On the Spectral Properties of Single-Sideband Angle-Modulated Signals

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The representation of single-sideband angle-modulated carriers as originally given by Bedrosian is generalized through the functional and spectral notions of distribution theory. In this treatment the class of related modulating signals is extended to rather general types of distributions, and spectral criteria and iteration algorithms are established by which such modulating signals can be recovered from bandlimited components of the modulated carriers.

I. INTRODUCTION

Among the more recent signal transmission techniques for conserving spectral bandwidth is single-sideband angle modulation, first proposed and investigated by Bedrosian.[1] In this scheme a carrier wave is simultaneously angle modulated by an appropriate baseband (bandlimited) signal and amplitude modulated (multiplied) by the negative exponential of the Hilbert transform of the baseband signal, the combined modulation process resulting in an RF spectrum which vanishes identically on the low-frequency side of the carrier frequency and carrier axis crossings which coincide exactly with those of a conventional angle-modulated carrier modulated by the same baseband signal. The single-sideband and axis-crossing properties, although suggesting means with which to obtain ideal bandwidth reduction and compatible detection, are only partially applicable to physical systems.* In general, the RF spectra under the combined and conventional modulation schemes are of infinite extent, and the nonvanishing portion of the spectrum under the former can have, according to any one of several common definitions, a larger effective bandwidth than that under the latter; consequently, single-sideband angle modulation does not necessarily lead to bandwidth reduction, and

* Detection compatibility is suggested by the fact that the output of an ideal limiter depends only on the axis crossings of the input.

the axis-crossing patterns of filtered versions of single-sideband and conventional angle-modulated carriers can differ appreciably. Nevertheless, Bedrosian has shown that the combined form of modulation, as prescribed, offers the possibility of both a reduction in the effective bandwidth over limited ranges of the angle-modulation index and detection compatibility. It is therefore of practical and theoretical interest to establish criteria relating either directly or indirectly to the spectral properties of such carrier waves. In the present paper we specify rather general signal conditions under which the Bedrosian scheme and the associated single-sideband property obtain, and determine spectral conditions under which knowledge of the RF spectrum over a frequency interval slightly wider than half the signal bandwidth provides enough information to recover the baseband signal up to an additive constant.* Signal recovery in this second category is effected by an iterative computation that cannot be carried out exactly in real time; however, the possibility of pure mathematical recovery based on a finite portion of the spectrum constitutes an important spectral property, indicating that the RF spectrum, although infinite in extent, can be viewed theoretically as having an effective bandwidth equal to half the signal bandwidth.† These qualitative results are now restated somewhat more explicitly.

In precise terms, single-sideband angle-modulated carriers are generally assumed to have the form

$$y_c(t) = \exp[-\hat{x}(t)] \cos[2\pi f_c t + x(t)]$$

where x, $\hat{x}$, and y_c represent respectively a specified angle-modulating signal, its Hilbert transform, and the modulated carrier, the first two functions being periodic or square-integrable, bounded, and bandlimited to some frequency interval $[-f_0, f_0]$. Modulated under these conditions, y_c exhibits the two previously mentioned properties with respect to bandwidth and detection; viz., the corresponding amplitude spectrum (Fourier transform) vanishes over $(-f_c, f_c)$, and the axis crossings as well as the effects that they produce at the output of an ideal limiter coincide exactly with those of the usual angle-modulated carrier

* Contrary to established usage, the term "bandwidth" refers here and throughout to the total frequency spread of the spectrum of the baseband signal over both positive and negative frequencies (cf. Section 2.2).

† Other problems and criteria pertaining to the recoverability of signals subject to nonlinear and bandlimiting operations have received considerable attention recently.[2,3] Beurling's theorem, directly applicable to instantaneous compandors, is perhaps the principal result along these lines.[4] In unpublished work, H. O. Pollak shows by means of Fredholm equation methods that under special conditions the baseband signal of a conventional FM carrier can be recovered mathematically from knowledge of the RF spectrum over an interval of twice the signal bandwidth.

$$y_1 = \cos [2\pi f_c t + x(t)].^*$$

To deal with more general modulating signals, i.e., signals which are neither periodic nor square-integrable yet to which the spectral concepts of Fourier transforms and the results above still apply, requires the theory of temperate distributions (generalized functions).[5,6,7] In this paper we treat both x and y_c as special types of distributions and investigate the feasibility of recovering the former from bandlimited components of the latter. Specifically, we: (*i*) generalize the definitions, concepts, and methods of classical Hilbert transform theory to incorporate arbitrary distributions (cf. Section II, Definition 2 and Theorem 3);† (*ii*) extend the class of modulating signals to include all bounded, bandlimited distributions with bounded generalized Hilbert transforms (cf. Section III, Theorem 4); and (*iii*) establish through a standard fixed-point theorem related subclasses for which the spectrum of y_c over any open interval containing $[f_c, f_c + f_0]$ furnishes sufficient information for reconstructing derivative $x'(t)$ by iteration (cf. Section IV, Theorems 7–9).‡ It is intended also that this development illustrate the distribution-theoretic approach to be generally employed in connection with other modulation schemes.

II. PRELIMINARIES

As noted above, characterizing the amplitude spectra and spectral properties of the signals considered in this paper requires the theory of temperate distributions.[6] We discuss here four aspects of this theory: notation and terminology, bandlimited distributions, convolution, and generalized Hilbert transforms.

