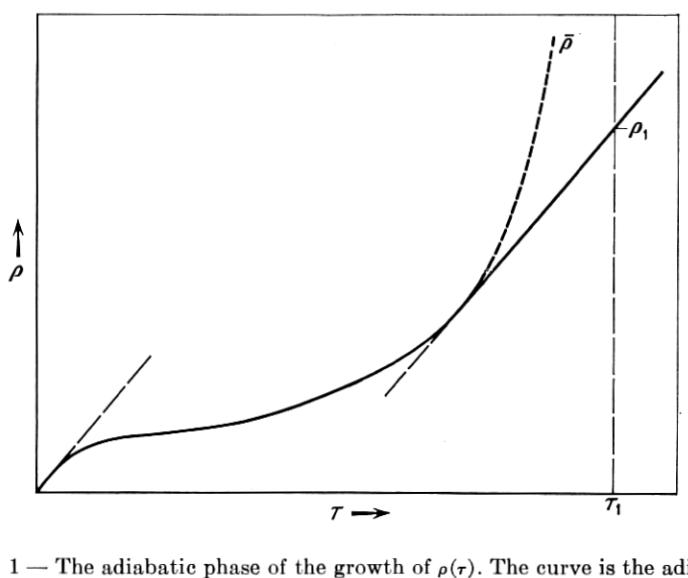

Fig. 1 — The adiabatic phase of the growth of $\rho(\tau)$. The curve is the adiabatic solution (17), and the line segments of slope σ are constructions to approximate $\rho(\tau)$ near $\tau = 0$ and $\tau = \tau_1$ where $\eta = \eta_1 = 1$.

ing up and down and eventually settling down to its steady-state value $\eta_\infty = 1$. Thus we make the approximations

$$\eta - 1 \approx \ln \eta, \tag{21}$$

and

$$\frac{\xi}{\eta} - 1 \approx \beta + \xi(1 - \eta). \tag{22}$$

The rate equations (11), (12) can now be written

$$\frac{d}{d\tau} \ln \rho = \ln \eta \tag{23}$$

$$\frac{d}{d\tau} \ln \eta = \omega(\beta - \xi \ln \eta - \rho). \tag{24}$$

Since the inversion is not directly observed we eliminate $\ln \eta$ from (23), (24); the result is most conveniently written

$$\ddot{\Psi} = \omega\beta(1 - e^{\Psi}) - \omega\xi\dot{\Psi} \tag{25}$$

in terms of the *logarithmic light output*

$$\Psi = \ln (\rho/\beta). \tag{26}$$

The discussion of (25) is greatly facilitated by a mechanical analogy which is shown in Fig. 2. We regard Ψ as the coordinate of a particle of unit mass moving in a one dimensional potential field.

$$\begin{aligned} V(\Psi) &= -\omega\beta \int_0^{\Psi} (1 - e^{\Psi})\, d\Psi \\ &= \omega\beta(e^{\Psi} - \Psi - 1). \end{aligned} \tag{27}$$

There is also a dissipative resistive force $\omega\xi\dot{\Psi}$ as if the particle were moving through a viscous medium. For the moment let us disregard the viscous force, in which case the total energy E of the particle is conserved

$$E = V(\Psi) + \tfrac{1}{2}\dot{\Psi}^2. \tag{28}$$

The particle executes a periodic motion between extreme points $\Psi_m < 0$ and $\Psi_M > 0$ such that

$$V(\Psi_m) = V(\Psi_M) = E. \tag{29}$$

In the spiking phase it is permissible to neglect exp (Ψ_m). From (27)

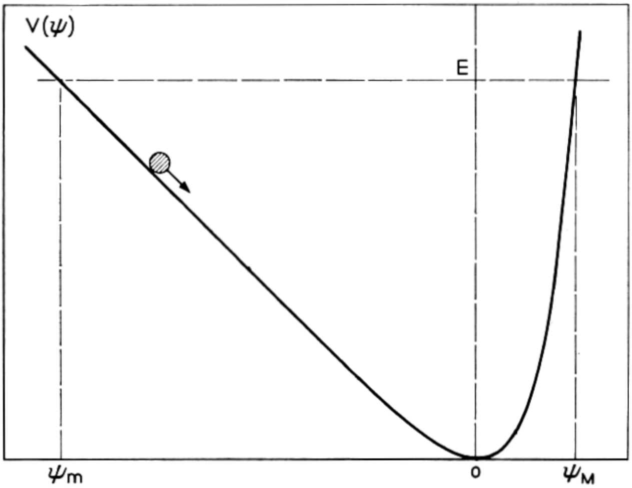

Fig. 2 — The spiking phase of the time development of $\rho(\tau)$ considered in terms of a mechanical analogy in which a particle with coordinate $\Psi = \ln (\rho/\beta)$ moves in a potential $V(\Psi)$ given by (27). The extremes of the motion Ψ_m and Ψ_M are determined by the total energy E of the particle.

and (29) we then obtain a simple relation between Ψ_m and Ψ_M

$$\Psi_M - \Psi_m = \exp (\Psi_M). \tag{30}$$

This is our second important result.

