On the $\mathcal{L}_2$-Boundedness of Solutions of Nonlinear Functional Equations

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Let $\mathcal{E}_N$ denote the set of N-vector-valued functions of t defined on $[0, \infty)$ such that for any real positive number y, the square of the modulus of each component of any element is integrable on $[0, y]$, and let $\mathfrak{L}_{2N}(0, \infty)$ denote the subset of $\mathcal{E}_N$ with the property that the square of the modulus of each component of any element is integrable on $[0, \infty)$.

In the study of nonlinear physical systems, attention is frequently focused on the properties of one of the following two types of functional equations

$$g = f + \mathbf{KQ}f$$

$$g = \mathbf{K}f + \mathbf{Q}f$$

in which $\mathbf{K}$ and $\mathbf{Q}$ are causal operators, with $\mathbf{K}$ linear and $\mathbf{Q}$ nonlinear, $g \,\varepsilon\, \mathcal{E}_N$, and f is a solution belonging to $\mathcal{E}_N$. Typically, f represents the system response and g takes into account both the independent energy sources and the initial conditions at $t = 0$.

It is often important to determine conditions under which a physical system governed by one of the above equations is stable in the sense that the response to an arbitrary set of initial conditions approaches zero (i.e., the zero vector) as $t \to \infty$. In a great many cases of this type, g belongs to $\mathfrak{L}_{2N}(0, \infty)$ and approaches zero as $t \to \infty$ for all initial conditions, and, in addition, it is possible to show that if $f \,\varepsilon\, \mathfrak{L}_{2N}(0, \infty)$, then $f(t) \to 0$ as $t \to \infty$.

In this paper we attack the stability problem by deriving conditions under which $g \,\varepsilon\, \mathfrak{L}_{2N}$ $(0, \infty)$ and $f \,\varepsilon\, \mathcal{E}_N$ imply that $f \,\varepsilon\, \mathfrak{L}_{2N}(0, \infty)$. From an engineering viewpoint, the assumption that $f \,\varepsilon\, \mathcal{E}_N$ is almost invariably a trivial restriction.

As a specific application of the results, we consider a nonlinear integral equation that governs the behavior of a general control system containing linear time-invariant elements and an arbitrary finite number of time-varying nonlinear elements. Conditions are presented under which every solution of this equation belonging to $\mathcal{E}_N$ in fact belongs to $\mathfrak{L}_{2N}(0, \infty)$ and approaches zero as $t \to \infty$.

I. NOTATION AND DEFINITIONS

Let M denote an arbitrary matrix. We shall denote by M', M^*, and M^{-1}, respectively, the transpose, the complex-conjugate transpose, and the inverse of M. The positive square-root of the largest eigenvalue of M^*M is denoted by $\Lambda\{M\}$.

The set of complex measurable N-vector-valued functions of the real variable t defined on $[0,\infty)[(-\infty,\infty)]$ is denoted by $\mathcal{H}_N(0,\infty)[\mathcal{H}_N(-\infty,\infty)]$, and

$$\mathcal{L}_{2N}(0, \infty) = \left\{f \mid f \varepsilon \mathcal{H}_N(0, \infty), \int_0^\infty f^*f\, dt < \infty\right\}.$$

In order to be consistent with standard notation, we let $\mathcal{L}_2(0,\infty) = \mathcal{L}_{2N}(0,\infty)$ when $N = 1$. We shall not distinguish between elements of $\mathcal{H}_N(0,\infty)[\mathcal{H}_N(-\infty,\infty)]$ that agree almost everywhere on $[0,\infty)$ $[(-\infty,\infty)]$. The range of any operator considered in this article is assumed to be contained in either $\mathcal{H}_N(0,\infty)$ or $\mathcal{H}_N(-\infty,\infty)$.

The inner product of two elements of $\mathcal{L}_{2N}(0,\infty)$, $f = (f_1, f_2, \cdots, f_N)'$ and $g = (g_1, g_2, \cdots, g_N)'$, is denoted by $\langle f, g\rangle$ and is defined by

$$\langle f, g\rangle = \int_0^\infty f^*g\, dt.$$

The norm of $f \varepsilon \mathcal{L}_{2N}(0,\infty)$ is denoted by $\| f \|$ and is defined by

$$\| f \| = \langle f,f\rangle^{\frac{1}{2}}.$$

The norm of a linear operator $\mathbf{T}$ defined on $\mathcal{L}_{2N}(0,\infty)$ is denoted by $\| \mathbf{T} \|$.

Let $y \varepsilon (0,\infty)$, and define f_y by

$$f_y(t) = f(t) \qquad \text{for } t \varepsilon [0, y]$$
$$= 0 \qquad \text{for } t > y$$

for any $f \varepsilon \mathcal{H}_N(0,\infty)$, and let

$$\mathcal{E}_N = \{f \mid f \varepsilon \mathcal{H}_N(0,\infty), \qquad f_y \varepsilon \mathcal{L}_{2N}(0,\infty) \text{ for } 0 < y < \infty\}.$$

The set of real vector-valued functions is denoted by $\mathcal{R}$, and $\mathbf{I}$ and 1_N, respectively, denote the identity operator on $\mathcal{L}_{2N}(0,\infty)$ and the identity matrix of order N.

With A an arbitrary measurable $N \times N$ matrix-valued function of t with elements $\{a_{nm}\}$ defined on $[0,\infty)$, let $\mathcal{K}_{pN}$ $(p = 1,2)$ denote

$$\left\{A \mid \int_0^\infty | a_{nm}(t) |^p\, dt < \infty \qquad (n, m = 1, 2, \cdots, N)\right\}.$$

Let $\psi[f(t),t]$ denote

$$(\psi_1[f_1(t),t],\ \psi_2[f_2(t),t],\ \cdots,\ \psi_N[f_N(t),t])', \qquad f \in \mathcal{R} \cap \mathcal{K}_N(0,\infty)$$

where $\psi_1(w,t)$, $\psi_2(w,t)$, $\cdots$, $\psi_N(w,t)$ are real-valued functions of the real variables w and t for $-\infty < w < \infty$ and $0 \leqq t < \infty$ such that

(*i*) $\psi_n(0,t) = 0$ for $t \in [0,\infty)$ and $n = 1, 2, \cdots, N$

(*ii*) there exist real numbers α and β with the property that

$$\alpha \leqq \frac{\psi_n(w,t)}{w} \leqq \beta \qquad (n = 1, 2, \cdots, N)$$

for $t \in [0,\infty)$ and all real $w \neq 0$.

(*iii*) $\psi_n[w(t),t](n = 1, 2, \cdots, N)$ is a measurable function of t whenever $w(t)$ is measurable.

The symbol s denotes a scalar complex variable with $\sigma = \mathrm{Re}[s]$ and $\omega = \mathrm{Im}[s]$.

