Design of Wideband Sampled-Data Filters

By R. M. GOLDEN and J. F. KAISER

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A design procedure is presented for readily obtaining sampled-data filter representations of continuous filters. The procedure utilizes the bilinear z transformation and preserves the essential amplitude characteristics of the continuous filter over the frequency range between zero and one-half the sampling frequency. It is shown that the procedure can yield meaningful sampled-data filter designs for many of those filters where the standard z transform cannot be used directly.

I. INTRODUCTION

Sampled-data filter representations for continuous filters can be obtained using several different design procedures.[1] A particular design method utilizing the bilinear transformation is developed herein. The method is especially useful in designing wideband* sampled-data filters which exhibit relatively flat frequency-magnitude characteristics in successive pass and stop bands. Filters of this type are widely used in network simulation and data processing problems.[2] The design method possesses two chief advantages over the standard z transform.[3] The first is that the transformation used is purely algebraic in form. This means it can be applied easily to a continuous filter having a rational transfer characteristic expressed in either polynomial or factored form. The second advantage is the elimination of aliasing[4] errors inherent in the standard z transform. Thus, the sampled-data filter obtained by this design method exhibits the same frequency response characteristics as the continuous filter except for a nonlinear warping of the frequency scale. Compensation for this warping can be made by a suitable frequency scale modification. Some of the more common filter networks to which the design method can be applied effectively are the Butterworth, Bessel, Chebyshev, and elliptic filter structures.

The essential properties of the bilinear transformation are presented

* A sampled-data filter design will be termed "wideband" if the frequency range of useful approximation approaches half the sampling frequency.

in the next section. For comparison purposes, properties of the standard z transform are also given. This is followed by a detailed description of a filter design procedure using frequency transformations. Examples illustrating the design procedure are then presented. A short discussion concerning computer simulation of the obtained sampled-data filters is also included.

II. THE STANDARD AND BILINEAR z TRANSFORMATIONS

In this section it is assumed that a satisfactory rational expression is known for the transfer function of a continuous filter for which a sampled-data approximation is sought. What is then necessary is a transformation which will convert this transfer characteristic into a sampled-data transfer function rational in z^{-1}, the unit delay operator.

Two transformations applicable to this problem are the standard z transform and the bilinear z transformation.

The standard z transform applied to $H(s)$, the transfer function of the filter, is[3]

$$H^*(s) = \sum_{m=-\infty}^{\infty} H(s + jm\omega_s) \tag{1}$$

or equivalently in terms of the impulse response, $h(t)$, of the filter

$$\mathcal{H}^*(z) = T \sum_{l=0}^{\infty} h(lT) z^{-l} \tag{2}$$

where

$$\begin{aligned}
s &= \sigma + j\omega \\
H(s) &= \text{Laplace transform of } h(t) \\
\omega_s &= 2\pi/T = \text{radian sampling frequency} \\
H^*(s) &= \text{Laplace transform of the sampled filter impulse response} \\
z^{-1} &= \exp(-sT) = \text{the unit delay operator} \\
\mathcal{H}^*(z) &= H^*(s) \mid_{s = (\ln z)/T} = z \text{ transform of } h(t).
\end{aligned}$$

The behavior of $H(s)$ for s greater than some critical frequency $j\omega_c$ is assumed to be of the form

$$H(s) \mid_{\text{all } s > j\omega_c} = K/s^n, \qquad n > 0 \tag{3}$$

where K is a determined constant.

Equation (1) or (2) is the transfer function of a sampled-data filter which *approximates* the continuous filter. In the time domain, the impulse response of the sampled-data filter is the sampled impulse response of the continuous filter. This can be shown by taking the inverse transform of (2). Equation (1) shows that in the baseband

$$(-\omega_s/2 \leqq \omega \leqq \omega_s/2)$$

the frequency response characteristics of the sampled-data filter, $H^*(s)$, differ from those of the continuous filter, $H(s)$. The difference is the amount added or "aliased"[4] in through terms of the form

$$H(s + jm\omega_s),\ m \neq 0.$$

If $H(s)$ is bandlimited to the baseband, i.e., $| H(s) | = 0$ for $\omega > \omega_s/2$, then there is no aliasing error and the sampled-data filter frequency response is identical to that of the continuous filter. Unfortunately, when $H(s)$ is a rational function of s, it is not bandlimited and therefore $H(s) \neq H^*(s)$ in the baseband.

