Optimal Rearrangeable Multistage Connecting Networks

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A rearrangeable connecting network is one whose permitted states realize every assignment of inlets to outlets—that is, one in which it is possible to rearrange existing calls so as to put in any new call. In the effort to provide adequate telephone service with efficient networks it is of interest to be able to select rearrangeable networks (from suitable classes) having a minimum number of crosspoints. This problem is fully resolved for the class of connecting networks built of stages of identical square switches arranged symmetrically around a center stage: roughly, the optimal network should have as many stages as possible, with switches that are as small as possible, the largest switches being in the center stage; the cost (in crosspoints per inlet) of an optimal network of N inlets and N outlets is nearly twice the sum of the prime divisors of N, while the number of its stages is $2x - 1$, where x is the number of prime divisors of N, in each case counted according to their multiplicity. By using a large number of stages, these designs achieve a far greater combinatorial efficiency than has been attained heretofore.

I. INTRODUCTION

A study of rearrangeable connecting networks, begun in a previous paper,[1] is here continued; the object of the present work is to solve the synthesis problem of choosing, from a class of networks that are built of stages of square switches and satisfy some reasonable conditions on uniformity of switch size, a rearrangeable connecting network having a *minimum* number of crosspoints. Some of the terminology, notation, and results of Ref. 1 are used, and familiarity with it will be assumed from Section IV on.

Naturally, we do not pretend that minimizing the number of crosspoints (used to achieve a given end) is the only consideration relevant to the design of a connecting network. Other factors, like the number of memory elements, the amount and placing of terminal equipment, the ease with which a network is controlled (e.g., the possibility of reliable end-marking), etc., may be of overriding significance, depending on the technology used. Still, it is important to know the limits of the region of possible designs, and these are obtained by optimizing on one variable without attention to others.

The problem of designing a good rearrangeable network was (probably first) considered in a paper of C. E. Shannon[2] investigating memory requirements in a telephone exchange. On the networks that he considered he imposed the realistic "separate memory condition" to the effect that in operation a separate part of the memory can be assigned to each call in progress. This means that completion of a new call or termination of an old call will not disturb the state of memory elements associated with any call in progress. Shannon showed that under this assumption a two-sided rearrangeable network, with N inlets and N outlets, and N a power of 2, requires at least

$$2N \log_2 N$$

memory elements (e.g., relays). He gave a design which actually realized this lower bound using

$$4(2^N - 1)\log_2 N$$

crosspoints (e.g., relay contacts). His design had the disadvantage of having very large numbers of contacts on certain relays. It is to be noted that Shannon was concerned with minimizing the number of memory elements, without regard to the number of crosspoints.

Shannon's separate memory condition is actually met by modern connecting networks that are of current practical interest, viz., by the networks made of stages of crossbar switches, considered here. For indeed, an inlet relay on an $n \times n$ crossbar switch is used to close any and each of n crosspoints: the exact one that closes depends on what outlet relay is simultaneously activated.

In this paper we consider the problem of minimizing the number of crosspoints in a network built of square switches, without attention to the number of relays. The following result (a consequence of Theorem 8) then complements Shannon's: For N a power of 2 it is possible to design a rearrangeable network with N inlets and N outlets using $4N \log_2 N - 6N$ relays and $4N(\log_2 N - 2)$ crosspoints. The figure for relays is roughly twice Shannon's while that for crosspoints is much smaller than his, for N large. In our design, no relay controls more than 4 contacts.

II. SUMMARY AND DISCUSSION

In Section III we discuss the notion of the combinatory power or efficiency of a connecting network, and propose to define it as the fraction r of permutations it can realize. According to this definition the four-stage No. 5 crossbar type of network with 10×10 switches has efficiency r close to zero, although it turns out that for the same number ($\approx$1000) of terminals there are networks that achieve $r = 1$ with a smaller number of crosspoints.[1] This greater efficiency is obtained by using many more stages than four.