2.1 *Temperate Distributions — Notation and Terminology*

Let I denote a specified, open interval on the real line with I_∞, $I_{+\infty}$, and $I_{-\infty}$ signifying respectively the intervals $(-\infty, \infty)$, $(0, \infty)$, and $(-\infty, 0)$; $\bar{I}$, the closure of I; $C^k(I)$, the space of scalar functions of which the derivatives up to and including order k are continuous on I; and C_d, the space of "rapidly decaying" functions, viz., the linear vector space

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* In the first case the nonvanishing portion of the spectrum of y_c is generally so smeared out as to have an effective bandwidth greater than that of y_1.

† For detailed examples of Hilbert transform applications in modulation theory, the reader is referred to the expositions of Rowe,[8] Bennett,[9] and Dugundji.[10]

‡ Landau,[2] Miranker,[2] Sandberg,[3] and Beneš[11] have recently made extensive use of fixed-point theorems in a variety of system-theoretic problems relating to recovery and stability.

$$C_d = \{\varphi \mid \varphi \in C^\infty(I_\infty),\, t^j\varphi^{(k)}(t) \to 0(\,|\,t\,| \to \infty\,)\,\forall j,k \geqq 0\}. \tag{1}$$

Also, a topology in C_d is introduced by means of the metric

$$\rho_d(\varphi_1, \varphi_2) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{2^{-k}}{(k+1)} \frac{\mu(\varphi_1 - \varphi_2, j, k-j)}{[1 + \mu(\varphi_1 - \varphi_2, j, k-j)]} \qquad (2)^*$$

$$\varphi_1, \varphi_2 \in C_d$$

where

$$\mu(\varphi, j, k) = \sup_{t \in I_\infty} |\, t^j\varphi^{(k)}(t)\,|.$$

For convenience the convergence of a sequence $\{\varphi_n\}$ relative to this metric is expressed as $\varphi_n \to \varphi(\varphi_n, \varphi \in C_d)$. Since the series in (2) converges uniformly over such sequences, it follows that

$$\varphi_n \to 0 \Leftrightarrow \sup_{t \in I_\infty} |\, t^j\varphi_n^{(k)}(t)\,| \xrightarrow[n \to \infty]{} 0 \qquad \forall j,k.$$

As generally defined, temperate distributions are merely the elements of the conjugate space of C_d, i.e., the space of linear, continuous functionals on C_d.[5,12] In the treatment below we represent this space by D and the corresponding elements by $x\langle\cdot\rangle$. Although mathematically distinct, a distribution $x\langle\cdot\rangle$ and an ordinary function $x(\cdot)$ for which

$$\int_{-\infty}^{\infty} x(t)\varphi(t)\,dt = x\langle\varphi\rangle \qquad \forall \varphi \in C_d \tag{3}$$

are regarded as characterizing one another, either form being essentially determined by the other.† To extend this notion, we associate every element $x \in D$ with a "generalized function" $x(\cdot)$ (cf. Ref. 6), viz., the totality of sequences $\{x_n(\cdot)\}$ in C_d such that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} x_n(t)\varphi(t)\,dt = x\langle\varphi\rangle \qquad \forall \varphi \in C_d. \qquad (4)‡$$

As distributions and generalized functions are in one to one correspondence, it is common to employ all related terms and symbols interchangeably. Also, the ordinary and generalized functions relating to (3) are considered to be equivalent in that both define the same element of D.

In connection with the equality of distributions, let $N[\varphi]$ signify the null set of $\varphi \in C_d$, viz.,

* Space C_d constitutes a complete linear topological space in $\rho_d(\cdot,\cdot)$ (cf. Ref. 12, p. 49).

† "Essentially" is used here to indicate that $x\langle\cdot\rangle$ determines $x(\cdot)$ almost everywhere on I_∞ (cf. Ref. 5, pp. 1645–1646).

‡ Sequences satisfying this condition can be shown to exist for an arbitrary distribution (cf. Ref. 6, p. 183).

$$N[\varphi] = \{t \mid t \in \bigcup_{\alpha} I_{\alpha} ; \varphi(t) = 0 \ \forall t \in I_{\alpha}\},$$

and $S[\varphi]$, the support of φ, viz.,

$$S[\varphi] = I_{\infty} - N[\varphi].$$

Similarly, let $N_D[x]$ signify the null set of $x \in D$, viz.,

$$N_D[x] = \{t \mid t \in \bigcup_{\alpha} I_{\alpha} ; x\langle\varphi\rangle = 0 \ \forall \varphi \ni S[\varphi] \subseteq \bar{I}_{\alpha} , \varphi \in C_d\},$$

and $S_D[x]$, the support of x, viz.,

$$S_D[x] = I_{\infty} - N_D[x].$$

Accordingly, if for an interval $I \subseteq I_{\infty}$ and two elements x and y of D, $I \subseteq N_D[x - y]$, we say $x\langle\cdot\rangle = y\langle\cdot\rangle$ and $x(\cdot) = y(\cdot)$ on I. This definition also allows one to equate generalized and ordinary functions on arbitrary intervals; that is, if

$$x\langle\varphi\rangle = \int_I v(t)\varphi(t)\,dt \tag{5}$$

for some v and all $\varphi \in C_d$ such that $S[\varphi] \subseteq \bar{I}$, then $I \subseteq N_D[x - v]$ and $x = v$ on I.