We shall say that a (not necessarily linear) operator $\mathbf{T}$ with domain $\mathfrak{D}(\mathbf{T}) \subset \mathcal{K}_N(0,\infty)$ is *causal* if an only if for an arbitrary $\delta > 0$,

$$(\mathbf{T}f)(t) = (\mathbf{T}g)(t) \quad \text{a.e. on } (0,\delta)$$

whenever $f,g \in \mathfrak{D}(\mathbf{T})$ and $f(t) = g(t)$ a.e. on $(0,\delta)$.

II. INTRODUCTION

In the study of nonlinear physical systems, attention is frequently focused on the properties of one of the following two types of functional equations

$$g = f + \mathbf{KQ}f \tag{1}$$

$$g = \mathbf{K}f + \mathbf{Q}f \tag{2}$$

in which $\mathbf{K}$ and $\mathbf{Q}$ are causal operators, with $\mathbf{K}$ linear and $\mathbf{Q}$ nonlinear, $g \in \mathcal{E}_N$, and f is a solution belonging to $\mathcal{E}_N$. Typically, f represents the system response and g takes into account both the independent energy sources and the initial conditions at $t = 0$.

It is often important to determine conditions under which a physical system governed by one of the above equations is stable in the sense that the response to an arbitrary set of initial conditions approaches zero (i.e., the zero vector) as $t \to \infty$. In a great many cases of this type, g belongs to $\mathcal{L}_{2N}(0,\infty)$ and approaches zero as $t \to \infty$ for all initial conditions, and, in addition, it is possible to show that if $f \in \mathcal{L}_{2N}(0,\infty)$, then $f(t) \to 0$ as $t \to \infty$.

In this paper we attack the stability problem by deriving conditions under which $g \in \mathfrak{L}_{2N}(0,\infty)$ and $f \in \mathcal{E}_N$ imply that $f \in \mathfrak{L}_{2N}(0,\infty)$. From an engineering viewpoint, the assumption that $f \in \mathcal{E}_N$ is almost invariably a trivial restriction.

As a specific application of the abstract results of Section III, we consider, in Section IV, the following integral equation which governs the behavior of a general control system containing linear time-invariant elements and an arbitrary finite number of time-varying nonlinear elements:

$$g(t) = f(t) + \int_0^t k(t-\tau)\psi[f(\tau),\tau]\,d\tau, \qquad t \geqq 0 \tag{3}$$

in which $k \in \mathcal{K}_{1N} \cap \mathcal{K}_{2N}$, $\psi[\cdot,\cdot]$ is as defined in Section I, and $g \in \mathfrak{L}_{2N}(0,\infty)$. We prove that every solution f of (3) belonging to $\mathcal{R} \cap \mathcal{E}_N$ in fact belongs to $\mathfrak{L}_{2N}(0,\infty)$ and approaches zero as $t \to \infty$ if, with

$$K(s) = \int_0^\infty k(t)e^{-st}\,dt \qquad \text{for} \qquad \sigma \geqq 0,$$

(*i*) $\det\,[1_N + \frac{1}{2}(\alpha+\beta)K(s)] \neq 0$ for $\sigma \geqq 0$

(*ii*) $\frac{1}{2}(\beta-\alpha)\sup_\omega \Lambda\{[1_N + \frac{1}{2}(\alpha+\beta)K(i\omega)]^{-1}K(i\omega)\} < 1.$

An analogous result is proved for the integral equation

$$g(t) = \psi[f(t),t] + \int_0^t k(t-\tau)f(\tau)\,d\tau, \qquad t \geqq 0.$$

For $N = 1$, the key condition (*ii*) possesses a simple geometric interpretation: it is satisfied if and only if the locus of $[K(i\omega)]^{-1}$ for $-\infty < \omega < \infty$ lies outside the circle of radius $\frac{1}{2}(\beta-\alpha)$ centered in the complex plane at $[-\frac{1}{2}(\alpha+\beta),0]$.†

In Section V we consider two direct applications to nonlinear differential equations. One of our results asserts that if f is any real-valued function of t defined and twice-differentiable on $[0,\infty)$ such that

$$\frac{d^2f}{dt^2} + a\frac{df}{dt} + \psi[f,t] = g$$

for almost all $t \in [0,\infty)$, where $g \in \mathcal{R} \cap \mathfrak{L}_2(0,\infty)$, $\psi[\cdot,\cdot]$ is as defined in Section I with $N = 1$ and $\alpha > 0$, and a is a real constant such that $a > \sqrt{\beta} - \sqrt{\alpha}$, then $f \in \mathfrak{L}_2(0,\infty)$ and $f(t) \to 0$ as $t \to \infty$.

† For some earlier results concerned with frequency-domain conditions for the stability of nonlinear or time-varying systems, see Refs. 1–4.

III. KEY RESULTS

Assumption 1: It is assumed throughout that

(i) $\mathbf{K}$ is a linear causal operator with domain $\mathfrak{D}(\mathbf{K})$ such that $\mathfrak{L}_{2N}(0,\infty) \subset \mathfrak{D}(\mathbf{K}) \subset \mathfrak{K}_N(0,\infty)$

(ii) $\mathbf{K}$ maps $\mathfrak{L}_{2N}(0,\infty)$ into itself, and is bounded on $\mathfrak{L}_{2N}(0,\infty)$

(iii) $\mathbf{Q}$ is a (not necessarily linear) causal operator with domain $\mathfrak{D}(\mathbf{Q}) \subset \mathfrak{K}_N(0,\infty)$.

The following two theorems are the key results of the paper.

Theorem 1: Let $f \,\varepsilon\, \mathfrak{D}(\mathbf{Q}) \cap \mathcal{E}_N$ *such that* $\mathbf{Q}f \,\varepsilon\, \mathfrak{D}(\mathbf{K}) \cap \mathcal{E}_N$, $\mathbf{KQ}f \,\varepsilon\, \mathcal{E}_N$, *and* $g = f + \mathbf{KQ}f$, *where* $g \,\varepsilon\, \mathfrak{L}_{2N}(0,\infty)$. *Let* f *not be the zero-element of* $\mathcal{E}_N$, *and let* $y_0 = \inf \{y \mid y > 0, \| f_y \| \neq 0\}$.