Preliminaries are treated in Section IV. Particular attention is drawn to the class C_N of all two-sided networks having N inlets and N outlets, and built of stages of identical square switches symmetrically arranged around a center stage. The cost $c(\nu)$ of such a network ν is defined as the total number of crosspoints, divided by N. It is proposed to select rearrangeable networks ν from C_N that have minimal cost $c(\nu)$. This problem is attacked in Section V by defining (i) a map T from C_N to a special set A such that $c(\nu)$ is a function of $T(\nu) \in A$, and (ii) a partial ordering of A. It is then shown (Section VI) that (roughly) a network ν is optimal if and only if $T(\nu)$ is at the bottom of the partial ordering of A. This result allows one to identify (Section VII) the optimal networks in C_N. Their general characteristics are these: Except in some easily enumerated cases, the optimal network should have as many stages as possible, and switches that are as small as possible, the largest switches being in the middle stage; the cost $c(\nu)$ of an optimal network ν is very nearly twice the sum of the prime divisors of N, while the number of its stages is $2x - 1$, where x is the number of prime divisors of N.

Our chief conclusion is that by using many stages of small switches it is possible to design networks that are rearrangeable and cost less (in crosspoints per terminal) than networks in current use, which are far from being rearrangeable. The price paid for this great increase in combinatory power is the current difficulty of controlling networks of many stages. This difficulty is technological, though, and will decrease as improved circuits are developed.

III. THE COMBINATORY POWER OF A NETWORK

A principal reason why *rearrangeable* networks are of practical interest is (of course) that they can be operated as nonblocking networks. If the control unit of the connecting system using the rearrangeable network is made complex enough, it is in principle possible to rearrange calls in progress, repeatedly, in such a way that no call is ever blocked.

At present this possibility is being exploited in only a few special-purpose systems, because of the large amount of searching and data-processing it requires.

However, there is another reason why rearrangeable networks should evoke current interest. Even if we do not care to exploit it, the property of rearrangeability in a connecting network is an indication of its combinatory power or reach, and so can be used as a qualitative "figure of merit" for comparing networks. Other things being equal, a rearrangeable network is better than one which cannot realize all assignments of inlets to outlets. Rearrangeability expresses to some extent the efficiency with which crosspoints have been utilized in designing a connecting network for *distribution*, that is, for reaching many outlets from inlets.

If a numerical measure is called for, one can use the fraction of realizable maximal assignments. For a network ν with the same number N of inlets as outlets, and with inlets disjoint from outlets, this is just

$$r = \frac{\text{number of permutations realizable by } \nu}{N!}$$

$$= \text{combinatorial power of } \nu.$$

It is apparent that $0 \leqq r \leqq 1$, and that for a rearrangeable network $r = 1$. Also, r may be viewed as the chance that a permutation chosen at random will be realizable.

We shall calculate a bound on the combinatorial power r of the kind of connecting network most commonly found in modern telephone central offices. This is the network illustrated in Fig. 1. We choose the switch size $n = 10$ as a representative value; the network then has $N = 1000$ inlets, as many outlets, and 4×10^4 crosspoints. Clearly, the network can realize at most all the permutations that take exactly n terminals from each frame on the inlet side into each frame on the outlet side. Now a frame has n^2 inlets (outlets), and there are

$$\frac{n^2!}{(n!)^n}$$

ways of partitioning n^2 things into n groups of n each. Since there are $2n$ frames, there are

$$\left(\frac{n^2!}{(n!)^n}\right)^{2n}$$

ways of choosing n groups of n each on each frame, and assigning inlet groups to outlet groups (one-to-one and onto) in such a way that for

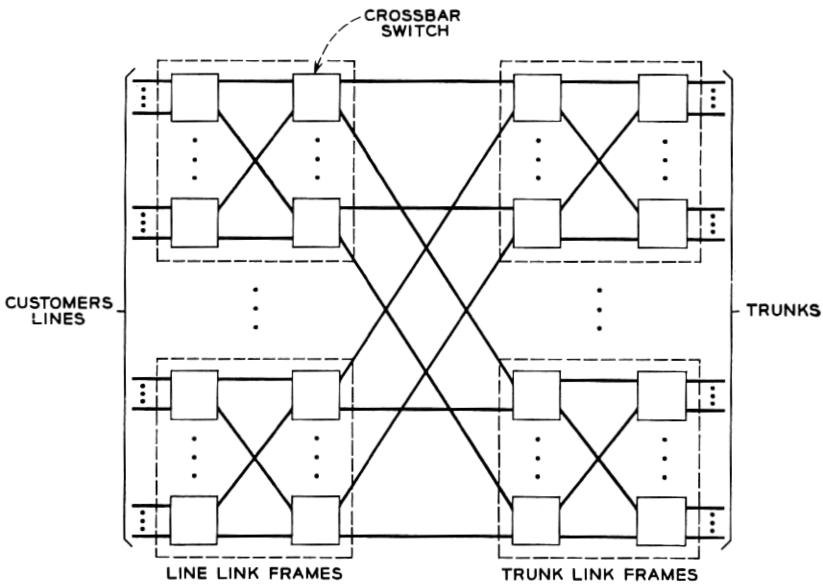

Fig. 1 — Structure of No. 5 crossbar network.