Let us define

$$\Psi_1 = \ln (\rho_1/\beta); \tag{31}$$

then it follows from (28) that

$$\Psi_m = \Psi_1 - (\sigma_1/\rho_1)^2/2\omega\beta, \tag{32}$$

since $\dot{\Psi}_1 = \sigma_1/\rho_1$. It will be clear from the numerical example in the next section that the second term can be neglected, and we can write

$$\Psi_m \approx \Psi_1 . \tag{33}$$

This says that the small kinetic energy of the particle at the start of the spiking phase can be neglected.

It is convenient to denote the successive times when $\eta(\tau) = 1$ by $\tau_1, \tau_2, \tau_3, \cdots$. From (23) we see that these times correspond to extrema in the motion of Ψ; according to this convention the successive minima and maxima of Ψ are

$$\begin{aligned} &\text{minima: } (\Psi_1), \Psi_3, \Psi_5, \Psi_7, \cdots \\ &\text{maxima: } \Psi_2, \Psi_4, \Psi_6, \cdots . \end{aligned} \tag{34}$$

We have placed Ψ_1 in parentheses, since it is a minimum only in the mechanical analogy and not in the observed light output. In the absence of damping, of course, we would have $\Psi_1 = \Psi_3 = \Psi_5 = \cdots$, and $\Psi_2 = \Psi_4 = \Psi_6 = \cdots$. If (16) is satisfied, the damping will be small enough so that the motion is approximately periodic, or quasi-periodic. Thus to a good approximation we can compute the *first maximum* Ψ_2 from Ψ_1 by means of (30). Before considering the damping we shall consider other quantities of experimental interest which are characteristic of the periodic motion.

The *maximum velocity* $\dot{\Psi}_{\max} = \dot{\Psi}_0$ is capable of being measured experimentally from spiking patterns in which the spikes are well resolved in time. It follows from (28), (29) and (33) that

$$\begin{aligned}\dot{\Psi}_{\max} = \dot{\Psi}_0 &= [2\omega\beta(e^{\Psi_1} - \Psi_1 - 1)]^{\frac{1}{2}} \\ &\approx [2\omega\beta(-\Psi_1 - 1)]^{\frac{1}{2}}.\end{aligned} \tag{35}$$

In numerical applications this can be used to ascertain the validity of (21), since according to (23)

$$\dot{\Psi}_0 = \ln \eta_{\max} \approx \eta_{\max} - 1. \tag{36}$$

In general we can write near Ψ_1

$$\begin{aligned}\dot{\Psi} &= \{2[E - V(\Psi)]\}^{\frac{1}{2}} \\ &\approx [2\omega\beta(\Psi - \Psi_1)]^{\frac{1}{2}};\end{aligned} \tag{37}$$

thus the time dependence near a minimum is given by

$$\Psi(\tau) \approx \Psi_1 + \tfrac{1}{2}\omega\beta(\tau - \tau_1)^2. \tag{38}$$

Let us denote by m the full width in time of the minimum measured between points e times the minimum in light output; this is the same as the full width of $\Psi(\tau)$ measured between points $\Psi_m + 1$. Thus the *duration of the minima* according to (38) is

$$m = (8/\omega\beta)^{\frac{1}{2}}. \tag{39}$$

This applies to all minima regardless of damping, which provides a very convenient means for determining almost by inspection whether or not a spike pattern is consistent with the assumption of constant $\omega\beta$. Even if the minima are not observed to have the same durations, it may be meaningful, in the sense that our theory provides an adiabatic approximation, to apply (39) to each minimum separately and deduce the variation of $\omega\beta$.

Near the maximum Ψ_2 we write instead of (37)

$$\begin{aligned}\dot{\Psi} &\approx [2\omega\beta(e^{\Psi_2} - 1)(\Psi_2 - \Psi)]^{\frac{1}{2}} \\ &\approx [2\omega\beta(\Psi_2 - \Psi_1 - 1)(\Psi_2 - \Psi)]^{\frac{1}{2}},\end{aligned} \tag{40}$$

where use has been made of (30); thus

$$\Psi(\tau) \approx \Psi_2 - \tfrac{1}{2}\omega\beta(\Psi_2 - \Psi_1 - 1)(\tau - \tau_2)^2. \tag{41}$$

The *duration M of a maximum* will be defined as the time interval measured between points at $1/e$ times the maximum in light output; this is the same as the interval between points at $\Psi_M - 1$. We see from the factor $(\Psi_2 - \Psi_1 - 1)$ in (41) that M will depend upon damping; it may be written in a general way

$$M = m(\Psi_M - \Psi_m - 1)^{-\frac{1}{2}}. \tag{42}$$

There is in this relation a certain ambiguity which is inherent in our method of regarding the motion as quasi-periodic. In applying the relation we may wonder whether the minimum is the one preceding or following the maximum. Within the accuracy of the quasi-periodic approximation it makes no difference: either may be used, or the average of the two.

