The Resistance of an Infinite Slab with a Disk Electrode

By G. F. FOXHALL and J. A. LEWIS

(Manuscript received April 17, 1964)

We consider the resistance of an infinite slab measured between an electrode covering one face and a circular electrode attached to the other face by a uniform resistive film representing a contact resistance. Upper and lower bounds are found on the difference between the total resistance and the resistance of the film alone. The bounds are obtained by a combination of analysis and experiment, using an electrolytic tank. The results may be applied to determine contact resistance from a measured value of total resistance and a knowledge of the bulk resistivity of the slab material.

I. INTRODUCTION

We consider the resistance of an infinite conducting slab as measured between an electrode entirely covering one face and a circular electrode affixed to the other face by a resistive film. This resistance can be imagined to be made up of two resistances in series: namely, the film contact resistance and a resistance which is due to the body to which the electrode is attached, but which depends on the film resistance.

Lewis[1] has derived general upper and lower bounds on this body resistance. In the present case the upper bound may be calculated analytically. The lower bound is the resistance which would exist between the electrodes in the absence of the film. Calculation of the last-mentioned resistance involves a classical potential problem treated by Weber in 1873, but still not completely solved today.

We treat this problem by a combination of analysis and experiment, the latter in effect being an analog computation using an electrolytic tank. An asymptotic solution is found which converges rapidly for slab thicknesses as small as one disk radius, while for smaller thicknesses experimentally determined values of resistance are used.

The upper and lower bounds for the body resistance, which differ only by 8 per cent for a thick slab and tend to the same value for a thin slab, provide a convenient estimate of slab resistance in the presence of a contact film. More important from the practical point of view, they provide a useful estimate of contact resistance for measured total resistance.

II. THE PROBLEM

We wish to determine the electrical resistance of the slab electrode configuration shown in Fig. 1. The entire base of the slab is in perfect contact with a highly conducting, grounded electrode, while the upper electrode, a circular disk of radius a, is separated from the slab by a thin

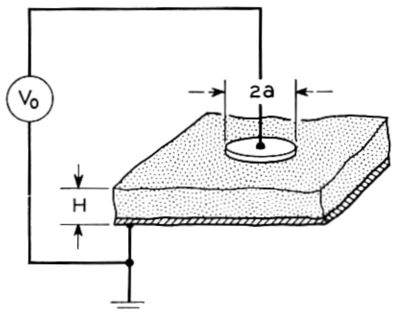

Fig. 1 — Infinite slab with disk electrode.

film of surface conductance c. The potential $V(R,Z)$ in the slab then satisfies Laplace's equation

$$\partial^2 V/\partial R^2 + \partial V/R\partial R + \partial^2 V/\partial Z^2 = 0, \tag{1}$$

for all R and $0 < Z < H$, and the boundary conditions

$$V(R,0) = 0, \tag{2}$$

$$\sigma \partial V(R,H)/\partial Z = \begin{cases} c[V_0 - V(R,H)], & \text{for } R < a \\ 0, & \text{for } R > a, \end{cases} \tag{3}$$

where σ is the conductivity of the slab.

The total input current I is given by

$$I = \int_0^a \sigma \frac{\partial V}{\partial Z} (R,H) 2\pi R \, dR, \tag{4}$$

so that the resistance measured between the electrodes is

$$R_m = \frac{V_0}{I} = V_0 \Big/ \int_0^a \sigma \frac{\partial V}{\partial Z} (R,H) 2\pi R \, dR. \tag{5}$$

If we set

$$r = R/a, \qquad z = Z/a, \qquad h = H/a, \qquad w(r,z) = V(R,Z)/V_0,$$

this becomes

$$\sigma a R_m = 1 \Big/ \int_0^1 \frac{\partial w}{\partial z}(r,h) 2\pi r \, dr, \tag{6}$$

where

$$\nabla^2 w = \partial^2 w/\partial r^2 + \partial w/r\partial r + \partial^2 w/\partial z^2 = 0, \tag{7}$$

in $0 < z < h$,

$$w(r,0) = 0, \tag{8}$$

for all r, and

$$\partial w(r,h)/\partial z = \begin{cases} (ca/\sigma)[1 - w(r,h)], & \text{for } r < 1 \\ 0, & \text{for } r > 1. \end{cases} \tag{9}$$

It is easy to show that R_m may be written in the form

$$R_m = R_c + R(w), \tag{10}$$

where R_c is the film resistance ($1/\pi a^2 c$, in the present case) and $R(w)$ is the ratio of average potential difference between electrodes to total current, i.e.,

$$\sigma a R(w) = 2 \int_0^1 w(r,h) r \, dr \Big/ \int_0^1 \frac{\partial w}{\partial z}(r,h) 2\pi r \, dr. \tag{11}$$

Thus, if $R(w)$ has been calculated and R_m measured, the film resistance R_c may be determined.