Suppose that $\{f_y, 0 < y < \infty\} \subset \mathfrak{D}(\mathbf{Q})$ *and that there exists a real or complex number* x *such that*

(i) *on* $\mathfrak{L}_{2N}(0,\infty)$, $(\mathbf{I} + x\mathbf{K})^{-1}$ *exists and is causal*

$$(ii)\ \| (\mathbf{I} + x\mathbf{K})^{-1}\mathbf{K} \| \sup_{y>y_0} \frac{\| (\mathbf{Q}f_y)_y - xf_y \|}{\| f_y \|} < 1.$$

Then $f \,\varepsilon\, \mathfrak{L}_{2N}(0,\infty)$ *and*

$$\| f \| \leqq (1 - r)^{-1} \| (\mathbf{I} + x\mathbf{K})^{-1} g \|,$$

in which

$$r = \| (\mathbf{I} + x\mathbf{K})^{-1}\mathbf{K} \| \sup_{y>y_0} \frac{\| (\mathbf{Q}f_y)_y - xf_y \|}{\| f_y \|}.$$

Theorem 2: Let $f \,\varepsilon\, \mathfrak{D}(\mathbf{Q}) \cap \mathfrak{D}(\mathbf{K}) \cap \mathcal{E}_N$ *such that* $\mathbf{K}f \,\varepsilon\, \mathcal{E}_N$, $\mathbf{Q}f \,\varepsilon\, \mathcal{E}_N$, *and*

$$g = \mathbf{K}f + \mathbf{Q}f$$

where $g \,\varepsilon\, \mathfrak{L}_{2N}(0,\infty)$. *Let* f *not be the zero-element of* $\mathcal{E}_N$, *and let* $y_0 = \inf \{y \mid y > 0, \| f_y \| \neq 0\}$.

Suppose that $\{f_y, 0 < y < \infty\} \subset \mathfrak{D}(\mathbf{Q})$ *and that there exists a real or complex number* x *such that*

(i) *on* $\mathfrak{L}_{2N}(0,\infty)$, $(x\mathbf{I} + \mathbf{K})^{-1}$ *exists and is causal*

$$(ii)\ \| (x\mathbf{I} + \mathbf{K})^{-1} \| \sup_{y>y_0} \frac{\| (\mathbf{Q}f_y)_y - xf_y \|}{\| f_y \|} < 1.$$

Then $f \,\varepsilon\, \mathfrak{L}_{2N}(0,\infty)$ *and*

$$\| f \| \leqq (1 - q)^{-1} \| (x\mathbf{I} + \mathbf{K})^{-1} g \|,$$

in which

$$q = \| (x\mathbf{I} + \mathbf{K})^{-1} \| \sup_{y>y_0} \frac{\| (\mathbf{Q}f_y)_y - xf_y \|}{\| f_y \|}.$$

3.1 *Proof of Theorem 1*

It is convenient to introduce the operator $\mathbf{P}$ defined on $\mathcal{H}_N(0, \infty)$ by $\mathbf{P}f = f_y$, where y is an arbitrary real positive number.

From $g = f + \mathbf{KQ}f$, we clearly have

$$g_y = f_y + \mathbf{PKQ}f.$$

Since $\mathbf{K}$ is causal,

$$g_y = f_y + \mathbf{PKPQ}f.$$

Similarly, since $\mathbf{Q}$ is causal,

$$\begin{aligned} g_y &= f_y + \mathbf{PKPQP}f \\ &= f_y + \mathbf{PKPQ}f_y . \end{aligned}$$

Thus,

$$g_y = \mathbf{P}(\mathbf{I} + x\mathbf{K})f_y + \mathbf{PKP}(\mathbf{Q} - x\mathbf{I})f_y .$$

Since on $\mathcal{L}_{2N}(0, \infty)$, $(\mathbf{I} + x\mathbf{K})^{-1}$ exists and is causal,

$$\mathbf{P}(\mathbf{I} + x\mathbf{K})^{-1}\mathbf{P}(\mathbf{I} + x\mathbf{K})f_y = f_y ,$$

and hence,

$$f_y = -\mathbf{P}(\mathbf{I} + x\mathbf{K})^{-1}\mathbf{PKP}(\mathbf{Q} - x\mathbf{I})f_y + \mathbf{P}(\mathbf{I} + x\mathbf{K})^{-1}g_y .$$

It follows that

$$\| f_y \| \leqq \| \mathbf{P}(\mathbf{I} + x\mathbf{K})^{-1}\mathbf{PK} \| \cdot \| \mathbf{PQ}f_y - xf_y \| + \| \mathbf{P}(\mathbf{I} + x\mathbf{K})^{-1}g_y \|.$$

Moreover, in view of the causality of $(\mathbf{I} + x\mathbf{K})^{-1}$,

$$\mathbf{P}(\mathbf{I} + x\mathbf{K})^{-1}\,\mathbf{PK} = \mathbf{P}(\mathbf{I} + x\mathbf{K})^{-1}\mathbf{K},$$

and hence, using the fact that $\mathbf{P}$ is a projection on $\mathcal{L}_{2N}(0, \infty)$,

$$\| \mathbf{P}(\mathbf{I} + x\mathbf{K})^{-1}\mathbf{PK} \| \leqq \| (\mathbf{I} + x\mathbf{K})^{-1}\mathbf{K} \|.$$

Similarly,

$$\| \mathbf{P}(\mathbf{I} + x\mathbf{K})^{-1}g_y \| \leqq \| (\mathbf{I} + x\mathbf{K})^{-1}g \|.$$

Thus, with r as defined in the statement of the theorem,

$$\| f_y \| \leqq r \| f_y \| + \| (\mathbf{I} + x\mathbf{K})^{-1}g \|$$

or

$$\| f_y \| \leqq (1 - r)^{-1} \| (\mathbf{I} + x\mathbf{K})^{-1}g \|.$$

Since this inequality is valid for arbitrary positive y, it follows that $f \varepsilon \mathfrak{L}_{2N}(0, \infty)$ and

$$\| f \| \leqq (1 - r)^{-1} \| (\mathbf{I} + x\mathbf{K})^{-1} g \|.$$

3.2 *Proof of Theorem 2*

The argument is essentially the same as the one used in the proof of Theorem 1.

We have, with **P** as defined in the proof of Theorem 1,

$$\begin{aligned} g_y &= \mathbf{PK}f + \mathbf{PQ}f \\ &= \mathbf{PKP}f + \mathbf{PQP}f = \mathbf{PK}f_y + \mathbf{PQ}f_y \\ &= \mathbf{P}(x\mathbf{I} + \mathbf{K})f_y + \mathbf{P}(\mathbf{Q} - x\mathbf{I})f_y . \end{aligned}$$

Using the fact that on $\mathfrak{L}_{2N}(0, \infty)$, $(x\mathbf{I} + \mathbf{K})^{-1}$ exists and is causal

$$f_y = -\mathbf{P}(x\mathbf{I} + \mathbf{K})^{-1}\mathbf{P}(\mathbf{Q} - x\mathbf{I})f_y + \mathbf{P}(x\mathbf{I} + \mathbf{K})^{-1}g_y .$$

Thus, with q as defined in the statement of the theorem,

$$\| f_y \| \leqq q \| f_y \| + \| (x\mathbf{I} + \mathbf{K})^{-1} g \|,$$

or

$$\| f_y \| \leqq (1 - q)^{-1} \| (x\mathbf{I} + \mathbf{K})^{-1} g \|.$$

This inequality is valid for arbitrary positive y. Hence $f \varepsilon \mathfrak{L}_{2N}(0, \infty)$ and

$$\| f \| \leqq (1 - q)^{-1} \| (x\mathbf{I} + \mathbf{K})^{-1} g \|.$$

Remark: A moment's reflection concerning the proofs of Theorems 1 and 2 will show that by simply reinterpreting the symbols, analogous results can be obtained for other function spaces.