The most readily observed quantity in spiking experiments is the *interval* between spikes, which can be identified with the period of the quasi-periodic motion

$$I = \oint d\Psi/\dot{\Psi}, \tag{43}$$

where the integral is over one cycle. To evaluate I we use the approximations (37) and (40), which are accurate near the turning points Ψ_1 and Ψ_2 respectively where $1/\dot{\Psi}$ is large. Upon comparing (35) and (37) we see that (37) is reasonably accurate even at $\Psi = 0$ providing $|\Psi_1| \gg 1$. However, (40) is only accurate near Ψ_2, say in the range

$$\Psi_2 - 1 < \Psi \leqq \Psi_2 .$$

Thus we shall use (37) in the range $\Psi_1 \leqq \Psi \leqq \Psi_c$ and (40) in the range $\Psi_c < \Psi \leqq \Psi_2$, where Ψ_c is a crossover point which will be determined presently. The integral (43) can now be carried out to obtain

$$I(\Psi_c) \approx m\left[(\Psi_c - \Psi_1)^{\frac{1}{2}} + \frac{(\Psi_2 - \Psi_c)^{\frac{1}{2}}}{(\Psi_2 - \Psi_1 - 1)^{\frac{1}{2}}}\right]. \tag{44}$$

Since both our approximations tend to underestimate $1/\dot{\Psi}$, we must choose Ψ_c so as to maximize I, which gives the condition

$$\Psi_c = \Psi_2 - 1. \tag{45}$$

Thus the *interval I between spikes* is given by

$$\begin{aligned} I &= m[(\Psi_M - \Psi_m - 1)^{\frac{1}{2}} + (\Psi_M - \Psi_m - 1)^{-\frac{1}{2}}] \\ &= M(\Psi_M - \Psi_m). \end{aligned} \tag{46}$$

It is logical in this case to choose the minimum between the two maxima between which I is measured. There is still an ambiguity, however, in the choice of maxima. Since our approximation tends to underestimate I, it is good to use the larger of the two maxima.

We now return to the equation of motion (25) and consider the damping force $-\omega\xi\dot{\Psi}$. If (16) holds, the damping will be small, and can be computed from the work done per cycle against the damping force.

$$W = \omega\xi \oint \dot{\Psi} d\Psi. \tag{47}$$

With damping present, energy is no longer conserved, but decreases by W every cycle of the motion until the particle eventually settles down at its equilibrium position $\Psi = 0$, corresponding to the steady-state light output given by (15). Near Ψ_1 we can neglect e^{Ψ}, so that (27) gives

$$\omega\beta(\Psi_3 - \Psi_1) = W. \tag{48}$$

We evaluate (47) just as we did (43), using (37) and (40) with a cross-over point ψ_c, determined this time by the condition that W should be a minimum; the result is

$$W = (4\sqrt{2}/3)(\omega\xi)(\omega\beta)^{\frac{1}{2}}[(\Psi_M - \Psi_m - 1)^{\frac{3}{2}} + (\Psi_M - \Psi_m - 1)^{\frac{1}{2}}]. \tag{49}$$

Let us denote damping by the increment $\Delta\Psi_m$ between successive minima, or $\Delta\Psi_M$ between successive maxima. From (48) and (49) the general formula for damping of minima is

$$\Delta\Psi_m = (4\sqrt{2}/3)\omega\xi(\omega\beta)^{-\frac{1}{2}}[(\Psi_M - \Psi_m - 1)^{\frac{3}{2}} + (\Psi_M - \Psi_m - 1)^{\frac{1}{2}}]. \tag{50}$$

The choice of Ψ_m is ambiguous, but Ψ_M refers to the maximum between the two minima of $\Delta\Psi_m$. It is obvious from the shape of the potential $V(\Psi)$ shown in Fig. 2 that the maxima will be less damped than the minima. For small damping we have

$$\omega\beta(e^{\Psi_2} - 1)(\Psi_2 - \Psi_4) = W, \tag{51}$$

where W is now computed by integrating (47) from Ψ_2 around the cycle and back to Ψ_2. The result is

$$\Delta\Psi_M = (4\sqrt{2}/3)\omega\xi(\omega\beta)^{-\frac{1}{2}}[(\Psi_M - \Psi_m - 1)^{\frac{1}{2}} + (\Psi_M - \Psi_m - 1)^{-\frac{1}{2}}] \quad (52)$$

with the familiar ambiguity in choice of Ψ_M. From (52), (46) and (39) we obtain the very convenient formula

$$\Delta\Psi_M = \tfrac{2}{3}(\omega\xi)I \quad (53)$$

relating the damping directly to the interval. We note that all ambiguity has disappeared from (53).

We now have a complete arsenal of formulas with which to attack experimental spiking patterns. Our formulas give all of the minima and maxima of Ψ as well as the durations and intervals and the maximum of $\dot{\Psi}$ in terms of the dimensionless rate equation parameters ω, ξ, σ and $\beta = \xi - 1$. These formulas are valid in the spiking phase where

$$\Psi_M - \Psi_m - 1 \gg 1. \quad (54)$$

As the spikes damp out the solution finally enters the *small-signal phase*

$$\Psi_M - \Psi_m \to 0. \quad (55)$$

The small-signal solution is well known,[5,6,13,22,29,31] so there is no need to discuss it here, but we give it for ready reference:

$$\begin{aligned}\eta(\tau) &= 1 + Ae^{-\frac{1}{2}\omega\xi\tau}[\Omega \cos(\Omega\tau + \varphi) - \tfrac{1}{2}\omega\xi \sin(\Omega\tau + \varphi)] \\ \rho(\tau) &= \beta[1 + Ae^{-\frac{1}{2}\omega\xi\tau} \sin(\Omega\tau + \varphi)] \\ \Omega^2 &= \omega\beta - \tfrac{1}{4}(\omega\xi)^2\end{aligned} \quad (56)$$

where A is an arbitrary real small amplitude and φ is an arbitrary real phase. Exactly the same small-signal solution is obtained from (25), thereby justifying the approximation (22). It is now easy to see that when (16) breaks down the frequency Ω of the small-signal solution becomes imaginary and there are no oscillations. This may be the case in the gas laser, where we may have $\xi \sim 1$, $\beta \ll 1$, and $\omega > 4\beta$.