However, the calculation of $R(w)$ in general involves the solution of a difficult mixed boundary value problem. Furthermore, since w depends on the film conductance c, which is essentially the quantity to be determined by combined calculation and measurement, $R(w)$ must be calculated for a large number of values of c to make certain that the experimental range is covered. These difficulties can be circumvented, with only a moderate loss in accuracy in the present case, by the use of certain upper and lower bounds on $R(w)$.

III. UPPER AND LOWER BOUNDS

The bounds on $R(w)$ have the form[1]

$$R(u) \leqq R(w) \leqq R(v), \tag{12}$$

where

$$\sigma a R(u) = 1 \Big/ \int_0^1 \frac{\partial u}{\partial z}\,(r,h) 2\pi r\,dr, \tag{13}$$

$$\sigma a R(v) = (2/\pi) \int_0^1 v(r,h) r\,dr, \tag{14}$$

u and v satisfy (7) and (8), and

$$u(r,h) = \partial v(r,h)/\partial z = 1, \qquad \text{for } r < 1 \tag{15}$$

$$\partial u(r,h)/\partial z = \partial v(r,h)/\partial z = 0, \qquad \text{for } r > 1. \tag{16}$$

The lower bound $R(u)$ is the resistance as usually defined, i.e., the reciprocal of the total current for a unit potential difference applied uniformly between the electrodes. On the other hand, $R(v)$ is the *average* potential difference required to give unit total current, distributed uniformly over the input electrode. The former case may be realized by letting the film conductance c become very large; the latter by letting c tend to zero and V_0 tend to infinity in such a way that cV_0 tends to a finite value.

The calculation of $R(v)$ is straightforward, involving only the solution of an unmixed boundary value problem ($\partial v/\partial z$ specified all over $z = h$), but $R(u)$ involves the solution of a *mixed* boundary value problem and ultimately the solution of dual integral equations. An asymptotic solution of these integral equations, valid for large thickness ($h \gg 1$), has been obtained by Tranter.[2] The corresponding expression for $R(u)$ is rapidly convergent, so that it appears to be usable down to values of h of order unity. The range $h < 1$ is covered by measurements on an electrolytic tank analog.

IV. THE UPPER BOUND

If we set

$$v(r,z) = \int_0^\infty f(p) \frac{\sinh pz}{\cosh ph} J_0(pr) dp, \tag{17}$$

the conditions

$$\nabla^2 v = v(r,0) = 0$$

are satisfied and the remaining conditions become

$$\partial v(rh)/\partial z = \int_0^\infty p f(p) J_0(pr) dp = \begin{cases} 1, & r < 1 \\ 0, & r > 1. \end{cases} \tag{18}$$

Now

$$\int_0^\infty J_0(pr)J_1(p)dp = \begin{cases} 1, & r < 1 \\ 0, & r > 1, \end{cases} \tag{19}$$

so that, if we set

$$f(p) = J_1(p)/p,$$

we obtain the complete solution. In particular,

$$v(r,h) = \int_0^\infty \frac{\tanh ph}{p} J_1(p)J_0(pr)dp \tag{20}$$

and

$$\sigma a R(v) = (2/\pi) \int_0^\infty [J_1(p)/p]^2 \tanh ph \, dp. \tag{21}$$

For small h, $\sigma a R(v) \approx h/\pi$ or $R(v) \approx H/\pi a^2\sigma$, the resistance of a circular cylinder, while for large h, $\sigma a R(v) \approx 8/3\pi^2$, a result derived in Carslaw and Jaeger.[3]

Miss M. C. Gray has obtained an expansion in powers of h^{-1} whose first two terms, i.e.,

$$\sigma a R(v) \approx \frac{8}{3\pi^2} - \frac{\log 2}{2\pi h} \tag{22}$$

give reasonable accuracy down to $h = 1$. She has also evaluated the integral for $R(v)$ numerically for $0.1 < h < 10$. This is the curve labeled $R(v)$ in Fig. 2.

V. THE ASYMPTOTIC VALUE OF $R(u)$

If we assume $u(r,z)$ to have the same form as $v(r,z)$ in the previous section, the function $f(p)$ must now satisfy the dual integral equations

$$u(r,h) = \int_0^\infty f(p) \tanh ph J_0(pr)dp = 1, \qquad r < 1, \tag{23}$$

$$\partial u(r,h)/\partial z = \int_0^\infty pf(p)J_0(pr)dp = 0, \qquad r > 1. \tag{24}$$

We can no longer solve these equations by inspection, but for large h an approximate solution, due to Tranter,[2] is available. Tranter gives

$$f(p) \approx \left(\frac{2}{\pi}\right) A(h) \frac{\sin p}{p}, \tag{25}$$

where

$$A(h) = 1 + \frac{2 \log 2}{\pi h} + \left(\frac{2 \log 2}{\pi h}\right)^2 + O(h^{-3}). \tag{26}$$

For $A = 1$ $(h = \infty)$ we obtain the classical result

$$[\sigma a R(u)]_\infty = \tfrac{1}{4},$$

and for large h

$$\sigma a R(u) \approx 1/4A(h). \tag{27}$$

This function is shown in Fig. 2 for $h \geqq 1$.