Among the standard operations associated with distributions, five require special notation:

(*i*) *Products.* With respect to any two distributions x and y of which at least one, say y, characterizes an ordinary function $y(\cdot)$ such that $y(\cdot)\varphi \in C_d \ \forall \varphi \in C_d$, let $xy\langle\cdot\rangle$ (and $yx\langle\cdot\rangle$) denote the product of x and y given by

$$xy\langle\varphi\rangle = yx\langle\varphi\rangle \equiv x\langle y\varphi\rangle \qquad \forall \varphi \in C_d , \tag{6}$$

and let $x(\cdot)y(\cdot)$ (and $y(\cdot)x(\cdot)$) denote the related generalized function.[6]

(*ii*) *Derivatives.* For any $x \in D$, let $p^n x\langle\cdot\rangle$ denote the nth order derivative of x given by

$$p^n x\langle\varphi\rangle \equiv (-1)^n x\langle\varphi^{(n)}\rangle \qquad \forall \varphi \in C_d , \tag{7}$$

and $p^n x(\cdot)$, or $(d^n/d(\cdot)^n)x(\cdot)$, the related generalized function.[6]

(*iii*) *Antiderivatives.* For any $x \in D$, let

$$\int_n x\langle\cdot\rangle$$

denote any nth order antiderivative of x satisfying

$$p^n \int_n x\langle\varphi\rangle = x\langle\varphi\rangle \qquad \forall\varphi \in C_d,$$

and

$$\int_n x(\cdot),$$

the related generalized function. All nth order antiderivatives of a particular element $x \in D$ can differ by only additive polynomials of degree $n - 1$ (cf. Ref. 7, p. 8).

(*iv*) *Limits.* Distribution limits of the form

$$\lim_{\lambda} x_\lambda\langle\varphi\rangle = x\langle\varphi\rangle \in D \qquad \forall\varphi \in C_d$$

are represented in terms of generalized functions by

$$\lim_{\lambda}{}^{(\mathrm{D})} x_\lambda(\cdot) = x(\cdot).$$

(*v*) *Fourier Transforms.* For any $x \in D$, let $\tilde{x}\langle\cdot\rangle$ denote the generalized Fourier transform of x, viz., the distribution given by

$$\tilde{x}\langle\varphi\rangle \equiv x\langle F\cdot\varphi\rangle \qquad \forall\varphi \in C_d \tag{8}$$

where

$$F\cdot\varphi = \int_{-\infty}^{\infty} \varphi(\xi)e^{-2\pi i f\xi}d\xi \qquad i = \sqrt{-1},$$

and let $\tilde{x}(\cdot)$, or $F\cdot x(\cdot)$, denote the related generalized function (cf. Ref. 6, p. 188). For the right-hand functional in definition (8) to exist, it is required that $F\cdot\varphi \in C_d$, a condition which holds for all $\varphi \in C_d$. Rewriting this relation yields the more suggestive form

$$F\cdot x\langle\varphi\rangle = x\langle F\cdot\varphi\rangle. \tag{9}$$

Similarly,

$$F^{-1}\cdot x\langle\varphi\rangle = x\langle F^{-1}\cdot\varphi\rangle$$

where

$$F^{-1}\cdot\varphi = \int_{-\infty}^{\infty} \varphi(\xi)e^{2\pi i t\xi}d\xi.$$

One property pertaining to operators $\lim^{(\mathrm{D})}$, p^n, and F is of paramount importance in applications of distribution theory: the last two commute with the first.[6] The reader is referred to the previously mentioned literature for a detailed discussion of these and other operations as well as the various terms outlined above.

2.2 *Bandlimited Distributions*

Let f and $\bar{J}$ denote respectively a point and a compact set on the real frequency line I_∞. A distribution x for which $S_D[F\cdot x] \subseteq \bar{J}$ (i.e., $F\cdot x = 0$ on any I disjoint from $\bar{J}$) is defined to be bandlimited to $\bar{J}$, the space of such elements being designated as

$$B(\bar{J}) = \{x \mid x \in D,\ S_D[F\cdot x] \subseteq \bar{J}\}.$$

Defining, in addition, the space

$$C_g = \{v \mid v \in C^\infty(I_\infty); \forall k \exists j \ni (1+t^2)^{-j} v^{(k)}(t) \to 0(\,|t| \to \infty)\}, \quad (10)$$

we establish the following

Lemma 1: If $x \in B(\bar{J})$, then $x(\cdot) \in C_g$.

Proof: Construct a real, positive function $\zeta(f) \in C_d$ satisfying the conditions

$$\zeta(f) = \begin{cases} 1 & f \in I_1 \supset \bar{J} \\ 0 & f \notin I_2 \supset \bar{I}_1,\ I_2 \subset I_\infty, \end{cases}$$

and set

$$v(t) = F\cdot x\langle \zeta(f) e^{2\pi i f t}\rangle. \quad (11)$$

We consider first representing $F\cdot x\langle\cdot\rangle$ on I_2 by an integral. For this it is necessary to employ the well known result that on any finite interval an arbitrary distribution can be characterized by a multiple derivative of some ordinary, continuous function; more specifically, there exist both a function $\psi \in C(I_2)$ and an integer $N \geqq 0$ such that

$$F\cdot x\langle\varphi\rangle = (-1)^N \int_{I_2} \psi(f)\varphi^{(N)}(f)\,df \quad (12)$$