IV. A NUMERICAL EXAMPLE

Before attempting to apply our formulas to the analysis of spiking patterns we wish to discuss their accuracy with the aid of a machine calculation. For numerical integration the rate equations (11), (12) are best written in the form

$$\begin{aligned}\dot{x} &= \omega(\xi - x - xe^{y}) \\ \dot{y} &= \sigma e^{-y} + x - 1,\end{aligned} \quad (57)$$

where

$$x = \eta$$
$$y = \ln \rho. \tag{58}$$

As already pointed out following (17), the initial condition on x,y is not critical; it is convenient to start the solution on the adiabatic solution (17), so we take

$$y(0) = \ln \sigma, \qquad x(0) = 0. \tag{59}$$

The constants will have the values

$$\omega = 7.12 \times 10^{-6}$$
$$\xi = 5 \tag{60}$$
$$\sigma = 2 \times 10^{-9}$$

which are typical for a ruby laser. The numerical integration of (57) has been performed on the IBM 7090 computer using Hamming's[32] predictor-corrector method. The program provides for automatic halving and doubling of the integration interval under the control of the fractional error of the increment and a preselected tolerance (5×10^{-5}). The stability and accuracy of the program for this problem were checked using initial conditions which lead to the small-signal solution (56).

The results are shown in Fig. 3. The solid curve and scale on the right give $y(\tau)$, while the dotted curve and scale on the left give $x(\tau)$ for τ in the range 3×10^4 to 4×10^4. The origin for τ is completely arbitrary. From (26), (58) and (60) we have

$$y = \Psi + \ln \beta = \Psi + 1.387. \tag{61}$$

We are concerned primarily with y, since the inversion is not ordinarily observed experimentally. The main features of the computed results are summarized in Table I, which lists the values of τ_n, y_n, and m_n or M_n for $n = 1, \cdots, 9$ for the first nine extrema in the notation of (34). These values were obtained from the computed points by fitting a parabola to the three points nearest the extremum. The spacing of the computed points was sufficiently small ($\Delta\tau = 0.005 \times 10^4$) that in all cases all three points lay well within the validity of the parabolic approximations (38) or (41).

We shall now attempt to calculate the information of Table I by the formulas of the preceding section. In every case we shall indicate the

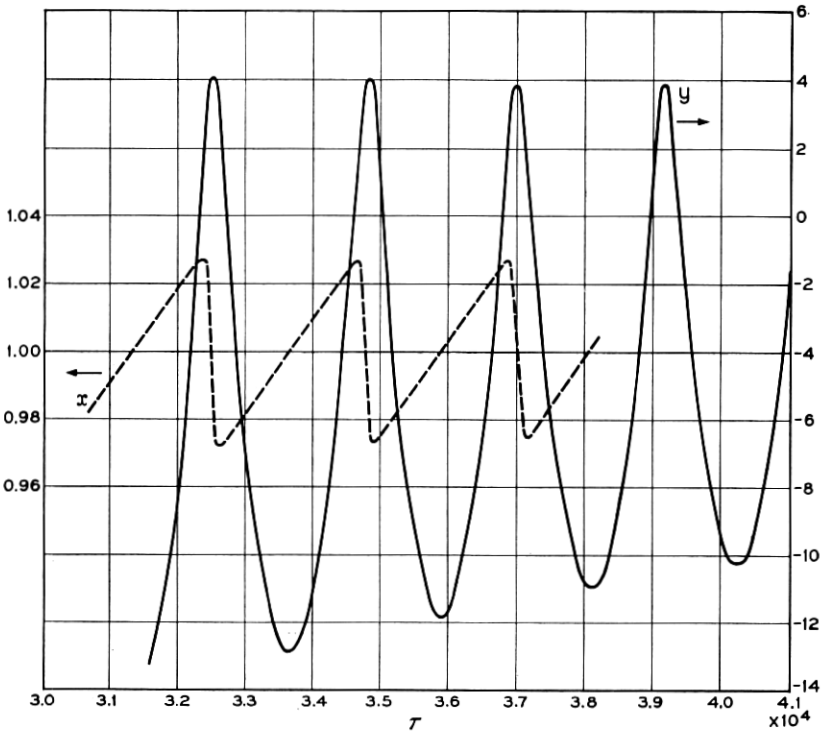

Fig. 3 — A machine calculation of $y(\tau)$, solid curve with scale on the right, and $x(\tau)$, dotted curve with scale on the left, satisfying the rate equations (57) for $\omega = 7.12 \times 10^{-6}$, $\xi = 5$, $\sigma = 2 \times 10^{-9}$. The important results are summarized in Table I. Here $x(\tau)$ represents the inversion and $y(\tau)$ the logarithmic light output.

