A Frequency-Domain Condition for the Stability of Feedback Systems Containing a Single Time-Varying Nonlinear Element

By I. W. SANDBERG

(Manuscript received May 6, 1964)

It is proved that a condition similar to the Nyquist criterion guarantees the stability (in an important sense) of a large class of feedback systems containing a single time-varying nonlinear element. In the case of principal interest, the condition is satisfied if the locus of a certain complex-valued function (a) is bounded away from a particular disk located in the complex plane, and (b) does not encircle the disk.

I. INTRODUCTION

The now well-known techniques introduced by Lyapunov have led to many very interesting results concerning the stability of time-varying nonlinear feedback systems governed by systems of differential equations. However, these methods have by no means led to a definitive theory of stability for even the simplest nontrivial time-varying nonlinear feedback systems. The general problem is, of course, one of considerable difficulty.

The unparalleled utility of the Nyquist stability criterion for single-loop, linear, time-invariant feedback systems is directly attributable to the fact that it is an explicit frequency-domain condition. The Nyquist locus not only indicates the stability or instability of a system, it presents the information in such a way as to aid the designer in arriving at a suitable design. The criterion is useful even in cases in which the system is so complicated that a sufficiently accurate analysis is not feasible, since experimental measurements can be used to construct the loop-gain locus.

The primary purpose of this article is to point out that some recently obtained mathematical results,[1] not involving the theory of Lyapunov, imply that a condition similar to, and possessing the advantages of, the Nyquist criterion guarantees the stability (in an important sense) of feedback systems containing a single time-varying nonlinear element.*,†

II. THE PHYSICAL SYSTEM AND DEFINITION OF $\mathfrak{L}_2$-STABILITY

Consider the feedback system of Fig. 1. We shall restrict our discussion throughout to cases in which g_1, f, u, and v denote real-valued measurable functions of t defined for $t \geqq 0$.

The block labeled ψ is assumed to represent a memoryless time-varying (not necessarily linear) element that introduces the constraint $u(t) = \psi[f(t),t]$, in which $\psi(x,t)$ is a function of x and t with the

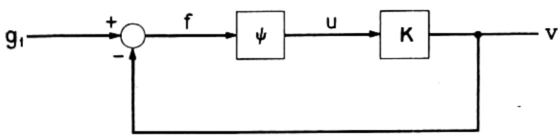

Fig. 1 — Nonlinear feedback system.

properties that $\psi(0,t) = 0$ for $t \geqq 0$ and there exist a positive constant β and a real constant α such that

$$\alpha \leqq \frac{\psi(x,t)}{x} \leqq \beta, \qquad t \geqq 0$$

for all real $x \neq 0$. In particular, we permit the extreme cases in which $\psi(x,t)$ is either independent of t or linear in x [i.e., $\psi(x,t) = \psi(1,t)x$].

The block labeled **K** represents the linear time-invariant portion of the forward path. It is assumed to introduce the constraint

$$v(t) = \int_0^t k(t - \tau)u(\tau)d\tau - g_2(t), \qquad t \geqq 0$$

in which k and g_2 are real-valued functions such that

$$\int_0^\infty |k(t)|\,dt < \infty, \qquad \int_0^\infty |g_2(t)|^2\,dt < \infty. \tag{1}$$

* The results of Ref. 1 relate to feedback systems containing an arbitrary finite number of time-varying nonlinear elements, but, with the exception of the case discussed here, they do not admit of a simple geometric interpretation.

† For results concerned with frequency-domain conditions for the global asymptotic stability (a sense of stability that is different from the one considered here) of nonlinear systems, see, for example, Refs. 2–4.

The function g_2 takes into account the initial conditions at $t = 0$. Our assumptions regarding $\mathbf{K}$ are satisfied, for example, if, as is often the case, u and v are related by a differential equation of the form

$$\sum_{n=0}^{N} a_n \frac{d^n v}{dt^n} = \sum_{n=0}^{N-1} b_n \frac{d^n u}{dt^n}, \qquad t \geqq 0$$

in which the a_n and the b_n are constants with $a_N \neq 0$, and

$$\sum_{n=0}^{N} a_n s^n \neq 0 \quad \text{for} \quad \operatorname{Re}[s] \geqq 0.$$

However, we *do not* require that u and v be related by a differential equation (or by a system of differential equations).