3.3 *Conditions under Which the Hypotheses of Theorems 1 or 2 Concerning x Are Satisfied*

The following theorem asserts that the hypotheses of Theorems 1 or 2 concerning x are satisfied in certain special but very important cases. The implications of the theorem are of direct interest in the theory of passive nonlinear electrical networks.

Theorem 3: Let f be as defined in Theorem 1 [Theorem 2]. Suppose that

(i) there exist a nonnegative constant k_1 and a positive constant k_2 such that

$$Re \langle \mathbf{Q}f_y , f_y \rangle \geqq k_1 \| f_y \|^2, \qquad \| \mathbf{Q}f_y \|^2 \leqq k_2 \| f_y \|^2 \qquad \text{for } 0 < y < \infty$$

(ii) **K** *maps* $\mathfrak{L}_{2N}(0, \infty)$ *into itself such that there exists a nonnegative constant c with the property that*

$$Re \langle \mathbf{K}h, h \rangle \geqq c \| h \|^2$$

for all $h \in \mathfrak{L}_{2N}(0, \infty)$.

Then the hypotheses concerning x of Theorem 1 [Theorem 2] are satisfied if either:

$$k_1 > 0 \quad and \quad c \geqq 0,$$

or

$$k_1 = 0 \quad and \quad c > 0.$$

3.4 *Proof of Theorem 3*

Lemmas 1, 2, and 3 (below) imply that for real positive x the operators $(\mathbf{I} + x\mathbf{K})$ and $(x\mathbf{I} + \mathbf{K})$ possess causal inverses on $\mathfrak{L}_{2N}(0, \infty)$ and

$$\| (\mathbf{I} + x\mathbf{K})^{-1}\mathbf{K} \|^2 \leqq \frac{2(\| \mathbf{K} \| - c) + x \| \mathbf{K} \|^2}{x(1 + x \| \mathbf{K} \|)^2}$$

$$\| (x\mathbf{I} + \mathbf{K})^{-1} \|^2 \leqq \frac{1}{x(x + 2c)} \cdot$$

With x real and positive,

$$\begin{aligned} \| (\mathbf{Q}f_y)_y - xf_y \|^2 &\leqq \| \mathbf{Q}f_y - xf_y \|^2 \\ &\leqq \| \mathbf{Q}f_y \|^2 - 2x \operatorname{Re} \langle \mathbf{Q}f_y , f_y \rangle + x^2 \| f_y \|^2 \\ &\leqq (k_2 - 2xk_1 + x^2) \| f_y \|^2. \end{aligned}$$

It is a simple matter to verify that if $(k_1 + c) > 0$, there exist positive values of x such that

$$\frac{2(\| \mathbf{K} \| - c) + x \| \mathbf{K} \|^2}{x(1 + x \| \mathbf{K} \|)^2} (k_2 - 2xk_1 + x^2) < 1,$$

and there exist positive values of x such that

$$\frac{k_2 - 2xk_1 + x^2}{x(x + 2c)} < 1.$$

Hence, it remains to prove Lemmas 1, 2, and 3.

Lemma 1: Let **T** *be a bounded linear mapping of* $\mathfrak{L}_{2N}(0, \infty)$ *into itself* *such that there exists a constant* $c_1 > -1$ *with the property that*

$$Re \langle \mathbf{T}f,f \rangle \geqq c_1 \| f \|^2$$

for all $f \varepsilon \mathfrak{L}_{2N}(0, \infty)$. *Then* $(\mathbf{I} + \mathbf{T})$ *possesses an inverse on* $\mathfrak{L}_{2N}(0, \infty)$.
Proof:

Since $c_1 > -1$, it is evident that there exists a positive constant k_1 such that

$$\mathrm{Re} \langle (\mathbf{I} + \mathbf{T})f,f \rangle \geqq k_1 \| f \|^2$$

for all $f \varepsilon \mathfrak{L}_{2N}(0, \infty)$. This, together with the boundedness of $\mathbf{T}$, implies that $(\mathbf{I} + \mathbf{T})^{-1}$ exists (see Ref. 5, for example).

Lemma 2: Let $\mathbf{T}$ *be an invertible bounded linear mapping of* $\mathfrak{L}_{2N}(0, \infty)$ *into itself such that* $\mathbf{T}$ *is causal and* $Re \langle \mathbf{T}h,h \rangle \geqq 0$ *for all* $h \varepsilon \mathfrak{L}_{2N}(0, \infty)$. *Then* $\mathbf{T}^{-1}$ *is causal.*
Proof:

A bounded linear mapping $\mathbf{A}$ of $\mathfrak{L}_{2N}(0, \infty)$ into itself is causal if and only if[6]

$$\mathrm{Re} \int_0^y (\mathbf{A}f)^* f \, dt \geqq - \| \mathbf{A} \| \int_0^y f^* f \, dt$$

for all real $y \geqq 0$ and all $f \varepsilon \mathfrak{L}_{2N}(0, \infty)$. Thus, to prove the lemma it suffices to point out that the causality of $\mathbf{T}$ implies that

$$\mathrm{Re} \int_0^y (\mathbf{T}g)^* g \, dt = \mathrm{Re} \langle \mathbf{T}g_y , g_y \rangle \geqq 0,$$

for all real $y \geqq 0$ and all $g \varepsilon \mathfrak{L}_{2N}(0, \infty)$, and hence that

$$\mathrm{Re} \int_0^y h^* \mathbf{T}^{-1} h \, dt \geqq 0$$

for all real $y \geqq 0$ and all $h \varepsilon \mathfrak{L}_{2N}(0, \infty)$.