Table I—Computed Results of Numerical Example

Summary of results of machine solution to the rate equations for $\omega = 7.12 \times 10^{-6}$, $\xi = 5$, $\sigma = 2 \times 10^{-9}$. Also obtained were $x_{\max} - 1 = 0.02830$, $\dot{y}_{\max} = 0.0282$.

n	τ_n	y_n	m_n	M_n
1	3.1344×10^4	-14.570	—	
2	3.2521	4.240		0.01416×10^4
3	3.3642	-12.815	0.0532	
4	3.4806	4.179		0.01415
5	3.5906	-11.795	0.0529	
6	3.7013	4.053		0.0151
7	3.8100	-10.986	0.0531	
8	3.9186	4.064		0.0152
9	4.0239	-10.303	0.0531	

correct value as computed by machine on the right in parentheses. According to (19)

$$\begin{aligned} \rho_1 &= 7.50 \times 10^5 \\ y_1 &= \ln \rho_1 = -14.09 \qquad (-14.57). \end{aligned} \tag{62}$$

Thus we *overestimate* y_1 by 0.48 or 3.3 per cent. From (61) the theoretical value for Ψ_1 is

$$\Psi_1 = -15.48. \tag{63}$$

Putting $\Psi_m = \Psi_1$ and $\Psi_M = \Psi_2$ in (30) gives

$$\Psi_2 = 2.91, \tag{64}$$

or

$$y_2 = 4.30 \qquad (4.240). \tag{65}$$

Thus we *overestimate* y_2 by 0.06 or 1.4 per cent. For the excursion of y we obtain

$$y_2 - y_1 = \Psi_2 - \Psi_1 = 18.39 \qquad (18.81), \tag{66}$$

which is an *underestimate* by 0.42 or 2.2 per cent. Henceforth we need not consider Ψ, but only y. From (50) using $\Psi_M - \Psi_m = y_2 - y_1$ we obtain

$$y_3 - y_1 = 0.965 \qquad (1.355) \tag{67}$$

$$y_3 = -13.12 \qquad (-12.815). \tag{68}$$

We are *underestimating* y_3 by 0.30 or 2.2 per cent. From (52) using $\Psi_M - \Psi_m = y_2 - y_3$ we obtain

$$y_2 - y_4 = 0.0541 \qquad (0.061) \tag{69}$$

$$y_4 = 4.25 \qquad (4.179). \tag{70}$$

By repeating steps (67) and (69) we could obtain values for any number of succeeding maxima and minima

$$y_5 - y_3 = 0.881 \qquad (1.020) \tag{71}$$

$$y_5 = -12.24 \qquad (-11.80) \tag{72}$$

$$y_4 - y_6 = 0.0523 \qquad (0.126) \tag{73}$$

$$y_6 = 4.19 \qquad (4.053). \tag{74}$$

We observe from Table I that $y_4 - y_6$ is unusually large (0.126), while $y_6 - y_8$ is negative (-0.011); the average of these is $\frac{1}{2}(y_4 - y_8) = (0.058)$ in better agreement with the machine-computed $y_2 - y_4$ and with theory. Thus we suspect that the discrepancy in (73) is not significant. There is apparently a slight inaccuracy in the computed solution on the third and fourth spike. One might expect that noise of various kinds would have a similar effect on real lasers, making the damping from spike to spike unreliable. Averaging over several spikes, however, would still give a meaningful value for Δy_M, as it does in our computed spike pattern.

The durations of all the minima should be given by (39)

$$m = 0.0526 \times 10^4 \qquad (0.0531 \times 10^4). \tag{75}$$

We see in Table I that m is constant within 0.4 per cent and the average over the four minima is 0.0531×10^4. The durations of the maxima will be calculated from (42), using for $\Psi_M - \Psi_m$ the values of the excursions just calculated, with the minimum preceding the maximum

$$\begin{aligned} M_2 &= 0.0127 \times 10^4 \qquad && (0.01416 \times 10^4) \\ M_4 &= 0.0130 && (0.01415) \\ M_6 &= 0.0134 && (0.0152). \end{aligned} \tag{76}$$

Here we have *underestimated* the spike durations by 11–12 per cent. The intervals are given by (46)

$$\begin{aligned} I_{2\text{-}4} &= 0.226 \times 10^4 \qquad && (0.2285 \times 10^4) \\ I_{4\text{-}6} &= 0.220 \times 10^4 && (0.2207 \times 10^4). \end{aligned} \tag{77}$$

The agreement here may be considered perfect. According to (53) we should have

$$\Delta y_M / I = 2.37 \times 10^{-5} \tag{78}$$

for all consecutive spikes; the computed results are as follows:

$$\begin{aligned} 2 &\to 4 \qquad && (2.66 \times 10^{-5}) \\ 4 &\to 6 && (5.70 \times 10^{-5}) \\ 6 &\to 8 && (0.51 \times 10^{-5}) \\ \tfrac{1}{2}[(4 \to 6) &+ (6 \to 8)] && (2.63 \times 10^{-5}). \end{aligned} \tag{79}$$

The last line is considered more significant than the discrepancies in the third and fourth lines. Thus our theory underestimates $\Delta y_M / I$ by

11 per cent. Finally, from (35)

$$\begin{aligned} \dot{y}_{\max} &= x_{\max} - 1 = 0.0288 \\ &(0.0282) \qquad (0.0283), \end{aligned} \tag{80}$$

which shows that the approximation (21) is excellent.

We may conclude from this confrontation between our formulas and a machine computation that the accuracy is probably adequate for the analysis of experimental spike patterns.