The magnitude of the errors resulting from aliasing is directly related to the high-frequency asymptotic behavior of $H(s)$ as defined in (3). If n is large and $\omega_c \ll \omega_s/2$, then the aliasing errors will be small and the standard z transform generally will yield a satisfactory sampled-data filter design. However, in wideband designs, ω_c is usually an appreciable fraction of $\omega_s/2$. Furthermore, many continuous filter designs result in transfer functions in which n is no greater than 1. These two conditions, namely $\omega_c \approx \omega_s/2$ and $n = 1$, can create large aliasing errors in the frequency response characteristics, thus yielding an unusable result.

Fortunately, even when $\omega_c \approx \omega_s/2$ and $n = 1$, a design method employing the bilinear z transformation* may provide satisfactory wideband designs. This z form is defined from the mapping transformation,

$$s = (2/T) \tanh (s_1 T/2) \qquad (4)$$

where

$$s_1 = \sigma_1 + j\omega_1 .$$

The right-hand side of (4) is periodic in ω_1 with period $2\pi/T$. Considering only the principal values of ω_1, $-\pi/T < \omega_1 < \pi/T$, it is seen that the transformation given by (4) maps the *entire* complex s plane into the strip in the s_1 plane bounded by the lines $\omega_1 = -\pi/T$ and $\omega_1 = +\pi/T$. For this reason the bilinear transformation can be looked upon as a bandlimiting transformation. Therefore, when this transformation is applied to a transfer function $H(s)$, the entire s-plane frequency characteristics of $H(s)$ are uniquely carried over into the frequency characteristics of $H(s_1)$.

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* This transformation will be referred to as the bilinear z form or z form.

With the substitution

$$z^{-1} = e^{-s_1 T},$$

(4) can be written immediately as

$$s = \frac{2}{T} \frac{(1 - z^{-1})}{(1 + z^{-1})}. \tag{5}$$

Thus,

$$\mathcal{H}(z) \equiv H(s) \Big|_{s = \frac{2}{T} \frac{(1-z^{-1})}{(1+z^{-1})}} \tag{6}$$

where $\mathcal{H}(z)$ denotes a sampled-data transfer function obtained by the use of the bilinear z form.*

The transfer function $\mathcal{H}(z)$ obtained by means of the bilinear z form and the function $\mathcal{H}^*(z)$ obtained by means of the standard z transform are both rational in z^{-1} and of the same denominator order as the continuous filter. These two functions, $\mathcal{H}(z)$ and $\mathcal{H}^*(z)$, become essentially equal to each other as the sampling frequency becomes large compared to the moduli of each of the poles of the continuous filter function, $H(s)$. When the sampling frequency is not large, representation of some filters by the standard z transform can be quite unsatisfactory because of aliasing errors. However, the bilinear z form, with its absence of aliasing errors, may give a satisfactory representation for these filters. A particular set of filters to which the bilinear z form always can be applied successfully are those which exhibit relatively flat frequency-magnitude characteristics in successive pass and stop bands. This follows directly from (6).

Thus, sampled-data filters designed by using the bilinear z form preserve the essential amplitude characteristics of the continuous filter. In the baseband $(-\omega_s/2 \leqq \omega \leqq \omega_s/2)$, the frequency characteristics of the sampled-data filter are identical to those of the continuous filter *except* for a nonlinear warping of the frequency scale.

This warping is found from (4) upon substituting $j\omega$ for s and is

$$\omega = (2/T) \tan (\omega_1 T/2). \tag{7}$$

For small values of ω_1, the relation is essentially linear, producing

* It should be noted that the bilinear transform is used here in a distinctly different way than it is commonly used in sampled-data control system design. In the control system literature it is used to transform the sampled-data function $\mathcal{H}^*(z)$ from the discrete domain back to the continuous domain for conventional stability and frequency response analysis. See for example J. T. Tou, *Digital and Sampled-Data Control Systems*, McGraw-Hill, New York, 1959, pp. 244–247 and pp. 466–470.