for all $\varphi \in C_d$ for which $S[\varphi] \subseteq \bar{I}_2$ (cf. Ref. 7, pp. 11–12). Inasmuch as $S[\zeta] \subseteq \bar{I}_2$, expression (11) becomes

$$v(t) = \int_{I_2} (-1)^N \psi(f) \frac{\partial^N}{\partial f^N} [\zeta(f) e^{2\pi i f t}]\,df,$$

which in turn gives

$$v^{(k)}(t) = \int_{I_2} (-1)^N \psi(f) \frac{\partial^N}{\partial f^N} [(2\pi i f)^k \zeta(f) e^{2\pi i f t}] df$$
$$= F \cdot x\langle (2\pi i f)^k \zeta(f) e^{2\pi i f t} \rangle = 0(t^N) \qquad (|t| \to \infty)$$

for all k; therefore, $v \in C^\infty(I_\infty)$. In order to identify v further, observe that for any k

$$(1 + t^2)^{-j} v^{(k)}(t) \to 0 \qquad (|t| \to \infty)$$

for some integer j; consequently, by (10), $v \in C_g$.

Regarding the relationship between v and x we note that since

$$\int_{-\infty}^{\infty} v(t)\varphi(t)dt = \int_{-\infty}^{\infty} \int_{I_2} (-1)^N \psi(f) \frac{\partial^N}{\partial f^N} [\zeta(f)\varphi(t)e^{2\pi i f t}] df\, dt$$
$$= (-1)^N \int_{I_2} \int_{-\infty}^{\infty} \psi(f) \frac{\partial^N}{\partial f^N} [\zeta(f)\varphi(t)e^{2\pi i f t}] dt\, df$$
$$= \int_{I_2} (-1)^N \psi(f) \frac{\partial^N}{\partial f^N} \left[\zeta(f) \int_{-\infty}^{\infty} \varphi(t) e^{2\pi i f t} dt\right] df$$
$$= F \cdot x\langle \zeta F^{-1} \cdot \varphi \rangle \qquad \forall \varphi \in C_d \qquad *$$

and since $S[(\zeta - 1)\theta] \cap S_D[F \cdot x]$ is empty for all $\theta \in C_d$,

$$x\langle\varphi\rangle = F \cdot x\langle F^{-1} \cdot \varphi\rangle + F \cdot x\langle(\zeta - 1)F^{-1} \cdot \varphi\rangle$$
$$= F \cdot x\langle \zeta F^{-1} \cdot \varphi \rangle = \int_{-\infty}^{\infty} v(t)\varphi(t)dt.$$

Hence, in accordance with (3) et seq., $v(\cdot) = x(\cdot)$ with $v \in C_g$.

2.3 *Convolution*

A convolution operation sufficiently general for most applications in signal theory is given by

Definition 1: For any two distributions x and y of which at least one, say y, is such that $F \cdot y(\cdot) \in C_g$ we define a distribution $x*y$, termed the convolution of x and y, by the relation

$$x*y = y*x = F^{-1} \cdot [\,(F \cdot x)\,(F \cdot y)\,]. \tag{13}$$

As to the consistency of this definition, observe that with $F \cdot y(\cdot) \in C_g$, $\varphi F \cdot y(\cdot) \in C_d$ for all $\varphi \in C_d$; therefore, according to (6) et seq., both the product $(F \cdot x)(F \cdot y)$ and corresponding convolution exist as dis-

* Interchanging the order of integration in this relation is justified by means of the Tonelli-Hobson theorem (cf. Ref. 13, p. 3).

tributions, and their factors commute. The associative and distributive properties of this operation depend in general on the factors involved, the results in any given case being determined directly from (13). One important consequence of Definition 1 is stated as

Theorem 1: For any two distributions x and y of which at least one, say y, is such that $y(\cdot) \in C_g$

$$F\cdot(xy) = (F\cdot x)*(F\cdot y).$$

Proof: From

$$x\langle\varphi(t)\rangle = x\langle F\cdot F\cdot\varphi(-t)\rangle = F\cdot F\cdot x\langle\varphi(-t)\rangle \qquad \forall x \in D, \quad \forall\varphi \in C_d\,,$$

(6), and (13) it follows that

$$\begin{aligned} F\cdot(xy)\ \langle\varphi(f)\rangle &= xy\langle F\cdot\varphi\rangle = x\langle y(t)F\cdot\varphi\rangle \\ &= F\cdot F\cdot x\langle y(-t)F^{-1}\cdot\varphi\rangle = F\cdot F\cdot x\langle\ [F\cdot F\cdot y(t)]F^{-1}\cdot\varphi\rangle \\ &= [\ (F\cdot F\cdot x)\ (F\cdot F\cdot y)\ ]\ \langle F^{-1}\cdot\varphi\rangle \ = \ F^{-1}[(F\cdot F\cdot x)\ (F\cdot F\cdot y)\ ]\ \langle\varphi\rangle \\ &= [\ (F\cdot x)*(F\cdot y)\ ]\ \langle\varphi\rangle. \end{aligned}$$

We show at this point that Definition 1 relates to a more common but less general form of convolution (cf. Ref. 7, p. 31).