Lemma 3:† *Let* $\mathbf{T}$ *be a bounded linear mapping of* $\mathfrak{L}_{2N}(0, \infty)$ *into itself such that* $(\mathbf{I} + \mathbf{T})$ *is invertible and there exists a real constant* c_2 *with the property that*

$$Re \langle \mathbf{T}f, f \rangle \geqq c_2 \| f \|^2$$

for all $f \varepsilon \mathfrak{L}_{2N}(0, \infty)$. *Then, for* $c_2 \geqq -\frac{1}{2}$,

† Lemmas 1 and 3, and their proofs, remain valid if $\mathfrak{L}_{2N}(0, \infty)$ is replaced with an arbitrary Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.

$$\| (\mathbf{I} + \mathbf{T})^{-1}\mathbf{T} \| \leqq [1 - (2c_2 + 1)(1 + \| \mathbf{T} \|)^{-2}]^{\frac{1}{2}}$$

and, for $c_2 > -\frac{1}{2}$,

$$\| (\mathbf{I} + \mathbf{T})^{-1} \| \leqq (1 + 2c_2)^{-\frac{1}{2}}.$$

Proof:

In order to establish the first inequality, let $g = (\mathbf{I} + \mathbf{T})^{-1}\mathbf{T}f$ and, using the fact that $g = f - (\mathbf{I} + \mathbf{T})^{-1}f$, observe that

$$\langle g,g \rangle = \langle f,f \rangle - 2 \operatorname{Re} \langle \mathbf{T}z,z \rangle - \langle z,z \rangle,$$

where $z = (\mathbf{I} + \mathbf{T})^{-1}f$. Since

$$2 \operatorname{Re} \langle \mathbf{T}z,z \rangle + \langle z,z \rangle \geqq (2c_2 + 1) \| z \|^2$$

and

$$\| z \| \geqq \| (\mathbf{I} + \mathbf{T}) \|^{-1} \| f \| \geqq (1 + \| T \|)^{-1} \| f \|,$$

it follows that

$$\langle g,g \rangle \leqq [1 - (2c_2 + 1)(1 + \| \mathbf{T} \|)^{-2}]\langle f,f \rangle$$

for all $f, g \in \mathfrak{L}_{2N}(0, \infty)$ such that $g = (\mathbf{I} + \mathbf{T})^{-1}\mathbf{T}f$. Thus

$$\| (\mathbf{I} + \mathbf{T})^{-1}\mathbf{T} \| \leqq [1 - (2c_2 + 1)(1 + \| \mathbf{T} \|)^{-2}]^{\frac{1}{2}}.$$

The second inequality follows directly from the fact that if $g = (\mathbf{I} + \mathbf{T})^{-1}f$,

$$\| f \|^2 = \| g \|^2 + 2 \operatorname{Re} \langle \mathbf{T}g,g \rangle + \| \mathbf{T}g \|^2 \geqq (1 + 2c_2) \| g \|^2.$$

IV. APPLICATIONS TO NONLINEAR INTEGRAL EQUATIONS

In this section our primary objective is to prove the following two theorems.

Theorem 4: Let $k \in \mathcal{K}_{1N}$ *and let*

$$g(t) = f(t) + \int_0^t k(t - \tau)\psi[f(\tau), \tau] \, d\tau, \qquad t \geqq 0$$

where $g \in \mathfrak{L}_{2N}(0, \infty)$ *and* $f \in \mathfrak{R} \cap \mathcal{E}_N$. *Let*

$$K(s) = \int_0^\infty k(t)e^{-st} \, dt, \qquad \sigma \geqq 0.$$

Suppose that

(*i*) *det* $[1_N + \frac{1}{2}(\alpha + \beta)K(s)] \neq 0$ *for* $\sigma \geqq 0$

(*ii*) $\frac{1}{2}(\beta - \alpha) \sup_\omega \Lambda\{[1_N + \frac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}K(i\omega)\} < 1.$

Then $f \,\varepsilon\, \mathfrak{L}_{2N}(0, \infty)$.

Theorem 5: Let $k \,\varepsilon\, \mathcal{K}_{1N}$ *and let*

$$g(t) = \int_0^t k(t - \tau)f(\tau)\, d\tau + \psi[f(t), t], \qquad t \geqq 0$$

where $g \,\varepsilon\, \mathfrak{L}_{2N}(0, \infty)$ *and* $f \,\varepsilon\, \mathfrak{R} \cap \mathcal{E}_N$. *Let*

$$K(s) = \int_0^\infty k(t)e^{-st}\, dt, \qquad \sigma \geqq 0.$$

Suppose that

(*i*) *det* $[\frac{1}{2}(\alpha + \beta)1_N + K(s)] \neq 0$ *for* $\sigma \geqq 0$

(*ii*) $\frac{1}{2}(\beta - \alpha)\, \sup_\omega \Lambda\{[\frac{1}{2}(\alpha + \beta)1_N + K(i\omega)]^{-1}\} < 1.$

Then $f \,\varepsilon\, \mathfrak{L}_{2N}(0, \infty)$.

4.1 *Proof of Theorems 4 and 5*

In Theorems 1 and 2 let **Q** denote the operator defined by

$$(\mathbf{Q}g)(t) = \psi[g(t), t], \qquad 0 \leqq t < \infty$$

where g is an arbitrary element of $\mathfrak{R} \cap \mathcal{K}_N(0, \infty)$. This operator maps $\mathfrak{R} \cap \mathfrak{L}_{2N}(0, \infty)$ into itself and possesses the property that for any real x

$$\| \mathbf{Q}h - xh \| \leqq \eta(x)\, \| h \|, \qquad h \,\varepsilon\, \mathfrak{R} \cap \mathfrak{L}_{2N}(0, \infty)$$

where

$$\eta(x) = \max\, [(x - \alpha), (\beta - x)].$$

Thus, with **K** defined on $\mathfrak{L}_{2N}(0, \infty)$ by[7]

$$\mathbf{K}h = \int_0^t k(t - \tau)h(\tau)\, d\tau, \qquad h \,\varepsilon\, \mathfrak{L}_{2N}(0, \infty),$$

condition (*ii*) of Theorem 1 and the corresponding condition of Theorem 2, respectively, are satisfied if there exists a real x such that

$$\| (\mathbf{I} + x\mathbf{K})^{-1}\mathbf{K} \|\, \eta(x) < 1,$$

and

$$\| (x\mathbf{I} + \mathbf{K})^{-1} \|\, \eta(x) < 1.$$

Lemmas 4 and 5 (below) imply at once that if the assumptions of Theorem 4 (Theorem 5) are met, then hypotheses (*i*) and (*ii*) of Theorem 1 (Theorem 2) are satisfied with $x = \frac{1}{2}(\alpha + \beta)$. It can be shown[8] (with the aid of Lemma 4) that this choice of x is optimal in the sense that if there exists a real x such that $(\mathbf{I} + x\mathbf{K})$ possesses a causal inverse on $\mathfrak{L}_{2N}(0, \infty)$ and

$$\| (\mathbf{I} + x\mathbf{K})^{-1}\mathbf{K}\| \, \eta(x) < 1,$$

then $[\mathbf{I} + \frac{1}{2}(\alpha + \beta)\mathbf{K}]$ possesses a causal inverse on $\mathfrak{L}_{2N}(0, \infty)$ and

$$\| (\mathbf{I} + x\mathbf{K})^{-1}\mathbf{K} \| \, \eta(x) \geqq \| [\mathbf{I} + \tfrac{1}{2}(\alpha + \beta)\mathbf{K}]^{-1}\mathbf{K} \| \, \eta[\tfrac{1}{2}(\alpha + \beta)].$$

This choice of x is similarly optimal with regard to the statement of the conditions in Theorem 5.