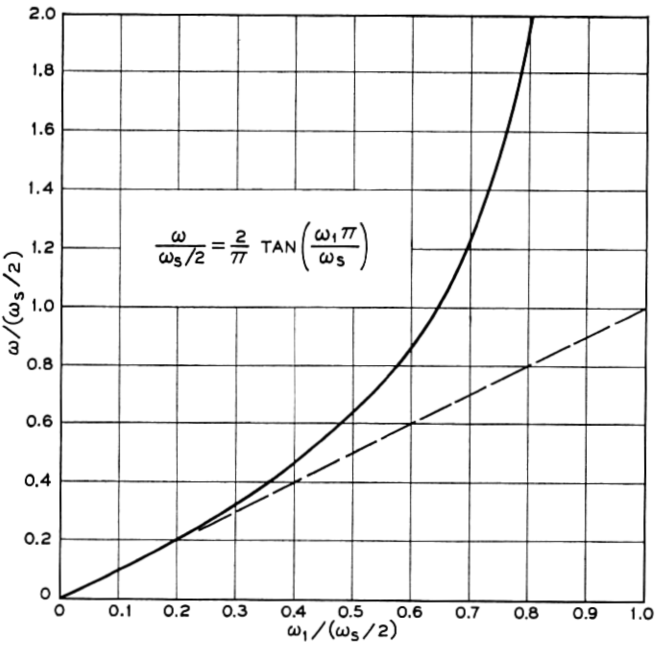


Fig. 1 — The frequency scale warping of the bilinear z transformation.

negligible warping at the lower end of the frequency scale. Fig. 1 shows the nature of this warping. Compensation for the effect of warping can be made by prewarping the band-edge frequencies of the continuous filter in such a way that application of the z-form transformation will shift the band-edge frequencies back to the desired values. The incorporation of this prewarping compensation into a sampled-data filter design procedure is discussed in the next section.

III. A SAMPLED-DATA FILTER DESIGN METHOD

A sampled-data filter design may be obtained by applying the z-form transformation of (6) to the rational transfer characteristic for a continuous filter. However, in order to compensate for the frequency warping imposed by the z form, the frequency characteristics of the continuous filter first must be altered or prewarped. Hence the transfer characteristic for the continuous filter must be redesigned such that the band-edge (cutoff) and maximum loss frequencies are computed according to,

$$\omega_c = (2/T) \tan (\omega_d T/2) \tag{8}$$

where:

ω_c = computed cutoff or loss frequency
ω_d = desired cutoff or loss frequency.

The redesign of the continuous transfer characteristic cannot be accomplished simply by applying (8) to each pole and zero of the original transfer characteristic. On the contrary a completely new transfer characteristic must be obtained for the continuous filter. It is then possible to obtain the desired sampled-data filter by applying the z-form of (6) directly to the redesigned transfer characteristic of the continuous filter. The sampled-data filter so obtained will then have the desired magnitude-frequency characteristics.

Compensation for frequency warping becomes especially simple to apply if the original continuous filter design was obtained by applying a frequency-band transformation to a suitable low-pass design such as Butterworth, Chebyshev, etc. Thus the extensive literature available on tabulated low-pass filter designs can be used to great advantage to simplify the filter design problem. The well-known frequency transformations which convert a normalized low-pass filter to a low-pass, a bandpass, a bandstop, or a high-pass design are

$$s_n = s/\omega_u \text{ low-pass to low-pass} \tag{9}$$

$$s_n = \frac{(s^2 + \omega_u\omega_l)}{s(\omega_u - \omega_l)} \text{ low-pass to bandpass} \tag{10}$$

$$s_n = \frac{s(\omega_u - \omega_l)}{(s^2 + \omega_u\omega_l)} \text{ low-pass to bandstop} \tag{11}$$

$$s_n = \omega_u/s \text{ low-pass to high-pass} \tag{12}$$

where:

s_n = the complex variable of the normalized low-pass filter transfer function
s = the complex variable of the desired filter transfer function
ω_u = the upper radian cutoff frequency
ω_l = the lower radian cutoff frequency.

When continuous filters are designed with the aid of these transformations, prewarping is accomplished by properly choosing the cutoff frequencies used in the frequency transformations. The choice of these cutoff frequencies is determined from the desired cutoff frequencies by means of (8). Using these values, the new prewarped transfer function is determined by applying (9), (10), (11) or (12) to the original low-pass function. Transformation is made to a sampled-data filter by applying (5) to the prewarped continuous function. This sampled-data filter will now have the correct cutoff frequencies. The transfer function thus obtained can be used directly in a digital computer simulation.