every inlet frame and every outlet frame exactly one group on the inlet frame is assigned to a group on the outlet frame. There are n^2 groups on a side (inlet or outlet), and within each group (at most) $n!$ permutations can be made, i.e., each inlet group can be mapped, terminal by terminal, in at most $n!$ ways onto its assigned outlet group. Hence at most

$$\left(\frac{(n^2!)}{(n!)^n}\right)^{2n} (n!)^{n^2}$$

permutations can be realized. There are $N = n^3$ terminals on a side, and a total of $n^3!$ possible permutations in all. Thus

$$r \leqq \frac{(n^2!)^{2n}}{n!^{n^2} n^3!} .$$

For $n = 10$, with

$$20 \log (100!) = 3159.4000$$

$$100 \log (10!) = 655.976$$

$$\tfrac{1}{2} \log 2\pi = 0.39959$$

$$\log (x!) \sim \tfrac{1}{2} \log 2\pi + (x + \tfrac{1}{2}) \log x - x \log_{10} e$$

we find roughly

$$r \leqq 10^{-64}.$$

Thus only a vanishingly small fraction of all possible permutations can actually be achieved by the No. 5 crossbar network (illustrated in Fig. 1) for $n = 10$, a reasonable switch size.

In the example calculated, the network has a "cost" of 40 crosspoints per terminal on a side. Much of the force of the example would be lost if it were in fact impossible to achieve high values of r (i.e., near 1) without incurring a great increase in the cost in crosspoints per terminal. This, however, is not the case. It follows from our Theorem 8 that a *rearrangeable* network ($r = 1$) can be designed for $N = 1024$ terminals on a side using only

$$4(\log_2 N - 2) = 32$$

crosspoints per terminal. Thus it is actually possible to *achieve* $r = 1$ for *more* than 1000 lines with *fewer* than 40 crosspoints per line. The network that does this turns out to have 17 stages instead of 4, an illustration of the way that allowing many stages can lead to vastly more combinatorially efficient network designs. The middle stage of this network consists of a column of 256 4×4 switches, and each of the other 16 stages, arranged symmetrically, consists of a column of 512 2×2 switches. For $k = 1, \cdots, 8$, the kth stage is connected to the $(k + 1)$th as follows: the first outlet of the first switch of stage k goes to the first switch of stage $(k + 1)$, the second outlet of the first switch of stage k goes to the second switch of stage $(k + 1)$, the first outlet of the second switch of stage k goes to the third switch of stage $(k + 1)$, etc., as in Fig. 2 with $1 \leqq k \leqq 7$; when each switch of stage $(k + 1)$ has 1 link on it the process starts over again with the first switch, and continues cyclically until all the links from stage k are assigned. The connections between stages k and $k + 1$ for $k = 9, \cdots, 17$ are the inverses of those for $k = 1, \cdots, 8$, so that the network is symmetric about the middle stage.[1]

IV. PRELIMINARIES

The symbol C_N, $N \geqq 2$, is used to denote the class of all connecting networks ν with the following properties:*

(1) ν is two-sided, with N terminals on each side

(2) ν is built of an odd number s of stages $\mathcal{S}_k$, $k = 1, \cdots, s$, of

* Familiarity with Ref. 1 is assumed henceforth.