VI. THE EXPERIMENT

In order to verify and supplement the computed values of resistance, an experiment using an electrolytic solution as the conducting slab was devised. The apparatus, shown in Fig. 3, consisted of a 10 × 14 inch plastic tank into which a gold-plated brass plate was fitted. A 0.01 normal KCl solution was used to simulate the conducting slab, while the end of a 0.564-inch diameter gold-plated brass rod served as the disk electrode.

The experiment consisted of two parts. First resistance measurements were made for the slab configuration at various solution levels. Then a glass sleeve, closely fitted to the upper electrode, was used to constrain the current in the solution to a simple cylindrical geometry. This provided a measurement of solution conductivity and also of contact resist-

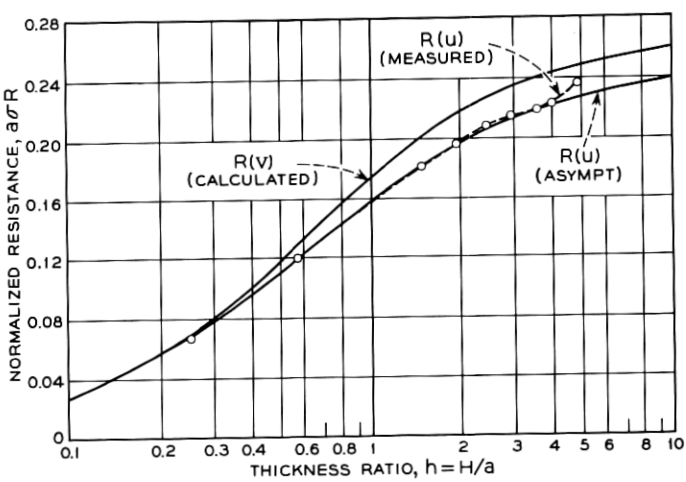

Fig. 2 — The resistance of an infinite slab.

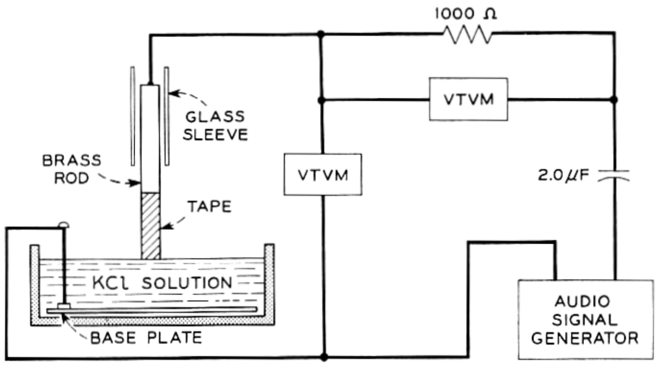

Fig. 3 — Electrolytic tank analog.

ance. The resulting measured values are shown in Fig. 4 as a function of solution depth. The slope gives a conductivity of 1.33×10^{-3} (ohm-cm)$^{-1}$, in close agreement with the tabulated value[4] for the solution temperature of 22°C. Extrapolation to zero solution depth indicates a negligible contact resistance.

As seen in Fig. 2, the measured values of resistance for the slab fall

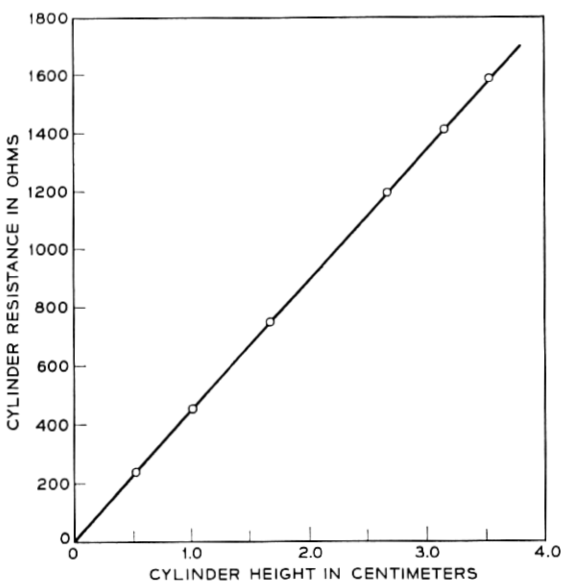

Fig. 4 — The resistance of a cylinder of electrolyte.

very close to the computed values in the range $2 \leqq h \leqq 4$, where the asymptotic form for $R(u)$ can be expected to be accurate. At smaller values of h, the accuracy of the asymptotic form decreases and the measured values may be taken as a good approximation to $R(u)$. At large values of h, on the other hand, the asymptotic form becomes very accurate, while the experimental model becomes less so. This may be seen in Fig. 2 for $h \approx 5$.