Theorem 2: If at least one of two distributions x and y, say y, has a finite support (i.e., $S_D[y] \subseteq \bar{I} \subset I_\infty$), then $x*y$ exists, and

$$x*y\langle\varphi\rangle = x\langle y\langle\varphi(t+\bar{t})\rangle\rangle \qquad \forall\varphi \in C_d\,.$$

Proof: Reversing the roles of t and f in Lemma 1 demonstrates that with $S_D[y]$ finite, i.e., with y time-limited to $\bar{I}$, $F\cdot y(\cdot) \in C_g$; hence, by Definition 1, $x*y$ exists. In addition, from (13), (9), and (6) there obtains

$$\begin{aligned} x*y\langle\varphi(t)\rangle &= F\cdot(x*y)\ \langle F^{-1}\cdot\varphi\rangle \ = \ [\ (F\cdot x)\ (F\cdot y)\ ]\ \langle F^{-1}\cdot\varphi\rangle \\ &= F\cdot x\langle\ (F\cdot y)\ (F^{-1}\cdot\varphi)\ \rangle = x\langle F\cdot[\ (F\cdot y)\ (F^{-1}\cdot\varphi)\ ]\ \rangle \\ &= x\left\langle\int_{-\infty}^{\infty} e^{-2\pi it\xi}(F\cdot y)_\xi(F^{-1}\cdot\varphi)_\xi d\xi\right\rangle \end{aligned}$$

where the subscripts indicate a function of ξ. As the integral of this last functional proves to be linear and continuous on C_d, i.e., as

$$\int_{-\infty}^{\infty} (F\cdot y)_\xi[e^{-2\pi it\xi}(F^{-1}\cdot\varphi)_\xi]d\xi = F\cdot y\langle(F^{-1}\cdot\varphi)_\xi e^{-2\pi it\xi}\rangle \in D,$$

then

$$\begin{aligned} x{*}y\langle\varphi(t)\rangle &= x\langle F\cdot y\langle(F^{-1}\cdot\varphi)_\xi e^{-2\pi i\bar{t}\xi}\rangle\rangle \\ &= x\langle y\langle F\cdot[(F^{-1}\cdot\varphi)_\xi e^{-2\pi i\bar{t}\xi}]\rangle\rangle \\ &= x\langle y\langle\varphi(t+\bar{t})\rangle\rangle. \end{aligned}$$

If defined by this expression instead of (13), convolution would not necessarily have commuting factors; e.g., with $y(\cdot)$ and $x(\cdot)$ equal to a Dirac delta function $\delta(\cdot)$ and a constant, respectively, $y\langle x\langle\varphi(t+\bar{t})\rangle\rangle$ is not defined because for this choice of x, $x\langle\varphi(t+\bar{t})\rangle$ is constant and not an element of C_d.*,†

2.4 *Generalized Hilbert Transforms*

In this subsection we extend the applicability of classical Hilbert transform properties and techniques to arbitrary distributions. Required initially are two lemmas relating to antiderivatives.

Lemma 2: Corresponding to all antiderivatives of an element $\mathrm{F}\cdot x \in D$ *the distribution limits*

$$\lim_{\lambda\to\infty}{}^{(\mathrm{D})}\left[\tan^{-1}\lambda f\int_N F\cdot x\right] \tag{14}$$

exist for some $N \geqq 0$.

Proof: Set $I_{\epsilon_1} = (-\epsilon_1, \epsilon_1)$ and $I_{\epsilon_2} = (-\epsilon_2, \epsilon_2)$ with $0 < \epsilon_1 < \epsilon_2 < \infty$, and construct a real, positive function $\eta(f) \in C_d$ satisfying the conditions

$$\eta(f) = \begin{cases} 1 & f \in I_{\epsilon_1} \\ 0 & f \notin I_{\epsilon_2} \supset \bar{I}_{\epsilon_1}. \end{cases}$$

It is convenient to consider first the same type of integral representation as was used in Lemma 1 [cf. (12)]; namely, there exist both a function $\psi \in C(I_{\epsilon_2})$ and an integer $N \geqq 0$ such that

$$F\cdot x\langle\varphi\rangle = (-1)^N\int_{I_{\epsilon_2}}\psi(f)\varphi^{(N)}(f)\,df \tag{15}$$

* The Dirac distribution is given formally by the equation $\delta\langle\varphi\rangle = \varphi(0)$.

† Commutativity can be forced in such cases by defining the convolution according to the form

$$x{*}y\langle\varphi\rangle = x\langle y\langle\zeta_0(\bar{t})\varphi(t+\bar{t})\rangle\rangle$$

where $\bar{t}$ corresponds to the distribution of finite support and where $\zeta_0 \in C_d$ equals unity over an open interval containing this support and vanishes outside some finite interval.

for all $\varphi \in C_d$ for which $S[\varphi] \subseteq \bar{I}_{\epsilon_2}$. In agreement with (7) and (5) et seq., relation (15) merely asserts that

$$F \cdot x = \frac{d^N}{df^N} \psi(f) \qquad \text{on } I_{\epsilon_2}$$

and that for all antiderivatives,

$$\int_N F \cdot x = \psi(f) + \sum_{n=0}^{N-1} \alpha_n f^n \qquad \text{on } I_{\epsilon_2} \tag{16}$$

where constants α_n are arbitrary. Since $S[\eta\varphi] \subseteq \bar{I}_{\epsilon_2}$ for all $\varphi \in C_d$, Eq. (16) can be written as

$$\int_N F \cdot x\langle \eta\varphi \rangle = \int_{I_{\epsilon_2}} \left[\psi(f) + \sum_{n=0}^{N-1} \alpha_n f^n \right] \eta(f)\varphi(f) df.$$