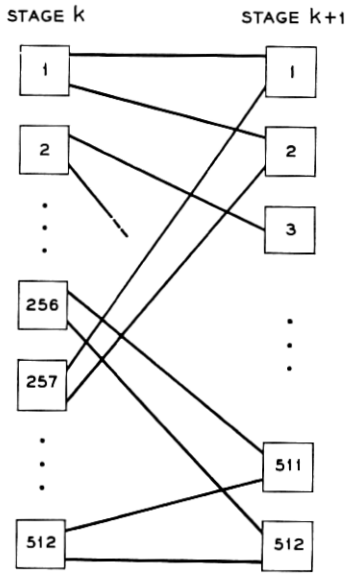

Fig. 2 — Link assignment.

square switches, i.e., there are permutations $\varphi_1, \cdots, \varphi_{s-1}$ such that

$$\nu = \mathcal{S}_1\varphi_1\mathcal{S}_2, \cdots, \varphi_{s-1}\mathcal{S}_s$$

(3) ν is *symmetric* in the sense that

$$\mathcal{S}_k = \mathcal{S}_{s-k+1} \qquad \text{for} \qquad k = 1, \cdots, \tfrac{1}{2}(s-1)$$

(4) With the notation

$$s = s(\nu) = \text{number of stages of } \nu$$

$$n_k = n_k(\nu) = \text{switch size in the } k\text{th stage of } \nu,$$

ν has N/n_k identical switches in stage k, i.e., each stage $\mathcal{S}_k$ is of the form

$$\bigcup_{A \in \Pi} A \times A$$

for some partition Π with $|A| = |B|$ for all $A, B \in \Pi$.

The defining conditions of C_N imply that

$$n_k = n_{s-k+1} \qquad \text{for} \qquad k = 1, \cdots, (s-1)/2$$

and that

$$\prod_{k=1}^{\frac{1}{2}(s+1)} n_k = N.$$

It is assumed throughout that $n_k(\nu) \geqq 2$ for all ν and all $k = 1, \cdots, s(\nu)$.

The *cost per terminal* (on a side) $c(\nu)$ of a network $\nu \in C_N$ is defined to be the total number of crosspoints of ν divided by the number N of terminals on a side. Since there are N/n_k $n_k \times n_k$ switches in stage k, the total number of crosspoints is (using the symmetry condition)

$$\sum_{k=1}^{s} (N/n_k) \cdot n_k^2 = N \sum_{k=1}^{s} n_k$$
$$= N \left(n_{\frac{1}{2}(s+1)} + 2 \sum_{k=1}^{\frac{1}{2}(s-1)} n_k \right)$$

and so

$$c(\nu) = n_{\frac{1}{2}(s+1)} + 2 \sum_{k=1}^{\frac{1}{2}(s-1)} n_k .$$

A network ν is called *optimal* if

$$c(\nu) = \min \{c(\mu) : \mu \in C_N\}.$$

It is clear that the cost per terminal of a network $\nu \in C_N$ depends only on the switch sizes, and not at all on the permutations that define the link patterns between stages.

Also, it is apparent from Theorem 3 of Ref. 1 that given any network $\nu_1 \in C_N$ there is another network $\nu_2 \in C_N$ that is rearrangeable and differs from ν_1 only in the fixed permutations that are used to connect the stages; in particular, ν_1 and ν_2 have the same number of crosspoints. Thus the problem of selecting an optimal rearrangeable network from C_N is equivalent to that of choosing an optimal network from C_N, rearrangeable or not. A network in C_N can be made rearrangeable by changing its link patterns at no increase in cost.

We make

Definition 1: $m = m(\nu) = [s(\nu) + 1]/2 =$ numerical index of the middle stage

$n = n(\nu) = n_{m(\nu)} =$ size of middle stage switches

Definition 2: $O(\nu) = \{n_1, \cdots, n_{m-1}\} =$ the set of switch sizes (with repetitions) in *outer* (i.e., nonmiddle) stages

Definition 3: $\omega(N) = \{O(\nu) : \nu \in C_N\}$.

Remark 1: $c(\nu) = n(\nu) + 2 \sum_{x \in O(\nu)} x.$

Theorem 1: Let (A,n) be a point (element) of

$$\omega(N) \times X$$

with

$$n \prod_{y \in A} y = N.$$

Then there exists a nonempty set $Y \subseteq C_N$ such that

$$T(\nu) = (A,n), \qquad \nu \in Y.$$

The ν's in Y differ only in the permutations between the stages and in the placing of the outer stages, and at least one of them is rearrangeable. This result follows from the definition of C_N and from Theorem 2 of Ref. 1.