IV. SIMULATION OF SAMPLED-DATA FILTERS

Application of either the standard z transform or the bilinear z form to a rational transfer function yields a transfer function rational in z^{-1} for the sampled-data filter. The programming or simulation of this sampled-data filter on a digital computer can be accomplished by either the direct, the cascade or the parallel form. These forms, as commonly defined, are shown in Fig. 2. In this figure $\mathcal{G}(z)$ and $\mathcal{F}(z)$ represent finite polynomials in z^{-1} for feed-forward and feedback transmissions re-

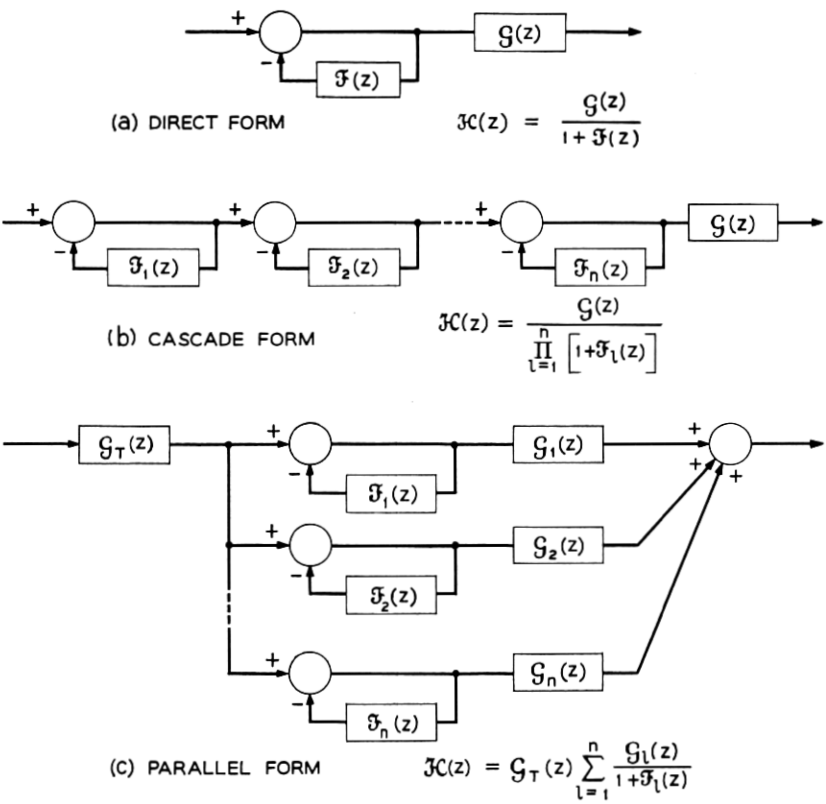


Fig. 2 — Some possible simulation forms for sampled-data filters.

spectively; where subscripted, the order of the polynomial is at most second.

The choice of which of the three forms to use for simulation of the sampled-data filter depends on the complexity of the filter function $H(s)$, on the form of $H(s)$, and on the particular z transformation used. Generally, simulation by the direct form requires considerably greater accuracy in the determination of the filter parameters than either of the other two forms. This is especially true when the order of $H(s)$ is large and when $H(s)$ has poles with real parts that are a very small fraction of the sampling frequency. For this reason either the cascade or parallel form may be preferred.

The choice between using the cascade or the parallel form depends largely on which z-transform method is used to obtain the sampled filter and how that particular method is applied. Realization in the cascade form requires calculation of the numerator polynomial, $\mathcal{G}(z)$, or its factors. This computation consists of a simple algebraic substitution when the bilinear z form is applied to a filter function $H(s)$ expressed in the form,

$$H(s) = \frac{G(s)}{\prod_{k=1}^{m} (s - \alpha_k)}. \tag{13}$$

Determination of $\mathcal{G}(z)$ in polynomial or product form respectively allows either of the following cascade realizations to be made:

$$\mathcal{H}(z) = \mathcal{G}(z) \prod_{k=1}^{n} \left(\frac{1}{1 + b_{1k}z^{-1} + b_{2k}z^{-2}} \right) \tag{14}$$

or

$$\mathcal{H}(z) = \prod_{k=1}^{n} \left(\frac{a_{0k} + a_{1k}z^{-1} + a_{2k}z^{-2}}{1 + b_{1k}z^{-1} + b_{2k}z^{-2}} \right). \tag{15}$$

If the numerator $G(s)$ is in polynomial form, considerable care must be taken in the calculation of the coefficients of the polynomial $\mathcal{G}(z)$, as this computation involves differencing nearly equal numbers.