Before proceeding to the statement and proofs of the lemmas, it is convenient to introduce a few definitions.

4.2 *Definitions*

With τ an arbitrary positive constant, let $\mathbf{S}_\tau$ denote the mapping of $\mathfrak{L}_{2N}(0, \infty)$ into itself defined by

$$\begin{aligned}(\mathbf{S}_\tau f)(t) &= 0, & t \,\varepsilon\, [0,\tau) \\ &= f(t - \tau), & t \,\varepsilon\, [\tau, \infty)\end{aligned}$$

for any $f \,\varepsilon\, \mathfrak{L}_{2N}\,(0, \infty)$.

Let

$$\mathfrak{L}_{2N}(-\infty, \infty) = \left\{ f \mid f \,\varepsilon\, \mathfrak{K}_N(-\infty, \infty), \int_{-\infty}^{\infty} f^* f \, dt < \infty \right\}.$$

We take as the definition of the Fourier transform of $f \,\varepsilon\, \mathfrak{L}_{2N}(-\infty, \infty)$:

$$\hat{f} = \text{l.i.m.} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} \, dt,$$

and consequently,

$$2\pi \int_{-\infty}^{\infty} f^* f \, dt = \int_{-\infty}^{\infty} \hat{f}^* \hat{f} \, d\omega.$$

By the Fourier transform of an $f \,\varepsilon\, \mathfrak{L}_{2N}(0, \infty)$ we mean simply

$$\text{l.i.m.} \int_0^{\infty} f(t) e^{-i\omega t} \, dt.$$

4.3 *Lemmas 4 and 5*

Lemma 4: Let $\mathbf{A}$ *be an invertible linear mapping of* $\mathfrak{L}_{2N}(0, \infty)$ *into itself* *such that for an arbitrary $f \varepsilon \mathfrak{L}_{2N}(0, \infty)$*

$$\mathbf{A}\mathbf{S}_\tau f = \mathbf{S}_\tau \mathbf{A} f, \qquad \tau > 0.$$

Then $\mathbf{A}^{-1}$ is causal.

Proof:

Suppose that, on the contrary, $\mathbf{A}$ possesses an inverse on $\mathfrak{L}_{2N}(0, \infty)$, but that it is not causal. Then there exist elements z_1, $z_2 \varepsilon \mathfrak{L}_{2N}(0, \infty)$, and a $\delta > 0$ such that $z_1(t) \neq 0$ on some positive-measure subset of $(0,\delta)$, and $\mathbf{A}z_1 = \mathbf{S}_\delta z_2$. Since $\mathbf{A}$ is assumed to possess an inverse, there exists a unique $z_3 \varepsilon \mathfrak{L}_{2N}(0, \infty)$ such that $z_2 = \mathbf{A}z_3$. Thus,

$$\mathbf{A}\mathbf{S}_\delta z_3 = \mathbf{S}_\delta \mathbf{A} z_3 = \mathbf{S}_\delta z_2 .$$

Clearly, $\mathbf{S}_\delta z_3 \neq z_1$, which contradicts the assumption that $\mathbf{A}$ possesses an inverse. This proves the lemma.

Lemma 5: Let $u \varepsilon \mathcal{K}_{1N}$ and let $\mathbf{U}$ be the mapping of $\mathfrak{L}_{2N}(0, \infty)$ into itself defined by

$$\mathbf{U}f = \int_0^t u(t - \tau) f(\tau)\, d\tau, \qquad f \varepsilon \mathfrak{L}_{2N}(0, \infty).$$

Let

$$U(s) = \int_0^\infty u(t) e^{-st}\, dt, \qquad \sigma \geqq 0.$$

Suppose that $\det [1_N + U(s)] \neq 0$ for $\sigma \geqq 0$. Then

(i) $(\mathbf{I} + \mathbf{U})$ possesses an inverse on $\mathfrak{L}_{2N}(0, \infty)$

(ii) $$\| (\mathbf{I} + \mathbf{U})^{-1} \mathbf{U} \| \leqq \sup_\omega \Lambda\{[1_N + U(i\omega)]^{-1} U(i\omega)\}$$

$$\| (\mathbf{I} + \mathbf{U})^{-1} \| \leqq \sup_\omega \Lambda\{[1_N + U(i\omega)]^{-1}\}.$$

Proof:

Consider first the invertibility of the operator $(\bar{\mathbf{I}} + \bar{\mathbf{U}})$ defined on $\mathfrak{L}_{2N}(-\infty, \infty)$ by

$$(\bar{\mathbf{I}} + \bar{\mathbf{U}}) f = f + \int_{-\infty}^t u(t - \tau) f(\tau)\, d\tau, \qquad f \varepsilon \mathfrak{L}_{2N}(-\infty, \infty).$$

The assumption that $u \varepsilon \mathcal{K}_{1N}$ implies that the elements of $U(i\omega)$ approach zero as $|\omega| \to \infty$, and that they are uniformly bounded and uniformly continuous for $\omega \varepsilon (-\infty, \infty)$. Thus, $\det [1_N + U(i\omega)]$ approaches unity as $|\omega| \to \infty$, and is uniformly continuous for

$\omega \,\varepsilon\, (-\infty, \infty)$. It follows that $\det [1_N + U(i\omega)] \neq 0$ for all ω implies that

$$\inf_{\omega} | \det [1_N + U(i\omega)] | > 0,$$

and hence that

$$\sup_{\omega} \Lambda\{[1_N + U(i\omega)]^{-1}\} < \infty.$$

Let $\hat{g}$ denote the Fourier transform of an arbitrary $g \,\varepsilon\, \mathfrak{L}_{2N}(-\infty, \infty)$. Then,

$$\int_{-\infty}^{\infty} \hat{g}^*[1_N + U(i\omega)]^{-1*}[1_N + U(i\omega)]^{-1}\hat{g}\, d\omega$$

$$\leqq \int_{-\infty}^{\infty} \Lambda^2\{[1_N + U(i\omega)]^{-1}\}\hat{g}^*\hat{g}\, d\omega$$

$$\leqq \sup_{\omega} \Lambda^2\{[1_N + U(i\omega)]^{-1}\} \int_{-\infty}^{\infty} \hat{g}^*\hat{g}\, d\omega < \infty,$$

and hence, by the Riesz-Fischer theorem, there exists an $f \,\varepsilon\, \mathfrak{L}_{2N}(-\infty, \infty)$ with Fourier transform

$$\hat{f} = [1_N + U(i\omega)]^{-1}\hat{g}.$$

This establishes the existence of $(\bar{\mathbf{I}} + \bar{\mathbf{U}})^{-1}$.