Therefore, by the Lebesgue convergence theorem[13,14]

$$\lim_{\lambda} \int_N F \cdot x\langle (\tan^{-1} \lambda f) \eta\varphi \rangle = \frac{\pi}{2} \int_{I_{\epsilon_2}} (\text{sgn } f) \, [\psi(f) + \sum_n \alpha_n f^n] \eta(f) \varphi(f) df \tag{17}$$

with

$$\text{sgn } f = \begin{cases} 1, & f > 0 \\ -1, & f < 0. \end{cases}$$

On the other hand, since

$$I_{\epsilon_1} \subseteq N\left[f^j \frac{d^k}{df^k} (1 - \eta)\varphi \right]$$

for all j, k, and $\varphi \in C_d$,

$$(\tan^{-1} \lambda f)(1 - \eta)\varphi \rightarrow (\pi/2)(\text{sgn } f)(1 - \eta)\varphi \qquad \lambda \rightarrow \infty,$$

and

$$\lim_{\lambda} \int_N F \cdot x\langle (\tan^{-1} \lambda f)(1 - \eta)\varphi \rangle = \frac{\pi}{2} \int_N F \cdot x\langle (\text{sgn } f)(1 - \eta)\varphi \rangle. \tag{18}$$

Finally, adding limits (17) and (18) yields

$$\frac{\pi}{2} \int_N F \cdot x\langle (\text{sgn } f)(1 - \eta)\varphi \rangle + \frac{\pi}{2} \int_{I_{\epsilon_2}} (\text{sgn } f) \, [\psi + \sum_n \alpha_n f^n] \eta\varphi \, df = \lim_{\lambda} \int_N F \cdot x\langle (\tan^{-1} \lambda f)\varphi \rangle. \tag{19}$$

As both terms on the left are elements of D, the distribution limits given by (14) exist.

Lemma 3: Corresponding to element x, integer N, and all antiderivatives of Lemma 2 the generalized functions

$$\frac{d^N}{df^N}\cdot \underset{\lambda\to\infty}{lim}^{(D)}\left[(tan^{-1}\lambda f)\int_N F\cdot x\right]$$

differ by only the additive combinations

$$\pi\sum_{n=0}^{N-1}\alpha_n n!\ \delta(f)^{(N-n-1)}$$

where $\delta(f)^{(n)}$ represents the nth order derivative of the Dirac function. Furthermore, $\delta(f)^{(N-1)}$ is the highest-order Dirac component which can exist at $= 0$.

Proof: From (19) and (7) there results

$$\begin{aligned} p^N\cdot\underset{\lambda}{\lim}^{(D)}\cdot\int_N F\cdot x\langle(\tan^{-1}\lambda f)\varphi\rangle &= \frac{\pi}{2}F\cdot x\langle(\text{sgn } f)(1-\eta)\varphi\rangle \\ &+(-1)^N\frac{\pi}{2}\int_{I_{\epsilon_2}}(\text{sgn } f)\psi(f)[\eta(f)\varphi(f)]^{(N)}df \\ &+(-1)^N\frac{\pi}{2}\int_{I_{\epsilon_2}}[(\text{sgn } f)\sum_n\alpha_n f^n][\eta\varphi]^{(N)}df, \end{aligned} \tag{20}$$

the last, only nonunique term reducing to

$$\pi\sum_n\alpha_n n!(-1)^{N-n-1}\varphi(0)^{(N-n-1)} = \pi\sum_n\alpha_n n!\delta\langle\varphi\rangle^{(N-n-1)}.$$

Inspection of the two remaining distributions on the right of (20) shows, in addition, that $\delta(f)^{(N-1)}$ is the highest-order Dirac function possible at $f = 0$; for the support of the first does not include the origin, and the second represents the Nth derivative of an ordinary, sectionally continuous function.

The preceding two lemmas lead immediately to

Definition 2: For any distribution x we define a distribution $\hat{x}$, termed the generalized Hilbert transform of x, by the relation

$$\hat{x}(\cdot) = -i\frac{2}{\pi}F^{-1}\cdot\left\{\frac{d^{N_0}}{df^{N_0}}\cdot\underset{\lambda\to\infty}{\lim}^{(D)}\left[(\tan^{-1}\lambda f)\int_{N_0}F\cdot x\right] + \sum_{n=0}^{N_0-1}\beta_n\delta(f)^{(N_0-n-1)}\right\} \tag{21}$$

where N_0 designates the smallest integer for which Lemma 2 holds and where constants β_n are constrained so as to eliminate from $F \cdot \hat{x}$ (or to prescribe) all Dirac distributions at $f = 0$.

As regards ordinary Hilbert transforms it is noted that if

$$F \cdot x(\cdot) \in L_2(I_\infty) \quad \text{(square-integrable)},$$

then

$$\lim_{\lambda}{}^{(\mathrm{D})} (\tan^{-1} \lambda f) F \cdot x = (\pi/2)\, (\operatorname{sgn} f) F \cdot x.$$

Consequently, $N_0 = 0$ and

$$\hat{x}(\cdot) = -iF^{-1} \cdot \{(\operatorname{sgn} f) F \cdot x\},$$

a formula which is in agreement with classical theory (cf. Ref. 15, pp. 119–120).