V. CONSTRUCTION OF THE BASIC PARTIAL ORDERING

The solution to the problem of synthesizing an optimal rearrangeable network from C_N will be accomplished as follows: we shall define a mapping T of C_N into $\omega(N) \times X$, with $X = \{1, \cdots, N\}$, and a *partial ordering* $\leqq$ of $T(C_N)$; the map T will have the property that $c(\nu)$ is a function of $T(\nu)$; then we shall prove that (roughly speaking) a network ν is optimal if and only if $T(\nu)$ is at the "bottom" of the partial ordering, i.e., that $c(\nu)$ is almost an isotone function of $T(\nu)$.

To define a partial ordering of a finite set, it is enough to specify consistently which elements *cover* which others. Let $Z, Z_0, Z_1, \cdots$ be sets of positive integers $\leqq N$ possibly containing repetitions.

Definition 4: Z_1 covers Z_2 if and only if there are positive integers j and k such that k occurs in Z_1, j divides k, and Z_2 is obtained from Z_1 by replacing an occurrence of k with one occurrence each of j and k/j.

Definition 5: $Z_0 \leqq Z$ if and only if there is an integer n and sets $Z_1, Z_2, \cdots, Z_n$ such that Z_{i+1} covers Z_i, $i = 0, 1, \cdots, n-1$ and $Z_n = Z$.

Definition 6: T: $\nu \rightarrow O(\nu), n(\nu)$.
A partial ordering $\leqq$ of $T(C_N)$ is defined by the following definition of covering:

Definition 7: Let μ, ν be elements of $C(N)$.
$T(\mu)$ covers $T(\nu)$ if and only if either

(*i*) $n(\nu) < n(\mu)$, $n(\nu)$ divides $n(\mu)$, and $O(\nu)$ results from $O(\mu)$ by adding an occurrence of $n(\mu)/n(\nu)$, or

(*ii*) $n(\nu) = n(\mu)$ and $O(\mu)$ covers $O(\nu)$.

VI. COST IS NEARLY ISOTONE ON $T(C_N)$

Theorem 2: If $T(\nu) \leqq T(\mu)$, *and* $n(\mu) > 6$, *then*

$$c(\nu) \leqq c(\mu).$$

Proof: It is enough to prove the result for μ and ν such that $T(\mu)$ covers $T(\nu)$.

Case (*i*): $n(\nu) < n(\mu)$, $n(\nu)$ divides $n(\mu)$, $n(\nu) \geqq 2$, and $O(\nu)$ results from $O(\mu)$ by adding an occurrence of $n(\mu)/n(\nu)$. Then

$$\begin{aligned} c(\nu) &= n(\nu) + 2 \sum_{x \epsilon O(\nu)} x \\ &= n(\nu) + \frac{2n(\mu)}{n(\nu)} + 2 \sum_{x \epsilon O(\mu)} x \\ &= c(\mu) + n(\nu) - n(\mu) + \frac{2n(\mu)}{n(\nu)} \\ &= c(\mu) + n(\nu)\left[1 - \frac{n(\mu)}{n(\nu)}\right] + \frac{2n(\mu)}{n(\nu)}. \end{aligned}$$

Thus $c(\nu) \leqq c(\mu)$ if and only if

$$n(\nu)\left(1 - \frac{n(\mu)}{n(\nu)}\right) + \frac{2n(\mu)}{n(\nu)} \leqq 0$$

that is, if

$$\frac{2y}{y-1} \leqq x$$

where $x = n(\nu)$ and $y = n(\mu)/n(\nu)$. Now $n(\mu) > 6$ implies that either

(*i*) $n(\nu) = 2$ and $\dfrac{n(\mu)}{n(\nu)} \geqq 4$

or

(*ii*) $n(\nu) = 3$ and $\dfrac{n(\mu)}{n(\nu)} \geqq 3$

or

(*iii*) $n(\nu) \geqq 3$.

The condition $2y/(y - 1) \leqq x$ is fulfilled in all three cases, and so $c(\nu) \leqq c(\mu)$.

Case (*ii*): $n(\mu) = n(\nu)$ and $O(\mu)$ covers $O(\nu)$. There exist integers j,k such that j divides k, $j \geqq 2$ in $O(\mu)$, and $O(\nu)$ results from $O(\mu)$ by replacing one occurrence of k with one each of j and k/j. Then

$$\begin{aligned} c(\nu) &= n(\nu) + 2 \sum_{x \in O(\nu)} x \\ &= n(\mu) - 2k + 2j + (2k/j) + 2 \sum_{x \in O(\mu)} x \\ &= c(\mu) - 2k + 2j + (2k/j). \end{aligned}$$

Since j divides k and $j \geqq 2$, $k \geqq 2j$ and $k \geqq 2k/j$, so

$$k \geqq 2 \max\left(j, \frac{k}{j}\right) > j + \frac{k}{j}$$

and $c(\nu) < c(\mu)$.