Since $\det [1_N + U(s)] \neq 0$ for $\sigma \geqq 0$, and $U(s) \to 0$ as $|s| \to \infty$ uniformly in the closed right-half plane, every element of $[1_N + U(s)]^{-1}$ is analytic and uniformly bounded for $\sigma > 0$. Thus, $(\bar{\mathbf{I}} + \bar{\mathbf{U}})^{-1}$ maps

$$\{f \mid f \,\varepsilon\, \mathfrak{L}_{2N}(-\infty, \infty), f(t) = 0 \quad \text{for} \quad t < 0\}$$

into itself,[9,10] and hence the operator $(\mathbf{I} + \mathbf{U})$ defined on $\mathfrak{L}_{2N}(0, \infty)$ possesses an inverse.

To establish the first of the inequalities stated in the lemma, let $f \,\varepsilon\, \mathfrak{L}_{2N}(0, \infty)$ and let

$$g = (\mathbf{I} + \mathbf{U})^{-1}\mathbf{U}f.$$

Then, with $\hat{g}$ and $\hat{f}$, respectively, the Fourier transforms of g and f,

$$\hat{g} = [1_N + U(i\omega)]^{-1}U(i\omega)\hat{f}.$$

Thus,

$$\int_{-\infty}^{\infty} \hat{g}^*\hat{g}\, d\omega = \int_{-\infty}^{\infty} \hat{f}^*U(i\omega)^*[1_N + U(i\omega)]^{-1*}[1_N + U(i\omega)]^{-1}U(i\omega)\hat{f}\, d\omega$$

$$\leqq \int_{-\infty}^{\infty} \Lambda^2\{[1_N + U(i\omega)]^{-1}U(i\omega)\}\hat{f}^*\hat{f}\, d\omega$$

$$\leqq \sup_\omega \Lambda^2\{[1_N + U(i\omega)]^{-1}U(i\omega)\} \int_{-\infty}^{\infty} \hat{f}^*\hat{f}\, d\omega,$$

from which, using Plancherel's identity,

$$\| g \| \leqq \sup_\omega \Lambda\{[1_N + U(i\omega)]^{-1}U(i\omega)\} \; \| f \|.$$

Thus,

$$\| (\mathbf{I} + \mathbf{U})^{-1}\mathbf{U} \| \leqq \sup_\omega \Lambda\{[1_N + U(i\omega)]^{-1}U(i\omega)\}.$$

By simply repeating this argument with $(\mathbf{I} + \mathbf{U})^{-1}\mathbf{U}$ and $[1_N + U(i\omega)]^{-1}\cdot U(i\omega)$, respectively, replaced with $(\mathbf{I} + \mathbf{U})^{-1}$ and $[1_N + U(i\omega)]^{-1}$, we find that

$$\| (\mathbf{I} + \mathbf{U})^{-1} \| \leqq \sup_\omega \Lambda\{[1_N + U(i\omega)]^{-1}\}.$$

This proves the lemma.

4.4 *Remarks*

It can easily be shown that conditions (*i*) and (*ii*) of Theorems 4 and 5 are satisfied if $\alpha > 0$ and

$$K(i\omega) + K(i\omega)^*$$

is nonnegative definite for all ω.

A moment's reflection concerning the proof of Theorems 4 and 5 will show that those theorems remain valid if $\psi[f(t),t]$ and f, respectively, are replaced with $(\mathbf{Q}f)(t)$ and $f \,\varepsilon\, \mathfrak{D}(\mathbf{Q})$, with the understanding that

(a) $\mathbf{Q}$ is a causal mapping of a subset $\mathfrak{D}(\mathbf{Q})$ of $\mathcal{E}_N$ into $\mathcal{E}_N$ such that $\mathbf{Q}h \,\varepsilon\, \mathfrak{L}_{2N}(0,\infty)$ whenever $h \,\varepsilon\, \mathfrak{D}(\mathbf{Q}) \cap \mathfrak{L}_{2N}(0,\infty)$, and there exist real constants α and β ($\beta > \alpha$) with the property that

$$\| \mathbf{Q}h - \tfrac{1}{2}(\alpha + \beta)h \| \leqq \tfrac{1}{2}(\beta - \alpha) \; \| h \|$$

for all $h \,\varepsilon\, \mathfrak{D}(\mathbf{Q}) \cap \mathfrak{L}_{2N}(0,\infty)$.

(b) if $h \,\varepsilon\, \mathfrak{D}(\mathbf{Q})$, $\{h_y\,, 0 < y < \infty\} \subset \mathfrak{D}(\mathbf{Q})$.

4.5 *Conditions under Which $f(t) \to 0$ as $t \to \infty$*

Theorem 6: Suppose that the hypotheses of Theorem 4 are satisfied, that

$g(t) \to 0$ as $t \to \infty$, and that $k \,\varepsilon\, \mathcal{K}_{2N}$. Then $f(t) \to 0$ as $t \to \infty$.

Proof:

Observe first that the N-vector-valued function with values

$$\psi[f(t),t], \qquad 0 \leqq t < \infty,$$

is an element of $\mathcal{R} \cap \mathcal{L}_{2N}(0,\infty)$. Thus it suffices to show that if $h \,\varepsilon\, \mathcal{L}_{2N}(0,\infty)$,

$$\int_0^t k(t-\tau)h(\tau)\,d\tau \to 0 \quad \text{as} \quad t \to \infty.$$

In terms of $K(i\omega)$ and $\hat{h}(i\omega)$, respectively, the Fourier transforms of k and h,

$$\int_0^t k(t-\tau)h(\tau)\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} K(i\omega)\hat{h}(i\omega)e^{i\omega t}\,d\omega.$$

Since $k \,\varepsilon\, \mathcal{K}_{2N}$, it follows that the modulus of each element of the N-vector $K(i\omega)\hat{h}(i\omega)$ is integrable on the ω-set $(-\infty,\infty)$. Thus, by the Riemann-Lebesgue lemma

$$\int_0^t k(t-\tau)h(\tau)\,d\tau \to 0 \quad \text{as} \quad t \to \infty.$$

This proves Theorem 6.

It is obvious that essentially the same argument suffices to prove the following corresponding result relating to Theorem 5.

Theorem 7: Suppose that the hypotheses of Theorem 5 are satisfied, that $g(t) \to 0$ as $t \to \infty$, and that $k \,\varepsilon\, \mathcal{K}_{2N}$. Then $\psi[f(t),t] \to 0$ as $t \to \infty$.

V. APPLICATIONS TO NONLINEAR DIFFERENTIAL EQUATIONS

Theorem 8: Let A be an $N \times N$ matrix of real constants, let $\psi[\cdot,\cdot]$ be as defined in Section I with α and β, respectively, replaced with $\hat{\alpha}$ and $\hat{\beta}$, and let f denote a real N-vector-valued function of t defined and differentiable on $[0,\infty)$ such that

$$\frac{df}{dt} + Af + \psi[f,t] = g$$

for almost all $t \,\varepsilon\, [0,\infty)$, where $g \,\varepsilon\, \mathcal{R} \cap \mathcal{L}_{2N}(0,\infty)$.

