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A Condition for the $\mathcal{L}_\infty$-Stability of Feedback Systems Containing a Single Time-Varying Nonlinear Element

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In the automatic control literature, a feedback system is frequently said to be stable if, regardless of the initial state of the system, each bounded input applied at $t = 0$ produces a bounded output. The purpose of this brief is to present a sufficient condition for the feedback system of Fig. 1 to be stable in this sense.

Our discussion is restricted to cases in which g_1, f, u, and v (in Fig. 1) denote real-valued measurable functions of t defined for $t \geqq 0$. The block labeled ψ is assumed to represent a memoryless time-varying element that introduces the constraint $u(t) = \psi[f(t),t]$, in which $\psi(x,t)$ is a function of x and t with the properties that $\psi(0,t) = 0$ for $t \geqq 0$ and there exist a positive constant β and a real constant α such that

$$\alpha \leqq \frac{\psi(x,t)}{x} \leqq \beta, \qquad t \geqq 0$$

for all real $x \neq 0$.

The block labeled **K** represents the linear time-invariant portion of the forward path, and is assumed to introduce the constraint

$$v(t) = \int_0^t k(t - \tau)u(\tau)d\tau - g_2(t), \qquad t \geqq 0 \tag{1}$$

in which k and g_2 are real-valued measurable functions such that

$$\int_0^\infty | k(t) | dt < \infty, \qquad \sup_{t \geqq 0} | g_2(t) | < \infty.$$

The function g_2 takes into account the initial conditions at $t = 0$. We do not require that u and v be related by a differential equation (or by a system of differential equations).

Assumption: We shall assume that the response $v(t)$ is well defined and such that for all finite $y > 0$

$$\sup_{0 \leqq t \leqq y} | v(t) | < \infty,$$

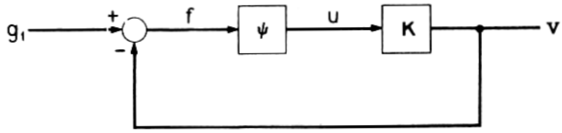

Fig. 1 — Nonlinear feedback system.

for each initial-condition function g_2 that meets the conditions stated above and each input g_1 such that

$$\sup_{t \geqq 0} | g_1(t) | < \infty .$$

Definition: We shall say that the feedback system of Fig. 1 is "$\mathfrak{L}_\infty$-stable" if and only if there exists a positive constant ρ with the property that the response v satisfies

$$\sup_{t \geqq 0} | v(t) | \leqq \rho \sup_{t \geqq 0} | g_1(t) + g_2(t) | + \sup_{t \geqq 0} | g_2(t) |$$

for every initial-condition function g_2 that meets the conditions stated above and every input g_1 such that

$$\sup_{t \geqq 0} | g_1(t) | < \infty .$$

Clearly, if the system is $\mathfrak{L}_\infty$-stable, then the response is bounded whenever the input is bounded.

Theorem: Suppose that

(i) with $K(s) = \int_0^\infty k(t) e^{-st} dt$ *for* $Re[s] \geqq 0$,

$1 + \frac{1}{2}(\alpha + \beta) K(s) \neq 0$ *for* $Re[s] \geqq 0$, *and*

(ii) with $h(\cdot)$ *the inverse Laplace transform of*

$$\frac{K(s)}{1 + \frac{1}{2}(\alpha + \beta) K(s)},^*$$

$$\tfrac{1}{2}(\beta - \alpha) \int_0^\infty | h(t) | \, dt < 1.$$

Then the feedback system of Fig. 1 is $\mathfrak{L}_\infty$*-stable.*

It is of interest to note that condition (*ii*) is satisfied whenever condition (*i*) is met, $\alpha > 0$, $h(t) \geqq 0$ for $t \geqq 0$, and

$$\int_0^\infty k(t) \, dt > 0,$$

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* Condition (*i*) and our assumption regarding $k(\cdot)$ imply that $h(\cdot)$ exists, and that its modulus is integrable on $[0, \infty)$ [see Ref. 1].

for then

$$\tfrac{1}{2}(\beta - \alpha) \int_0^\infty | h(t) | \, dt = \frac{\tfrac{1}{2}(\beta - \alpha)K(0)}{1 + \tfrac{1}{2}(\alpha + \beta)K(0)} < 1.$$

The theorem can be proved with the techniques discussed in Refs. 2 and 3. More specifically, consider the relation between f and $(g_1 + g_2)$:

$$g_1(t) + g_2(t) = f(t) + \int_0^t k(t - \tau)\psi[f(\tau),\tau]d\tau, \qquad t \geqq 0 \qquad (2)$$

and suppose that

$$\sup_{t \geqq 0} | g_1(t) + g_2(t) | < \infty .$$

Arguments very similar to those used to prove Theorem 1 of Ref. 2 and Theorem I of Ref. 3 show that if (a) f satisfies (2), (b)

$$\sup_{0 \leqq t \leqq y} | f(t) | < \infty$$

for all finite $y > 0$, and (c) conditions (i) and (ii) of our theorem are satisfied, then there exists a positive constant ρ_1 (which does not depend upon g_1 or g_2) such that

$$\sup_{t \geqq 0} | f(t) | \leqq \rho_1 \sup_{t \geqq 0} | g_1(t) + g_2(t) | .$$

Our theorem is a direct consequence of this fact, in view of (1) and the relation between u and f.

REFERENCES

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