This divergence, which at first was attributed to the finite diameter of the tank, is now believed to be due to polarization effects produced by the nonuniform field near the upper electrode. The abrupt upturn of resistance near $h = 5$ cannot be due to finite tank diameter, which would yield a resistance-depth curve with a slope decreasing from a value for a cylinder having the same diameter as the upper electrode to one for a long cylinder having the same diameter as the tank. On the other hand, the current density distribution over the upper electrode is nonuniform for any depth, being infinite at the electrode edge in the mathematical idealization. This nonuniformity increases with depth, for more current is drawn from the electrode center at small depth than at large depth. Thus for fixed total current (the experimental condition) the current density at the edge increases with increasing depth until a depth is reached ($h \approx 5$ in the experiments) at which local polarization effects become appreciable. This dependence on depth also accounts for the different behavior of measured values for the cylinder (Fig. 4) and the slab (Fig. 2).

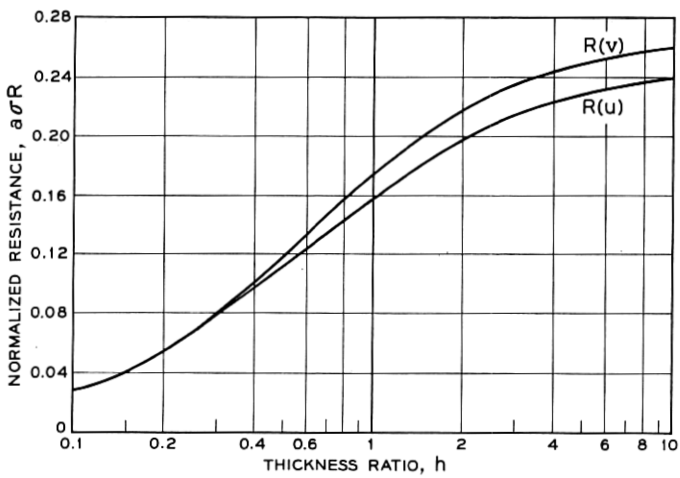

Fig. 5 — Upper and lower bounds on the resistance of an infinite slab.

VII. SUMMARY OF RESULTS

Fig. 5 summarizes the results of the analysis and the experiment. The lower curve, labeled $R(u)$, gives the resistance between the electrodes for zero contact resistance. It also gives a lower bound on the difference between the total resistance R_m with a resistive film between the upper electrode and the slab and the resistance $R_c = 1/\pi a^2 c$ of the film itself, while the upper curve, labeled $R(v)$, gives an upper bound on the same quantity. Thus, for all slab thicknesses and (constant) film conductances c,

$$R(u) \leqq R_m - R_c \leqq R(v), \tag{28}$$

or, assuming R_m to be determined by measurement,

$$R_m - R(v) \leqq R_c \leqq R_m - R(u) \tag{29}$$

giving an estimate of the contact resistance itself.

To supplement Fig. 3, the asymptotic forms

$$\sigma a R(u) \sim (0.250)/[1 + (0.441/h) + (0.441/h)^2], \tag{30}$$

$$\sigma a R(v) \sim (0.270) + (0.110/h), \tag{31}$$

may be used for thickness ratio $h \geqq 1$, while, for $h \leqq 0.1$, both $\sigma a R(u)$ and $\sigma a R(v)$ are closely approximated by the cylinder resistance h/π.

VIII. ACKNOWLEDGMENTS

The authors wish to acknowledge the help of Miss M. C. Gray with the evaluation of the integral for $R(v)$ and the advice of D. L. Klein on the experiment.

REFERENCES

1. Lewis, J. A., Generalized Thermal Resistance, Quart. Appl. Math., **19**, April, 1961, pp. 76–80.
2. Tranter, C. J., *Integral Transforms in Mathematical Physics*, 2nd Ed., Methuen, New York, 1956, pp. 117–120.
3. Carslaw, H. S., and Jaeger, J. C., *Conduction of Heat in Solids*, 2nd. Ed., Oxford, 1959, pp. 214–217.
4. *Handbook of Chemistry and Physics*, 32nd Ed., Chemical Rubber Publishing Company, Cleveland, 1950, p. 2147.