Suppose that

(*i*) $\det\,[s1_N + \tfrac{1}{2}(\hat{\alpha}+\hat{\beta})1_N + A] \neq 0$ for $\sigma \geqq 0$

(*ii*) $\tfrac{1}{2}(\hat{\beta}-\hat{\alpha}) \sup_{\omega} \Lambda\{[(i\omega)1_N + \tfrac{1}{2}(\hat{\alpha}+\hat{\beta})1_N + A]^{-1}\} < 1.$

Then $f \varepsilon \mathfrak{L}_{2N}(0, \infty)$ and $f(t) \to 0$ as $t \to \infty$.

Proof:

Clearly, $f \varepsilon \mathfrak{R} \cap \mathcal{E}_N$. Using the well-known expression for the solution of an inhomogeneous system of linear first-order differential equations in terms of the solution of the corresponding matrix homgeneous differential equation, and regarding

$$g + \tfrac{1}{2}(\hat{\alpha} + \hat{\beta})f - \psi[f,t]$$

as the "forcing function," we find that f satisfies

$$e^{-Bt}c + \int_0^t e^{-B(t-\tau)}g(\tau)d\tau$$

$$= f(t) + \int_0^t e^{-B(t-\tau)}\left\{\psi[f(\tau), \tau] - \frac{1}{2}(\hat{\alpha} + \hat{\beta})f(\tau)\right\} d\tau$$

for $t \varepsilon [0, \infty)$, in which $B = \frac{1}{2}(\hat{\alpha} + \hat{\beta})1_N + A$, and c is a real constant N-vector.

In view of (i), the matrix e^{-Bt} is an element of $\mathcal{K}_{1N} \cap \mathcal{K}_{2N}$. By the argument used in the proof of Theorem 6,

$$\int_0^t e^{-B(t-\tau)}g(\tau)\, d\tau \to 0 \quad \text{as} \quad t \to \infty,$$

a property which is obviously shared by $e^{-Bt}c$. Thus, using the fact that

$$-\frac{1}{2}(\hat{\beta} - \hat{\alpha}) \leqq \frac{\psi_n(w, t) - \frac{1}{2}(\hat{\alpha} + \hat{\beta})w}{w} \leqq \frac{1}{2}(\hat{\beta} - \hat{\alpha})$$

$$(n = 1, 2, \cdots, N)$$

for all $t \varepsilon [0, \infty)$ and all real $w \neq 0$, the theorem follows from a direct application of Theorems 4 and 6 [$-\frac{1}{2}(\hat{\beta} - \hat{\alpha})$ and $\frac{1}{2}(\hat{\beta} - \hat{\alpha})$, respectively, play the roles of α and β in Theorem 4].

5.1 *Generalization of an Earlier Theorem Concerning a Linear Differential Equation with Periodic Coefficients*[11]

Theorem 9: Let $\psi[\cdot,\cdot]$ be as defined in Section I with $N = 1$ and α and β, respectively, replaced with $\hat{\alpha}$ and $\hat{\beta}$. Let f denote a real-valued function of t defined and twice-differentiable on $[0, \infty)$ such that

$$\frac{d^2f}{dt^2} + a\frac{df}{dt} + \psi[f, t] = g$$

for almost all $t \varepsilon [0, \infty)$, where $g \varepsilon \mathfrak{R} \cap \mathfrak{L}_2(0, \infty)$ and a is a real constant. Then if $\hat{\alpha} > 0$ and $a > \sqrt{\hat{\beta}} - \sqrt{\hat{\alpha}}$, $f \varepsilon \mathfrak{L}_2(0, \infty)$ and $f(t) \to 0$ as $t \to \infty$.

Proof:

Proceeding as in the proof of Theorem 8, we find that f satisfies

$$h(t) + \int_0^t k(t-\tau)g(\tau)\,d\tau$$

$$= f(t) + \int_0^t k(t-\tau)\{\psi[f(\tau),\tau] - \frac{1}{2}(\hat{\alpha}+\hat{\beta})f(\tau)\}\,d\tau$$

for $t \varepsilon [0, \infty)$, in which h is a solution of

$$\frac{d^2h}{dt^2} + a\frac{dh}{dt} + \frac{1}{2}(\hat{\alpha}+\hat{\beta})h = 0,$$

and $k \varepsilon \mathcal{K}_{11} \cap \mathcal{K}_{21}$ with

$$K(s) = \int_0^\infty k(t)e^{-st}\,dt = \left[s^2 + as + \frac{1}{2}(\hat{\alpha}+\hat{\beta})\right]^{-1}, \qquad \sigma \geqq 0.$$

With $\alpha = -\frac{1}{2}(\hat{\beta} - \hat{\alpha})$ and $\beta = \frac{1}{2}(\hat{\beta} - \hat{\alpha})$, condition (i) of Theorem 4 is obviously satisfied, while condition (ii) reduces to

$$\tfrac{1}{2}(\hat{\beta} - \hat{\alpha}) < \inf_\omega \left| \tfrac{1}{2}(\hat{\alpha}+\hat{\beta}) - \omega^2 + ia\omega \right|.$$

It is a simple matter to show that this inequality is satisfied if $\hat{\alpha} > 0$ and $a > \sqrt{\hat{\beta}} - \sqrt{\hat{\alpha}}$. Hence $f \varepsilon \mathcal{L}_2(0, \infty)$.

Since $h(t) \to 0$ as $t \to \infty$ and, by the argument used to prove Theorem 6,

$$\int_0^t k(t-\tau)g(\tau)\,d\tau \to 0 \quad \text{as} \quad t \to \infty,$$

Theorem 6 implies that $f(t) \to 0$ as $t \to \infty$. This completes the proof of Theorem 9.

VI. FINAL REMARK

Some of the results and techniques of this paper are useful in establishing sufficient conditions for the existence and uniqueness of solutions of functional equations of the type that we have considered. The reader familiar with the contraction-mapping fixed-point theorem has probably recognized this fact.

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On the Theory of Linear Multi-Loop Feedback Systems, I. W. Sandberg, B.S.T.J., **42,** March, 1963, pp. 355–382.

On page 361, the expression $(y_1 + y_2)$, which appears twice, should be replaced in both positions with $(y_1 + \bar{y}_2)$, in which $\bar{y}_2$ denotes the value of y_2 when $y_1 = 0$.

On page 377, the left side of the first equation of Section 9.4 should be det $\mathbf{F}_{\bar{\mathfrak{y}}_1}$, not det $\mathbf{F}\ _1$.