V. APPLICATION OF THE FORMULAS

Experimental data on spiking are ordinarily obtained in the form of an oscilloscope trace proportional to the light output. The trace is proportional to $p(t)$, the number of photons in the laser mode of the cavity, but it is usually considered impractical to calibrate the equipment so as to obtain the absolute value of $p(t)$. And even if $p(t)$ could be measured absolutely it would not fix the absolute value of $\rho(\tau)$ defined in (9). We shall assume that only relative values of $\rho(\tau)$ can be measured. The most significant measurements of ρ are the *peak-to-valley ratios*

$$R = \rho_M/\rho_m , \tag{81}$$

the ratio of the maximum light output of a spike to the minimum output in a neighboring minimum. In analyzing a spike pattern with many spikes, some convention must be adopted on which minimum to choose. From (26) and (61)

$$\Psi_M - \Psi_m = y_M - y_m = \ln R. \tag{82}$$

Thus the excursions of the model particle in the potential $V(\Psi)$ are uniquely fixed by the data. It follows from (30) that both Ψ_M and Ψ_m are fixed by R

$$\begin{aligned} \Psi_M &= \ln(\ln R) \\ \Psi_m &= \ln(\ln R) - \ln R. \end{aligned} \tag{83}$$

Even Ψ_1, the hypothetical minimum not actually observed, can be determined by extrapolating the other minima Ψ_m. Thus all of the extrema may be regarded as immediately fixed by the data. From the extrema alone we obtain 2 relations satisfied by the three parameters ω,ξ,σ, namely (19) and (50) or (52).

It is now clear that a single spike pattern does not contain enough information to determine ω,ξ,σ. The reason for this is that we cannot make use of the measured durations or intervals, since we do not know the ratio t_p between real time t and dimensionless time τ. We might think that we could obtain information from time ratios such as (M/m) or (m/I). It turns out that we can, but it is a different kind of information, the kind we have called "qualitative" in Section I. Consider the three quantities

$$A = (M/I) \ln R \tag{84}$$

$$B = (m/I)[(\ln R - 1)^{\frac{1}{2}} + (\ln R - 1)^{-\frac{1}{2}}] \tag{85}$$

$$C = (\Delta\Psi_M/\Delta\Psi_m)(\ln R - 1), \tag{86}$$

all of which can be determined immediately from the data since they depend only on ratios. It is to be expected that C might have to be averaged over several spikes to get a meaningful value. Except for this difficulty, values of A, B, and C may be calculated for every spike in the data after adopting some convention to handle the usual ambiguity in the definitions (84), (85), (86). One such convention is illustrated by the following example:

$$\ln R_2 = \Psi_2 - \Psi_3$$

$$\left.\begin{aligned} A_2 &= (M_2/I_{2\text{-}4}) \ln R_2 \\ B_2 &= (m_3/I_{2\text{-}4})[(\ln R_2 - 1)^{\frac{1}{2}} + (\ln R_2 - 1)^{-\frac{1}{2}}] \\ C_2 &= [(\Psi_2 - \Psi_4)/(\Psi_3 - \Psi_1)](\ln R_2 - 1). \end{aligned}\right. \tag{87}$$

From (42), (46), (50) and (52) we find the simple relations

$$A = B = C = 1 \tag{88}$$

which do not involve the parameters ω,ξ,σ. Therefore (88) should hold even if ω,ξ are slowly-varying functions of time within our adiabatic approximation. We call the relations (88) *consistency relations*, since they can be used to determine whether data are consistent with the rate equations. If (88) is satisfied reasonably well, at least in an average sense over the spikes, it is a foregone conclusion that a reasonable fit to the data can be obtained from the rate equations. The converse is also true: if (88) is not satisfied the data cannot be fitted from the rate equations.

The importance of observing the minima (valleys) as well as the maxima (peaks) in a spike pattern is abundantly clear. Without the valleys we cannot determine R, m, or $\Delta\Psi_m$, all of which appear in the consistency relations. Therefore a great deal of information is lost unless the valleys can be seen above noise. This presents some difficulty, because ordinarily $R \gg 1$ is much too large to be measured from an oscilloscope trace that responds linearly to the light output and contains both the peaks and the valleys. In spike patterns the valleys usually just correspond to the noise level in the experiment. It is not our purpose to go into experimental details except to point out that with care the valleys should be observable. The basic experimental requirement is that the acceptance cone of the detector should correspond to the radiation cone of the laser so as to exclude extraneous light from the pump and the spontaneous emission of the laser medium. If it is not possible to measure the valleys, there is still one consistency relation that can be applied to the peaks alone. From (46), (82) and (83) we have

$$\begin{aligned} \ln R &= I/M \\ -\Psi_m &= (I/M) - \ln (I/M). \end{aligned} \tag{89}$$

The use of (46) is equivalent to assuming $A = 1$. Thus the minima can be calculated from I/M if we assume $A = 1$. We could obtain the durations m of the minima by putting $B = 1$. The damping of the minima would not be given very reliably by (89), but can be obtained from (86) by putting $C = 1$. Thus we can deduce R, m, and $\Delta\Psi_m$ from the peaks alone, but we lose all of our consistency relations (88). However, from (89), (42), (39) and (35) it follows that

$$D = 1, \tag{90}$$

where

$$D = (M\dot{\Psi}_{\max}/4)[(I/M) - 1]^{\frac{1}{2}}/[(I/M) - \ln (I/M) - 1]^{\frac{1}{2}} \tag{91}$$

can be determined from the peaks if the time resolution is good enough to give a good value for $(M\dot{\Psi}_{\max})$. If (90) is satisfied, it is probably good evidence that the laser obeys the rate equations, and a rate equation analysis is meaningful. However, if all that is desired is to apply the rate equations blindly to obtain quantitative information, it is not necessary to measure $M\dot{\Psi}_{\max}$. All of the quantitative information in a consistent spike pattern can be deduced from I, M, and $\Delta\Psi_M$.