Theorem 3: If $\nu \in C_N$ and $O(\nu)$ does not consist entirely of prime numbers (possibly repeated), then there exists a network μ in C_N of $s(\nu) + 2$ stages with $c(\mu) < c(\nu)$, and ν cannot be optimal in C_N.

Proof: There is a value of k in the range $1 \leqq k \leqq n(\nu) - 1$ for which n_k is not a prime, say $n_k = ab$. Define stages $\mathcal{S}_j(\mu)$, $j = 1, \cdots, s(\nu) + 2$ as follows:

$$\mathcal{S}_{j+1}(\mu) = \mathcal{S}_j(\nu), \qquad j = k + 1, \cdots, n(\nu);$$

let Π_a, Π_b be partitions of $X = \{1, \cdots, N\}$ with

$$| \Pi_a | = N/a \quad \text{and} \quad A \in \Pi_a \Rightarrow | A | = a$$

$$| \Pi_b | = N/b \quad \text{and} \quad B \in \Pi_b \Rightarrow | B | = b.$$

Set

$$\mathcal{S}_{k+1}(\mu) = \bigcup_{A \in \Pi_a} A^2$$

$$\mathcal{S}_k(\mu) = \bigcup_{B \in \Pi_b} B^2$$

$$\mathcal{S}_j(\mu) = \mathcal{S}_j(\nu) \qquad j = 1, \cdots, k - 1$$

$$\mathcal{S}_j(\mu) = \mathcal{S}_{s(\nu)-j+1}(\mu) \qquad \text{all } j = 1, \cdots, s(\nu) + 2$$

By Theorem 2 of Ref. 1 permutations $\varphi_1, \cdots, \varphi_{s(\nu)-1}$ can be found so that the network

$$\mu = \mathcal{S}_1\varphi_1, \cdots, \varphi_{s(\mu)-1}\mathcal{S}_{s(\mu)}$$

is in C_N and is rearrangeable. It is apparent that $n_{m(\mu)} = n_{m(\nu)}$ and that $O(\nu)$ covers $O(\mu)$. Hence the argument for case (*ii*) of Theorem 2 shows that μ has strictly lower cost than ν.

Corollary 1: If $N > 6$ and is not prime, then a network ν consisting of one square switch is not optimal.

VII. PRINCIPAL RESULTS

Definition 8: An element $T(\nu)$ of $T(C_N)$ is *ultimate* if there are no $\mu \in C_N$ such that $T(\nu)$ covers $T(\mu)$.

Remark 2: $T(\nu)$ is ultimate if and only if $n(\nu)$ is prime and $O(\nu)$ consists entirely of prime numbers.

Definition 9: An element $T(\nu)$ of $T(C_N)$ is *penultimate* if it covers an ultimate element.

Definition 10: p_n, $n = 1, 2, \cdots$, is the nth prime.

Definition 11: $\pi(n)$ is the prime decomposition of n, that is, the set of numbers (with repetitions) such that

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_l^{\alpha_l}$$

if and only if $\pi(n)$ contains exactly α_i occurrences of p_i, $i = 1, \cdots, l$, and nothing else.

Definition 12: p is the largest prime factor of N.

Lemma 1: If $p = 3$ and $N > 6$ is even, then the following conditions are equivalent:

(*i*) ν *is optimal*
(*ii*) $T(\nu)$ *is penultimate and* $n(\nu) = 6$ *or* 4
(*iii*) $T(\nu) = (\pi(N/)6,)6$ *or* $(\pi(N/4),4)$.