Denoting the linear mapping of (21) by H (i.e., $H : D \to D$), we list a few of the more significant properties of generalized Hilbert transforms:

(*i*) $H \cdot H \cdot x = -x$ provided there exist in $F \cdot x$ no Dirac components at $f = 0$.

(*ii*) $H \cdot x$ is real provided x is real.

(*iii*) $S_D[F \cdot H \cdot x] \subseteq S_D[F \cdot x]$.

These results follow directly from (20) and Definition 2. Of importance in single-sideband theory is the property given by

Theorem 3: For any distribution x,

$$S_D[F \cdot (x + i\hat{x})] \subseteq \bar{I}_{+\infty}$$

and

$$S_D[F \cdot (x - i\hat{x})] \subseteq \bar{I}_{-\infty}.$$

That is, $F \cdot (x + i\hat{x})$ and $F \cdot (x - i\hat{x})$ vanish on $I_{-\infty}$ and $I_{+\infty}$, respectively.

Proof: Consider all φ such that $S[\varphi] \subseteq \bar{I}_{-\infty}$; then,

$$(-1)^N \int_{I_{\epsilon 2}} (\operatorname{sgn} f)[\psi(f) + \sum_n \alpha_n f^n][\eta(f)\varphi(f)]^{(N)} df$$

$$= -p^N \int_N F \cdot x\langle \eta\varphi \rangle = -F \cdot x\langle \eta\varphi \rangle,$$

and from (20) and (21) there obtains

$$F \cdot x\langle \varphi \rangle + iF \cdot \hat{x}\langle \varphi \rangle = F \cdot x\langle \varphi \rangle - F \cdot x\langle (1 - \eta)\varphi \rangle - F \cdot x\langle \eta\varphi \rangle = 0.$$

Similarly, with $S[\varphi] \subseteq \bar{I}_{+\infty}$, $F \cdot x\langle \varphi \rangle - iF \cdot \hat{x}\langle \varphi \rangle = 0$.

III. SINGLE-SIDEBAND ANGLE MODULATION (SSBΘM)

The notions and results of the previous section apply directly to signals classified as single-sideband angle-modulated, namely, time functions of the form

$$y_c(t) = \exp[-\hat{x}(t)] \cos[2\pi f_c t + x(t)].$$

It is the intent here to show that if the modulating signals x correspond to elements of the space

$$S_0 = \{x \mid x \in B(\bar{I}_0),\ I_0 = (-f_0, f_0);\ |x|, |\hat{x}| < \infty,\ x \text{ real}\},$$

functions y_c characterize distributions, and have, as the term SSBΘM suggests, amplitude spectra (Fourier transforms) which vanish on the interval $(-f_c, f_c)$. We begin with three lemmas pertaining to exponentials and convolution.

Lemma 4: Elements of the spaces

$$S_1 = \{y \mid y = e^{iz},\ z = x + i\hat{x},\ x \in S_0\},$$

$$S_2 = \{v \mid v = z^N,\ z = x + i\hat{x},\ x \in S_0,\ N \geqq 0\}$$

are equivalent to generalized functions [*cf.* (4) *et seq.*].

Proof: Clearly, since $S_D[F \cdot \hat{x}] \subseteq S_D[F \cdot x]$, both x and $\hat{x}$ are bandlimited as well as bounded, and are, by Lemma 1, elements of Cg; hence, y is bounded on I_∞ and integrable over finite intervals, and

$$\left| \int_{-\infty}^{\infty} y(t)\varphi(t)\,dt \right| \leqq \sup_{t \in I_\infty} |(1 + t^2)\varphi(t)| \int_{-\infty}^{\infty} \left| \frac{y(t)}{1 + t^2} \right| dt \qquad (22)$$

$$\forall \varphi \in C_d .$$

This latter condition, however, implies that

$$\int_{-\infty}^{\infty} y(t)\varphi_n(t)\,dt \to 0 \qquad (23)$$

for $\varphi_n \to 0$. Therefore, the left-hand integral of (22) constitutes a continuous, linear functional on C_d, i.e., a distribution, and y is equivalent to a generalized function. Precisely the same argument applies to $z^N (N \geqq 0)$, showing that this function is also bounded, integrable over finite intervals, and equivalent to a generalized function.

Lemma 5: For $x \in S_0$ and $z = x + i\hat{x}$

$$e^{iz} = \lim_{N\to\infty}{}^{(D)} \sum_{n=0}^{N} \frac{(iz)^n}{n!} . \qquad (24)$$

Proof: Set

$$y = e^{iz},$$

$$y_N = \sum_{n=0}^{N} \frac{(iz)^n}{n!}.$$

Then, by Darboux's formula,[16]

$$y - y_N = \frac{(iz)^{N+1}}{N!} \int_0^1 e^{i\lambda z}(1 - \lambda)^N d\lambda,$$

and inasmuch as y and y_N represent generalized functions (cf. Lemma 4),

$$\begin{aligned} | y\langle\varphi\rangle - y_N\langle\varphi\rangle | &= \left| \int_{-\infty}^{\infty} [y(t) - y_N(t)]\varphi(t)dt \right| \\ &= \left| \frac{1}{N!} \int_{-\infty}^{\infty} z^{N+1}\varphi \int_0^1 e^{i\lambda z}(1 - \lambda)^N d\lambda \, dt \right| \\ &\leqq \frac{1}{N!} \sup_{t,\lambda} | z^{N+1} e^{i\lambda z} | \int_{-\infty}^{\infty} | \varphi(t) | \, dt \\ &\leqq \frac{1}{N!} (\sup_t | z |)^{N+1} \exp\, (\sup_t | \hat{x} |) \\ &\qquad \cdot \int_{-\infty}^{\infty} | \varphi(t) | \, dt \xrightarrow[N\to\infty]{} 0 \qquad \forall\varphi \in C_d, \end{aligned}$$

a result corresponding to (24).