For realization in the parallel form the partial fraction expansion of $H(s)$ must be known. Since in the standard z transform method obtaining the partial fraction expansion is a necessary step, simulation of filters designed by this method is most directly accomplished in the parallel form. Here the continuous filter transfer characteristic is represented by

$$H(s) = \sum_{k=1}^{N} \frac{P_{1k}s + P_{0k}}{Q_{2k}s^2 + Q_{1k}s + Q_{0k}}. \tag{16}$$

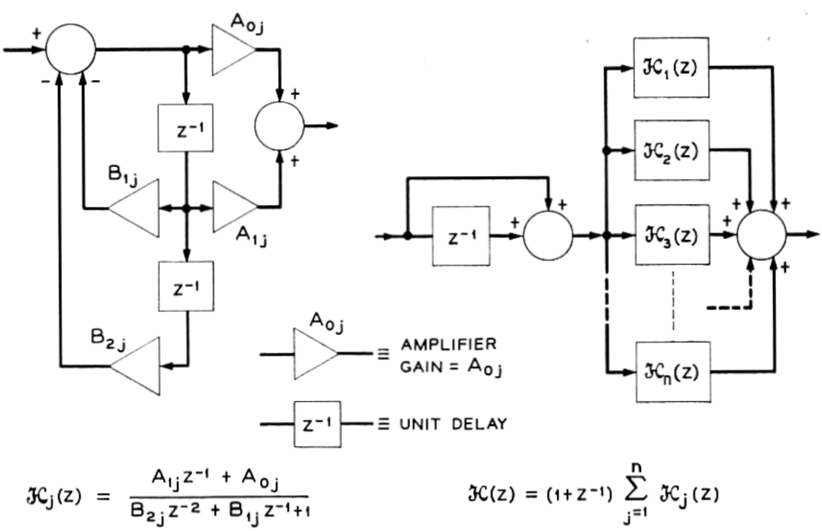
Transforming this expression by use of the standard z transformation yields

$$\mathcal{H}(z) = \sum_{k=1}^{N} \frac{A_{1k}z^{-1} + A_{0k}}{B_{2k}z^{-2} + B_{1k}z^{-1} + 1} \tag{17}$$

whereas transforming by use of the bilinear z form yields the similar expression

$$\mathcal{H}(z) = (1 + z^{-1}) \sum_{j=1}^{N} \frac{A_{1j}z^{-1} + A_{0j}}{B_{2j}z^{-2} + B_{1j}z^{-1} + 1}. \tag{18}$$

Each rational function in either of the above summations can be synthesized by the recursive structure shown in functional block diagram form in Fig. 3(a). This recursive structure uses only two delays, four multiplications, and five additions. The complete realization of (18) is shown in Fig. 3(b).



(a) TERM IN PARTIAL FRACTION EXPANSION

(b) COMPLETE TRANSFER FUNCTION EXPRESSED AS A PARTIAL FRACTION EXPANSION

Fig. 3 — Simulation in parallel form of a sampled-data filter obtained by the bilinear z transformation.

The programming of these sampled-data filters for computer simulation can be greatly simplified if a compiler such as the Block Diagram (BLODI) compiler[5] developed at Bell Telephone Laboratories is used. The compiler permits specification of a sampled-data system in functional block diagram form.

In the following section an example is presented for a filter designed, synthesized, and simulated by the foregoing method.