Proof: By Theorems 2, 3 only ν with $n(\nu) \leqq 6$ and $O(\nu)$ consisting entirely of primes can be optimal. Writing $N = 2^x 3^y$ with $x \geqq 1$ and $y \geqq 1$, it is seen that such ν must have a cost $c(\nu)$ having one of the forms

$$2 + 2[2(x - 1) + 3y] = 4x + 6y - 2,$$

$$3 + 2[2x + 3(y - 1)] = 4x + 6y - 3,$$

$$4 + 2[2(x - 2) + 3y] = 4x + 6y - 4 \quad \text{(only occurs if } x > 1),$$

$$6 + 2[2(x - 1) + 3(y - 1)] = 4x + 6y - 4.$$

The least of these is either of the last two, which correspond to $n(\nu) = 6$ if $x = 1$ or to $n(\nu) = 6$ or 4 if $x > 1$. It is apparent that (*ii*) is equivalent to (*iii*).

Lemma 2: If $p = 2$, and $N > 4$, then the following conditions are equivalent:

(*i*) *ν is optimal*
(*ii*) *$T(\nu)$ is penultimate and $n(\nu) = 4$*
(*iii*) *$T(\nu) = (\pi(N/4), 4)$.*

Proof: With $N = 2^x$ it can be seen as in Lemma 1 that only those ν can be optimal whose cost $c(\nu)$ has one of the forms

$$2 + 2[2(x - 1)],$$

$$4 + 2[2(x - 2)].$$

The second of these is the better, and corresponds to $n(\nu) = 4$.

Theorem 4: Let μ be a network such that a prime number $r > n(\mu)$ occurs in $O(\mu)$. Let M result from $O(\mu)$ by replacing one occurrence of r by $n(\mu)$. Then for any network ν with

$$T(\nu) = (M,r)$$

it is true that

$$c(\nu) < c(\mu)$$

i.e., ν is strictly better than μ. Among such ν, that is best for which r is largest.

Proof: Existence of a rearrangeable ν satisfying $T(\nu) = (M,r)$ is guaranteed by Theorem 1. For the rest of the proof, we observe that $r > n(\mu)$ and

$$\begin{aligned} c(\mu) &= n(\mu) + 2 \sum_{x \in O(\mu)} x \\ &= n(\mu) + 2r - 2n(\mu) + 2 \sum_{x \in M} x \\ &= r - n(\mu) + c(\nu). \end{aligned}$$

Theorem 5: If $n(\mu) \leqq 6$, $n(\mu) = 2^x 3^y 5^z$, some prime number $r > 3$ occurs in $O(\mu)$, and if M results from $O(\mu)$ by replacing one occurrence of r by x occurrences of 2, y occurrences of 3, and z occurrences of 5 then for any network $\nu \in C_N$ with

$$T(\nu) = (M,r)$$

it is true that

$$c(\nu) \leqq c(\mu)$$

i.e., ν is at least as good as μ. Among such ν, that is best for which r is largest.

Proof: Existence of a rearrangeable $\nu \in C_N$ satisfying $T(\nu) = (M,r)$ is given by Theorem 1. For the rest of the proof, we observe that $r \geqq 5$ and

$$\begin{aligned} c(\mu) &= n(\mu) + 2 \sum_{u \in O(\mu)} u \\ &= n(\mu) + 2r - 4x - 6y - 10z + 2 \sum_{u \in M} u \\ &= r + n(\mu) - 4x - 6y - 10z + c(\nu). \end{aligned}$$

Since x, y, and z can only assume the values 0 and 1, with $z = 1$ if and only if $x = y = 0$, we have $c(\mu) \geqq c(\nu)$, the best ν corresponding to the largest r.

Definition 13: $Q = \{(A,r): r \text{ a prime and } A = \pi(N/r)\}$.

Definition 14: $L = T^{-1}(Q)$.

Remark 2: Q consists of all the absolute minima in the partial ordering $\leqq$ of $T(C_N)$, i.e., $\nu \in L$ implies that there are no $\mu \in C_N$ for which

$$T(\mu) < T(\nu).$$

Theorem 6: If $p > 3$, then all optimal networks belong to L.

Proof: Let $\mu \in C_N - L$ be given. We show that there exists a $\nu \in L$ that is at least as good.

Case 1: There is a sequence $\mu = \mu_1, \mu_2, \cdots, \mu_n, \nu$ with $\mu_n \neq \nu$, $\nu \in L$, $n(\mu_n) > 6$,

$$T(\mu_1) \geqq T(\mu_2) \geqq \cdots \geqq T(\mu_n)$$

and such that $T(\mu_n)$ covers $T(\nu)$. Then the numbers $n(\mu_j)$, $j = 1, \cdots, n$ are all > 6, and the result follows from Theorem 2.