We now consider practical ways of obtaining quantitative information from spike patterns. The *one-pattern method* is to measure t_p by an independent experiment. The measurement of the cavity losses has been discussed by several authors.[26,33] Suffice it to say here that t_p can be measured from the dependence of the threshold flash energy for producing laser action on the temperature and on losses deliberately introduced into the laser mode. Once t_p is measured all of the times m, M and I become known in dimensionless time. From (39)

$$\omega\beta = \omega(\xi - 1) = 8/m^2, \tag{92}$$

and from (53)

$$\omega\xi = \tfrac{3}{2}\Delta\Psi_M/I. \tag{93}$$

These equations can be solved for ω,ξ

$$\begin{aligned} \xi &= \gamma m/(\gamma m - 1) \\ \omega &= 8(\gamma m - 1)/m^2, \end{aligned} \tag{94}$$

where

$$\gamma = \tfrac{3}{16}(m/I)\Delta\Psi_M \tag{95}$$

is independent of t_p. Now σ can be obtained from (19) and (89)

$$\sigma = \tfrac{1}{2}\beta(\omega\beta)^{\frac{1}{2}}(I/M)e^{-(I/M)} \tag{96}$$

with $M = M_2$ and $I = I_{2\text{-}4}$. This procedure makes use of the duration m of the valleys but not the peak-to-valley ratio R. If only the peaks are observed we write instead of (92)

$$\omega\beta = \omega(\xi - 1) = (8/MI)/[1 - (M/I)]. \tag{97}$$

Solving (93) and (97) for ω,ξ gives

$$\begin{aligned} \xi &= \delta M/(\delta M - 1) \\ \omega &= 8(\delta M - 1)/(I - M)M, \end{aligned} \tag{98}$$

where

$$\delta = \tfrac{3}{16}[1 - (M/I)]\Delta\Psi_M \tag{99}$$

is independent of t_p. This kind of analysis can be applied to every spike in a spike pattern. It may be found that ξ and especially ω vary slowly from spike to spike, which is permissible within our adiabatic approximation. If the data satisfy the consistency relations (88), the same values of ω,ξ will be obtained from (94) and (98). The greatest weakness of this method is that (94) fails completely if $\gamma m \leqq 1$; (98) fails if $\delta M \leqq 1$. Thus a great deal depends on the accuracy of the formulas as well as that of the measurements of t_p and γ or δ. It follows that the method will fail whenever $\xi \gg 1$.

As an example we shall apply this method to the machine calculation considered in Section IV. We consider the solid curve of Fig. 3 to be the data, which implies that t_p is known and both the peaks and valleys have been studied. The relevant numbers have already been given in parentheses in (69), (75), (76) and (77); we repeat them here without parentheses

$$\begin{aligned} \Delta\Psi_M &= y_2 - y_4 = 0.061 \\ m &= 0.0526 \times 10^4 \\ M &= M_2 = 0.0142 \times 10^4 \\ I &= I_{2\text{-}4} = 0.2285 \times 10^4. \end{aligned} \tag{100}$$

From (94), (95) and (96) we obtain

$$\begin{aligned} \xi &= 3.4 && (5.0) \\ \omega &= 12 \times 10^{-6} && (7.12 \times 10^{-6}) \\ \sigma &= 10 \times 10^{-9} && (2 \times 10^{-9}). \end{aligned} \tag{101}$$

The lack of accuracy is primarily due to the fact that $\xi = 5$ is a little too large to give good results with this method. Using only the peaks, we obtain from (96), (98) and (99)

$$\begin{aligned} \xi &= 2.9 && (5.0) \\ \omega &= 14 \times 10^{-6} && (7.12 \times 10^{-6}) \\ \sigma &= 8 \times 10^{-9} && (2 \times 10^{-9}). \end{aligned} \tag{102}$$

We now describe a *two-pattern method* which does not require the measurement of t_p. In fact, it yields a value for t_p and may in some cases give better results for ω,ξ than the one just described. It is based upon observing the valleys in two or more spike patterns for which the *relative* values of ω and ξ are known. We must assume that the linewidth $\Delta\nu$ is known as a function of temperature. Let us suppose that we measure the valley durations m and m' in two spike patterns in which the ratios (ξ'/ξ) and (ω'/ω) are known. We know (ξ'/ξ) from the temperatures at which the patterns were obtained. We can obtain ω'/ω from the relative pump intensities at the times when spiking occurred. It immediately follows from (92) that

$$\xi = \frac{(\omega m^2/\omega' m'^2) - 1}{(\omega m^2/\omega' m'^2) - (\xi'/\xi)} . \tag{103}$$

This result is meaningful providing that the data give $(\omega m^2/\omega' m'^2)$ lying between (ξ'/ξ) and unity. Once ξ is determined ω can be calculated from

(92) and (93)

$$\omega = \frac{9}{32} \frac{(\xi - 1)}{\xi^2} \left(\frac{m}{I}\right)^2 (\Delta\Psi_M)^2, \tag{104}$$

and σ is again obtained from (96).