Assumption: We shall assume throughout that the response v is well defined and satisfies the inequality

$$\int_0^t |\, v(\tau)\, |^2\, d\tau < \infty \tag{2}$$

for all *finite* $t > 0$, for each initial-condition function g_2 that meets the conditions stated above and each input g_1 such that

$$\int_0^\infty |\, g_1(t)\, |^2\, dt < \infty.$$

Although this assumption plays an important role in the proof of the theorem to be presented, from an engineering viewpoint it is a trivial restriction (see Ref. 5).

Definition: We shall say that the feedback system of Fig. 1 is "$\mathfrak{L}_2$-stable" if and only if there exists a positive constant ρ with the property that the response v satisfies

$$\left(\int_0^\infty |\, v(t)\, |^2\, dt\right)^{\frac{1}{2}} \leqq \rho \left(\int_0^\infty |\, g_1(t) + g_2(t)\, |^2\, dt\right)^{\frac{1}{2}} + \left(\int_0^\infty |\, g_2(t)\, |^2\, dt\right)^{\frac{1}{2}}$$

for every initial-condition function g_2 that meets the conditions stated above, and every input g_1 such that

$$\int_0^\infty |\, g_1(t)\, |^2\, dt < \infty.$$

In particular, if the system is $\mathfrak{L}_2$-stable, then the response is square-integrable whenever the input is square-integrable.

It can be shown* that the response $v(t)$ approaches zero as $t \to \infty$ for any square-integrable input g_1, provided that the system is $\mathfrak{L}_2$-stable,

* See the proof of Theorem 6 of Ref. 1.

$g_2(t) \to 0$ as $t \to \infty$, and

$$\int_0^\infty | k(t) |^2 \, dt < \infty. \tag{3}$$

In addition, it follows at once from the Schwarz inequality that the response $v(t)$ is uniformly bounded on $[0, \infty)$ for any square-integrable input g_1, provided that the system is $\mathfrak{L}_2$-stable, $g_2(t)$ is uniformly bounded on $[0, \infty)$, and (3) is satisfied.

III. SUFFICIENT CONDITIONS FOR $\mathfrak{L}_2$-STABILITY

Theorem: Let

$$K(i\omega) = \int_0^\infty k(t) e^{-i\omega t} \, dt, \qquad -\infty < \omega < \infty.$$

The feedback system of Fig. 1 is $\mathfrak{L}_2$-stable if one of the following three conditions is satisfied:

(i) $\alpha > 0$; and the locus of $K(i\omega)$ for $-\infty < \omega < \infty$ (a) lies outside the circle C_1 of radius $\frac{1}{2}(\alpha^{-1} - \beta^{-1})$ centered on the real axis of the complex plane at $[-\frac{1}{2}(\alpha^{-1} + \beta^{-1}),0]$, and (b) does not encircle C_1 (see Fig. 2)

(ii) $\alpha = 0$, and $Re[K(i\omega)] > -\beta^{-1}$ for all real ω

(iii) $\alpha < 0$, and the locus of $K(i\omega)$ for $-\infty < \omega < \infty$ is contained within the circle C_2 of radius $\frac{1}{2}(\beta^{-1} - \alpha^{-1})$ centered on the real axis of the complex plane at $[-\frac{1}{2}(\alpha^{-1} + \beta^{-1}),0]$ (see Fig. 3).

Proof: Note first that

$$\int_0^\infty | u(t) |^2 \, dt \leqq \max(\beta^2, | \alpha |^2) \int_0^\infty | f(t) |^2 \, dt,$$

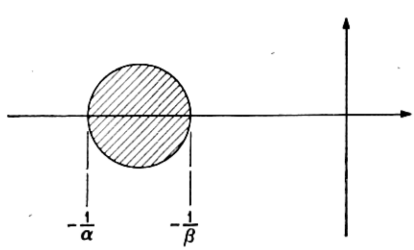

Fig. 2 — Location of the "critical circle" C_1 in the complex plane ($\alpha > 0$). The feedback system is $\mathfrak{L}_2$-stable if the locus of $K(i\omega)$ for $-\infty < \omega < \infty$ lies outside C_1 and does not encircle C_1.