Case 2: All sequences $\mu = \mu_1, \mu_2, \cdots, \mu_n, \nu$ with $\mu_n \neq \nu$, $\nu \in L$, $T(\mu_1) \geqq T(\mu_2) \geqq \cdots \geqq T(\mu_n)$, and such that $T(\mu_n)$ covers $T(\nu)$, are such that $n(\mu_n) \leqq 6$. Consider such a sequence. Let i be the smallest index j for which $n(\mu_j) \leqq 6, j = 1, \cdots, n$. Then Theorem 2 gives $c(\mu) \geqq c(\mu_i)$. Since $n(\mu_i) \leqq 6$ and $T(\mu_n)$ covers $T(\nu)$, it follows that $O(\mu_i)$ contains an occurrence of $p > 3$. Hence by Theorem 5 there exists a network $\eta \in C_N$ with $n(\eta) = p$ and

$$c(\eta) \leqq c(\mu_i) \leqq c(\mu).$$

Let $\xi \in L$ be such that $n(\xi) = p$ and $T(\eta)$ covers $T(\xi)$. Then $c(\xi) \leqq c(\eta)$ by case (ii) of Theorem 2. Hence

$$c(\xi) \leqq c(\mu)$$

$$\xi \in L.$$

Theorem 7: If $N \leqq 6$ and ν is optimal, then ν is a square switch and $c(\nu) = N$.

Proof: For prime N with $2 \leqq N < 6$ the result is obvious. If $N = 6$ and $\nu \in C_6$ then exactly one of the following alternatives obtains:

$$T(\nu) = (\theta, 6) \quad \text{and} \quad c(\nu) = 6$$

$$T(\nu) = (\{3\}, 2) \quad \text{and} \quad c(\nu) = 8$$

$$T(\nu) = (\{2\}, 3) \quad \text{and} \quad c(\nu) = 7.$$

The first alternative is optimal, and there is exactly one $\nu \in C_6$ such that $T(\nu) = (\theta, 6)$, viz., the 6×6 square switch. Similarly, if $N = 4$ and $\nu \in C_4$, then $T(\nu) = (\theta, 4)$ or $(\{2\}, 2)$; the former has cost 4, the latter 6.

Definition 15: For $n \geqq 2$, $D(n)$ is the sum of the prime divisors of n counted according to their multiplicity; thus if

$$n = 2^{\alpha_1}3^{\alpha_2} \cdots p_k^{\alpha_k}$$

then

$$D(n) = \sum_{j=1}^{k} p_j\alpha_j = \sum_{x \in \pi(n)} x.$$

Definition 16: $c(N) = \min\{c(\nu) : \nu \in C_N\}$.

Theorem 8:

$$c(N) = \begin{cases} N \text{ if } N \leqq 6 & \textit{or } N \textit{ is prime} \\ p + 2D(N/p) & \textit{if } N > 6 \textit{ and either } p > 3 \textit{ or } N \textit{ is odd} \\ 2D(N/2) & \textit{if } N > 6 \textit{ in all other cases (i.e., } p = 2, \textit{ or} \\ & \quad p = 3 \textit{ and } N \textit{ is even).} \end{cases}$$

Proof: Putting together Lemmas 1, 2 and Theorems 1, 2, 3, 4, 6, and 7 we obtain the following values for the minimal cost in crosspoints per terminal on a side for networks in C_N:

$$c(N) = \begin{cases} N \text{ if } N \leqq 6 \text{ or } N \text{ is prime} \\ p + 2 \sum_{x \in \pi(N/p)} & \textit{if } p > 3, N > 6 \\ 6 + 2 \sum_{x \in \pi(N/6)} x = 2 \sum_{x \in \pi(N/2)} x & \textit{if } p = 3, N > 6, N \textit{ even} \\ 3 + 2 \sum_{x \in \pi(N/3)} x = 3 + 6(\log_3 N - 1) \\ & \textit{if } p = 3, N > 6, N \textit{ odd} \\ 4 + 2 \sum_{x \in \pi(N/4)} x = 4(\log_2 N - 2) = 2 \sum_{x \in \pi(/2)} x \\ & \textit{if } p = 2, N > 6; \end{cases}$$

simplification gives Theorem 8.

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