V. DESIGN, SYNTHESIS, AND SIMULATION OF A WIDEBAND BANDSTOP SAMPLED-DATA FILTER

As an example of the application of the bilinear z-form to the simulation of a practical filter, consider the design of a particular bandstop filter. The filter is to exhibit at least 75 db loss in a rejection band which extends between 2596 cps and 2836 cps. Below 2588 cps and above 2844 cps, the loss is to be between 0 and +0.5 db. The sampling rate of the discrete filter is to be 10 kc. The complexity or order of the filter is to be held to a minimum. Fig. 4 shows a sketch of the amplitude response characteristics desired of the filter.

Minimum filter complexity and sharp transition between pass and stop bands suggest the use of an elliptic filter[6] (equiripple) as the basic low-pass type. However, before a suitable low-pass structure can be determined, the above specified critical frequencies must be prewarped by

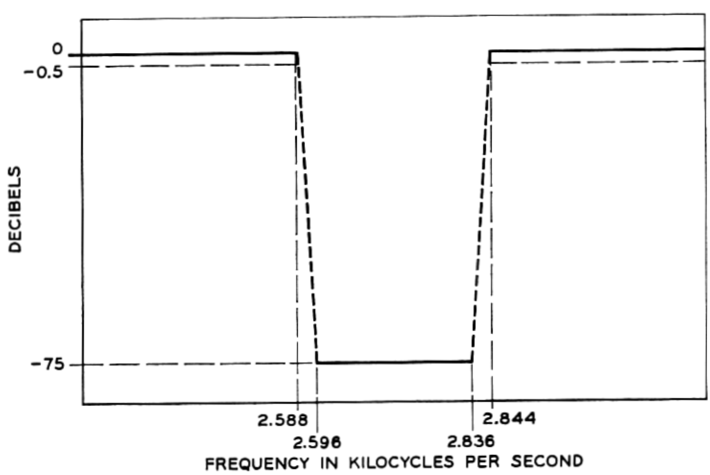


Fig. 4 — Desired amplitude response characteristic of a bandstop filter.

means of (8). The warped values are:

f_{lp} = lower cutoff frequency in passband (2588 cps) = 3364.15 cps
f_{lr} = lower cutoff frequency in rejection band (2596 cps)
= 3381.13 cps
f_{ur} = upper cutoff frequency in rejection band (2836 cps)
= 3937.54 cps
f_{up} = upper cutoff frequency in passband (2844 cps) = 3957.84 cps.

The warped values at the lower band edge require a low-pass filter with a transition ratio of 0.93792, while the values at the upper band edge require a transition ratio[6] of 0.93658. Therefore, to meet the original specifications, the larger of the two transition ratios must be chosen. Hence specifications required for the basic low-pass elliptic filter are:

in-band ripple = 0.5 db
out-of-band minimum attenuation = 75.0 db
transition ratio = 0.93792.

Application of elliptic filter design procedure with these specifications yields a basic low-pass structure of eleventh order. The poles and zeros for the transfer function of this low-pass filter are listed in Table I. The low-pass filter has been normalized to have a cutoff frequency of one radian per second and amplitude gain of unity at zero frequency.

TABLE I—POLES AND ZEROS OF NORMALIZED ELEVENTH-ORDER ELLIPTIC LOW-PASS FILTER

In-band ripple = 0.500 db
Minimum attenuation = 76.504 db
Transition ratio = 0.937917
Gain factor = 0.0011060
Poles
$-0.0069130 \pm j\ 1.0010752$
$-0.0257616 \pm j\ 0.9756431$
$-0.0615122 \pm j\ 0.9063786$
$-0.1269215 \pm j\ 0.7504391$
$-0.2142976 \pm j\ 0.4483675$
-0.2611853
Zeros
$\pm j\ 1.0695414$
$\pm j\ 1.1009005$
$\pm j\ 1.1946271$
$\pm j\ 1.4652816$
$\pm j\ 2.5031313$

The desired bandstop filter is obtained next by applying the low-pass-to-bandstop transformation, given in (11), to the normalized elliptic low-pass filter. The cutoff frequencies used are the warped values obtained above for f_{lp} (3364.15 cps) and f_{up} (3957.84 cps). The resulting bandstop filter is then transformed by the bilinear z form to yield the required sampled-data filter. Table II lists the coefficients of the resulting sampled-data filter needed for parallel realization of the form shown in Fig. 3. Fig. 5 shows the frequency response characteristics of this sampled-data filter. It is seen that the original filter specifications are met by the sampled-data filter. For comparison purposes, the standard z transform was applied directly to the twenty-second-order continuous filter. The frequency response characteristic of this filter is shown in Fig. 6. It is seen that the standard z transform has yielded an unusable result.