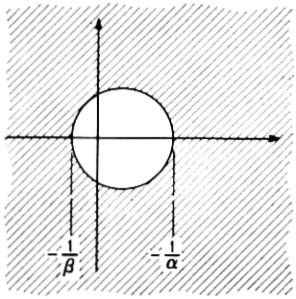

Fig. 3 — Location of the "critical circle" C_2 in the complex plane ($\alpha < 0$). The feedback system is $\mathfrak{L}_2$-stable if the locus of $K(i\omega)$ for $-\infty < \omega < \infty$ is contained within C_2.

and hence, by a well-known result,

$$\int_0^\infty \left| \int_0^t k(t - \tau)u(\tau)d\tau \right|^2 dt \leqq \left(\int_0^\infty | k(t) | dt\right)^2 \int_0^\infty | u(t) |^2 dt$$

$$\leqq \max\left(\beta^2, | \alpha |^2\right)\left(\int_0^\infty | k(t) | dt\right)^2 \int_0^\infty | f(t) |^2 dt.$$

Using Minkowski's inequality,

$$\left(\int_0^\infty | v(t) |^2 dt\right)^{\frac{1}{2}} \leqq \left(\int_0^\infty \left| \int_0^t k(t - \tau)u(\tau)d\tau \right|^2 dt\right)^{\frac{1}{2}}$$

$$+ \left(\int_0^\infty | g_2(t) |^2 dt\right)^{\frac{1}{2}} \leqq \max(\beta, | \alpha |) \int_0^\infty | k(t) | dt$$

$$\cdot\left(\int_0^\infty | f(t) |^2 dt\right)^{\frac{1}{2}} + \left(\int_0^\infty | g_2(t) |^2 dt\right)^{\frac{1}{2}}.$$

Consider now the relation between $(g_1 + g_2)$ and f:

$$g_1(t) + g_2(t) = f(t) + \int_0^t k(t - \tau)\psi[f(\tau),\tau]d\tau, \qquad t \geqq 0$$

and suppose that

$$\int_0^\infty | g_1(t) + g_2(t) |^2 dt < \infty.$$

According to the results of Ref. 1, our assumptions* imply that there

* In Ref. 1 it is assumed that

$$\int_0^t | f(\tau) |^2 d\tau < \infty$$

exists a positive constant ρ_1 (which does not depend upon g_1 or g_2) such that

$$\int_0^\infty |f(t)|^2\,dt < \rho_1 \int_0^\infty |g_1(t) + g_2(t)|^2\,dt$$

provided that, with

$$K(s) = \int_0^\infty k(t)e^{-st}\,dt$$

and $\omega = \text{Im}[s]$,

(i) $1 + \frac{1}{2}(\alpha + \beta)K(s) \neq 0$ for $\text{Re}[s] \geqq 0$, and

(ii) $\frac{1}{2}(\beta - \alpha) \max\limits_{-\infty<\omega<\infty} |K(i\omega)[1 + \frac{1}{2}(\alpha + \beta)K(i\omega)]^{-1}| < 1.$

Thus the feedback system of Fig. 1 is $\mathfrak{L}_2$-stable if conditions (i) and (ii) are satisfied.

According to the well-known theorem of complex-function theory that leads to the Nyquist criterion, condition (i) is satisfied if (and only if) the polar plot of $K(i\omega)$ for $-\infty < \omega < \infty$ does not encircle or pass through the point $[-2(\alpha + \beta)^{-1},0]$. It can easily be verified that condition (ii) is met if one of the following three conditions is satisfied.

(a) $\alpha > 0$, and the locus of $K(i\omega)$ for $-\infty < \omega < \infty$ lies outside the circle C_1 of radius $\frac{1}{2}(\alpha^{-1} - \beta^{-1})$ centered in the complex plane at $[-\frac{1}{2}(\alpha^{-1} + \beta^{-1}),0]$.

(b) $\alpha = 0$, and $\text{Re}[K(i\omega)] > -\beta^{-1}$ for all real ω.

(c) $\alpha < 0$, and the locus of $K(i\omega)$ for $-\infty < \omega < \infty$ is contained within the circle C_2 of radius $\frac{1}{2}(\beta^{-1} - \alpha^{-1})$ centered in the complex plane at $[-\frac{1}{2}(\alpha^{-1} + \beta^{-1}),0]$.

If $\alpha > 0$, the point $[-2(\alpha + \beta)^{-1},0]$ lies on the real-axis diameter of C_1, while if condition (b) or (c) is met, it is impossible for the polar plot of $K(i\omega)$ to encircle the point $[-2(\alpha + \beta)^{-1},0]$. Therefore, the conditions of the theorem guarantee that the feedback system is $\mathfrak{L}_2$-stable.