As an example of this method we shall apply it to two machine-calculated spike patterns, one of which is that of Fig. 3 with the parameters (60), and the other has the parameters

$$\begin{aligned} \omega' &= \omega \\ \xi' &= 4 \\ \sigma' &= \sigma. \end{aligned} \tag{105}$$

From the machine-calculated pattern we find

$$m' = m_3' = 0.0614 \times 10^4. \tag{106}$$

The value of $m = m_3$ is given in (75). We assume that the ratio

$$\xi'/\xi = 0.8 \tag{107}$$

is known from the temperatures at which the data were taken, and the ratio

$$(\omega m^2/\omega' m'^2) = 0.748 \tag{108}$$

is calculable from (ω/ω') and the two valleys. We now obtain from (103) the value

$$\xi = 4.85 \qquad (5.0). \tag{109}$$

From (104) we now obtain

$$\omega = 9.0 \times 10^{-6} \qquad (7.12 \times 10^{-6}) \tag{110}$$

using the values in parentheses from (75), (77) and (69). From (39), (109) and (110) we obtain the *absolute value* of m

$$m = 0.048 \times 10^4 \qquad (0.0531 \times 10^4). \tag{111}$$

It follows that t_p is given by

$$t_p = m_{\text{exp}}/m, \tag{112}$$

where m_{exp} is the measured duration in laboratory time. We conclude that the two-pattern method is to be preferred to the one-pattern method as a general approach to spike analysis. It should be mentioned that there seems to be no two-pattern method involving only the peaks which is sufficiently accurate to give meaningful results.

The most convenient experiments to perform involve changing the pump intensity $1/t_g$ while all other parameters remain fixed. If $t_g \ll t_r$, as is usually the case in flash-pumped lasers, ξ will be independent of t_g and ω will vary as $1/t_g$. Varying ω does not give quantitative information such as we obtained from varying ξ in the two-pattern method. Nevertheless, it is of interest to consider what effects are to be expected. From (30), (26) and (19) we have roughly

$$\begin{aligned} \rho_M &\sim -\beta\psi_m \\ &\sim \beta \ln (\beta\sqrt{\omega\beta}/2\sigma). \end{aligned} \tag{114}$$

Thus the maxima in ρ depend only logarithmically on pumping power. It follows from (9) that the observed light output proportional to p should vary as

$$p \propto 1/t_g . \tag{115}$$

The time intervals m, M, and I should vary as

$$m \propto M \propto I \propto t_g^{\frac{1}{2}}. \tag{116}$$

VI. SUMMARY

We have now outlined a comprehensive program for the study of lasers by means of their spiking patterns. The rate equations have been formulated in terms of the light output and the atomic inversion and five physical parameters in (1) and (2). We have written the equations in a general form valid for three- and four-level systems and taking into account statistical weights and thermalization of the upper laser level. These equations were then reduced to dimensionless form in terms of three parameters in (11) and (12). All of the properties of the spiking solutions of experimental interest were then deduced analytically by means of the mechanical analogy shown in Fig. 2, in which a particle moves in a potential well in a viscous medium.

The formulas obtained were illustrated and tested for accuracy against a machine-computed solution to the rate equations. The value of the formulas was confirmed by this comparison, and we went on to discuss their application to experimentally observed spiking patterns. Four relations were given whereby spike patterns can be tested for consistency with the rate equations. These consistency relations do not contain any parameters, only ratios of times and of light outputs.

Two methods were described for obtaining quantitative information. In the one-pattern method everything is deduced from a single spike pattern, but it is necessary to measure the photon lifetime t_p in the cavity by independent experiments. It is possible to apply this method to data in which only the peaks are observed. It is emphasized, however, that the observation of the valleys in light output should be perfectly feasible and very much worthwhile. In the two-pattern method all the parameters and t_p are deduced from two patterns obtained at different temperatures. In this method, the more accurate of the two, it is essential that the valleys be observed.

Our objective has been to provide the researcher with a set of tools for applying the rate equations to experimental spiking data. The spiking phenomenon in solid state lasers can be utilized in research now that it is beginning to come under good experimental control. We have tried here to point out what kind of spiking data is needed, and what additional information is needed, to get the maximum information from spiking. Our analysis applies to the sharp spike region of time in which the rate equations are highly nonlinear. It complements the well known small-signal analysis in which the rate equations become linear. The entire discussion is based upon the rate equations of Statz and De Mars, which we regard as reasonable, relevant and widely accepted. We have not gone into the derivation of these equations or the modifications that have been proposed or might be proposed. We prefer to leave that to the future, when we may reasonably expect that systematic spiking studies will have clearly revealed the adequacy or inadequacy of these equations, and indicated the direction which these modifications must take.

VII. ACKNOWLEDGMENTS

This work actually began several years ago in a joint effort with R. J. Collins, now of the University of Minnesota. Circumstances intervened to prevent the completion of our collaboration, but I am pleased to acknowledge the great benefits that accrued to me through it. For interest, encouragement, and informative discussions I am also indebted to D. F. Nelson, W. K. Kaiser and D. E. McCumber of Bell Telephone Laboratories.

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