VI. SUMMARY

The need for sampled-data filters in wideband simulations of many processing systems has led to a synthesis method which overcomes the shortcomings of the standard z transform. The method presented consists of directly transforming a suitable continuous transfer function to a sampled-data filter by means of the bilinear z form. For wideband filters the method is particularly suited to those filters that exhibit relatively constant magnitude-frequency characteristics in successive pass and stop bands. Conventional design techniques of continuous filters are used

TABLE II—PARTIAL-FRACTION EXPANSION COEFFICIENTS FOR PARALLEL REALIZATION OF SAMPLED-DATA BANDSTOP FILTER

Term	Numerator Coefficients		Denominator Coefficients	
	A_1	A_0	B_2	B_1
1	0.0001628	0.0008827	0.9987854	0.1106416
2	−0.0009283	−0.0001764	0.9989898	0.4285348
3	−0.0024098	−0.0027894	0.9956089	0.1063723
4	0.0031774	0.0026966	0.9957459	0.4317548
5	0.0102446	0.0026026	0.9879911	0.0940731
6	−0.0037799	−0.0112135	0.9883051	0.4414974
7	−0.0277640	0.0127415	0.9651789	0.0616261
8	−0.0108027	0.0289421	0.9661438	0.4663508
9	0.0272223	−0.1163873	0.8694592	−0.0204564
10	0.1206914	−0.0054765	0.8742300	0.5186036
11	0.2973946	−0.2973227	0.5283651	0.2074591

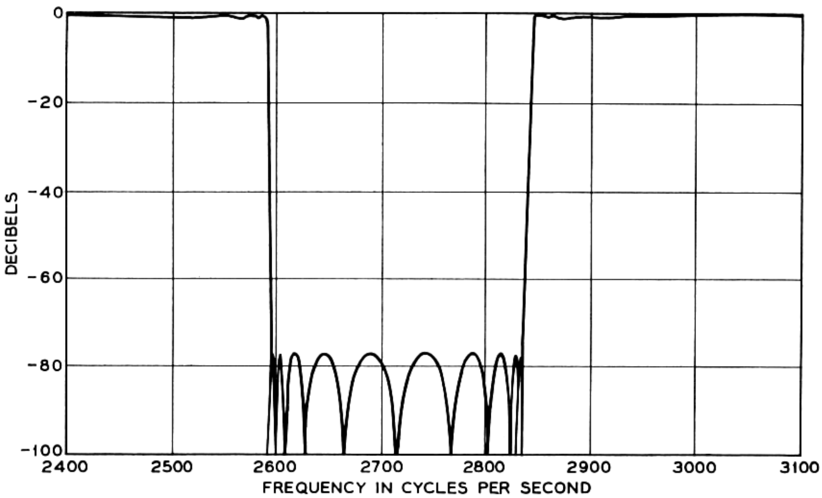


Fig. 5 — Frequency response characteristics of the sampled-data bandstop filter designed by the bilinear z transformation.

directly in the synthesis procedure of the sampled-data filters. (Thus the synthesized filters have frequency characteristics comparable to those of continuous filters.) An example has been presented of a filter function synthesized by this procedure and easily programmed for a simulation. Results obtained from this example demonstrate the usefulness and accuracy of the bilinear z-form method.

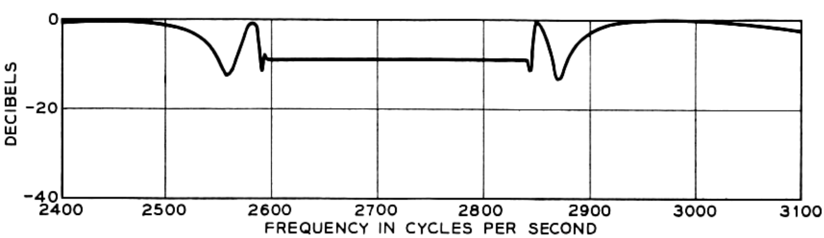


Fig. 6 — Frequency response characteristics of the sampled-data bandstop filter designed by the standard z transformation.

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