Remarks

With regard to the necessity of our sufficient conditions for $\mathfrak{L}_2$-stability, consider, for example, the case in which $\alpha > 0$ and suppose, for simplicity, that v and u are related by a differential equation of the type mentioned in Section II. Then, a moment's reflection shows that there exists a $\psi(x,t)$, in fact a $\psi(x,t)$ which is independent of t and linear in x,

---

for all finite $t > 0$. Our assumption that (2) is satisfied for all finite $t > 0$ implies that this condition is met.

that satisfies our assumptions and for which the feedback system is *not* $\mathfrak{L}_2$-stable, provided that for some value of ω, $K(i\omega)$ is a point on the real-axis diameter of C_1. This clearly shows that the condition is in the correct "ball park." Similar remarks can be made concerning our conditions for the cases in which $\alpha < 0$ and $\alpha = 0$.

IV. FURTHER PROPERTIES OF THE FEEDBACK SYSTEM OF FIG. 1

It is possible to say much more about the properties of the feedback system on the basis of frequency-domain information if our assumptions regarding $\psi(x,t)$ are strengthened.

For example, suppose that

$$\alpha \leqq \frac{\psi(x_1,t) - \psi(x_2,t)}{x_1 - x_2} \leqq \beta, \qquad \psi(0,t) = 0 \tag{4}$$

for $t \geqq 0$ and all real $x_1 \neq x_2$, and that one of the three conditions of our theorem is met. Let g_1 and $\hat{g}_1$ denote two arbitrary input functions such that

$$\int_0^t |g_1(\tau)|^2 \, d\tau < \infty \quad \text{and} \quad \int_0^t |\hat{g}_1(\tau)|^2 \, d\tau < \infty$$

for all finite $t > 0$, and

$$\int_0^\infty |g_1(\tau) - \hat{g}_1(\tau)|^2 \, d\tau < \infty.$$

Let v and $\hat{v}$, respectively, denote the (assumed well defined) responses due to g_1 and $\hat{g}_1$. Then if

$$\int_0^t |v(\tau)|^2 \, d\tau < \infty \quad \text{and} \quad \int_0^t |\hat{v}(\tau)|^2 \, d\tau < \infty$$

for all finite $t > 0$, and the assumptions of Section II are met, it follows* that

$$\int_0^\infty |v(\tau) - \hat{v}(\tau)|^2 \, d\tau < \infty$$

and that there exists a positive constant λ (which does not depend upon g_1 or $\hat{g}_1$) such that

$$\int_0^\infty |v(\tau) - \hat{v}(\tau)|^2 \, d\tau \leqq \lambda \int_0^\infty |g_1(\tau) - \hat{g}_1(\tau)|^2 \, d\tau.$$

* Consider Theorem 1 of Ref. 6 with $h_1(t) = f_1(t) = 0$ for $t < 0$.

Suppose now that $\psi(x,t)$ satisfies (4) and is either independent of t or periodic in t with period T for each x, and that one of the three conditions of our theorem is met. Assume that the initial-condition function $g_2(t)$ approaches zero as $t \to \infty$, and that the input $g_1(t)$ applied at $t = 0$ is a bounded periodic function with period T. Then it can be shown* that there exists a bounded periodic function p, with period T, which is independent of g_2 and such that the (assumed well defined) response $v(t)$ approaches $p(t)$ as $t \to \infty$, provided that the conditions of Section II are met, (2) is satisfied for all finite $t > 0$, and

$$\int_0^\infty \left| \int_t^\infty | k(\tau) | \, d\tau \right|^2 dt < \infty, \qquad \int_0^\infty | (1 + t)k(t) |^2 \, dt < \infty. \tag{5}$$

Observe that the conditions of (5) are satisfied if u and v are related by a differential equation of the form described in Section II.

REFERENCES

1. Sandberg, I. W., On the $\mathfrak{L}_2$-Boundedness of Solutions of Nonlinear Functional Equations, B.S.T.J., this issue, p. 1581.
2. Popov, V. M., Absolute Stability of Nonlinear Systems of Automatic Control, Avtomatika i Telemekhanika, **22**, Aug., 1961, pp. 961–978.
3. Kalman, R. E., Lyapunov Functions For the Problem of Lur'e in Automatic Control, Proc. Natl. Acad. Sci., **49**, Feb., 1963, pp. 201–205.
4. Rekasius, Z. V., A Stability Criterion for Feedback Systems with One Nonlinear Element, Trans. IEEE-PTGAC, **AC9**, Jan., 1964, pp. 46–50.
5. Tricomi, F. G., *Integral Equations*, Interscience Publishing, Inc., New York, 1957, p. 46.
6. Sandberg, I. W., and Beneš, V. E., On the Properties of Nonlinear Integral Equations That Arise in the Theory of Dynamical Systems, to be published.

* See Theorem 3 of Ref. 6.