Lemma 6: If two distributions g and h are such that

$$S_D[F \cdot g] \subseteq [0,f_1],$$

$$S_D[F \cdot h] \subseteq [0,f_2],$$

then

$$S_D[F \cdot (gh)] \subseteq [0,f_1 + f_2].$$

Proof: With respect to any $\varphi \in C_d$ for which $S[\varphi] \subseteq \bar{I}_{-\infty}$, set

$$\varphi_0(t) = F \cdot h\langle\varphi(t + \bar{t})\rangle.$$

As defined, $\varphi_0(t) = 0$ for all $t > 0$; i.e., $I_{+\infty} \subseteq N[\varphi_0]$ and $S[\varphi_0] \subseteq \bar{I}_{-\infty}$. Hence, by Theorems 1 and 2

$$\begin{aligned} [F \cdot (gh)]\langle\varphi\rangle &= [(F \cdot g) * (F \cdot h)]\langle\varphi\rangle \\ &= F \cdot g\langle F \cdot h\langle\varphi(t + \bar{t})\rangle\rangle = F \cdot g\langle\varphi_0(t)\rangle = 0, \end{aligned}$$

which yields

$$S_D[F\cdot(gh)] \subseteq \bar{I}_{+\infty}\,. \tag{25}$$

In a similar manner, consider any $\varphi \in C_d$ for which $S[\varphi] \subseteq [f_1 + f_2\,,\ \infty)$, and set

$$\varphi_1(t) = F\cdot h\langle\varphi(t+\bar{t})\rangle.$$

It follows that $\varphi_1(t) = 0$ for all $t \in (-\infty, f_1)$, i.e., that $S[\varphi_1] \subseteq [f_1\,,\ \infty)$. Therefore,

$$[F\cdot(gh)]\langle\varphi\rangle = F\cdot g\langle\varphi_1\rangle = 0,$$

which yields

$$S_D[F\cdot(gh)] \subseteq (-\infty, f_1 + f_2]. \tag{26}$$

Conditions (25) and (26) prove that

$$S_D[F\cdot(gh)] \subseteq [0, f_1 + f_2].$$

The main result of this section is stated as

Theorem 4: The amplitude spectra of generalized functions

$$y_c(t)\ exp\ [-\hat{x}(t)]\ cos\ [2\pi f_c t + x(t)] \qquad x \in S_0$$

vanish on the interval $(-f_c\,, f_c)$.

Proof: Again, for $x \in S_0$ and $z = x + i\hat{x}$, $S_D[F\cdot z] \subseteq \bar{I}_0$ and, by Theorem 3, $S_D[F\cdot z] \subseteq \bar{I}_{+\infty}$; consequently, $S_D[F\cdot z] \subseteq [0, f_0]$. This condition combined with Lemmas 5 and 6 leads to

$$\begin{aligned} S_D[F\cdot e^{iz}] &= S_D\left[F\cdot \lim_N{}^{(D)} \sum_{n=0}^{N} \frac{(iz)^n}{n!}\right] = S_D\left[\lim_N{}^{(D)} \sum_{n=0}^{N} \frac{1}{n!} F\cdot[(iz)^n]\right] \\ &\subseteq \bigcup_n S_D[F\cdot[(iz)^n]] \subseteq \bar{I}_{+\infty}\,. \end{aligned} \tag{27}$$

On the other hand, for $\bar{z} = x - i\hat{x}$

$$S_D[F\cdot e^{-i\bar{z}}] \subseteq \bar{I}_{-\infty}\,. \tag{28}$$

Finally, since $F\cdot[e^{\pm 2\pi i f_c t} y(t)] = \tilde{y}(f \mp f_c)$ for $F\cdot y \equiv \tilde{y}$, then (27) and (28) give

$$S_D[F\cdot y_c] = S_D[F\cdot(e^{iz}e^{2\pi i f_c t} + e^{-i\bar{z}}e^{-2\pi i f_c t})] \subseteq [f_c\,,\ \infty) \cup (-\infty, -f_c],$$

or, equivalently, $F\cdot y_c = 0$ on $(-f_c\,, f_c)$.

IV. SIGNAL RECOVERY FOR SSBΘM REPRESENTATIONS

In this section we treat the problem of reconstructing signals $x \in S_0$ from bandlimited versions of the associated $SSB\Theta M$ functions y_c. Specifically, it is demonstrated that for a large subclass of S_0, knowledge of the amplitude spectrum of y_c over any open interval containing $[f_c, f_c + f_0]$ proves sufficient to recover x up to an additive constant. As in the previous section, several lemmas involving the exponential e^{iz} are developed first. To collect notation, we set

$$z = x + i\hat{x} \qquad x \in S_0$$

$$g = iz', \qquad y_a = 1 - e^{iz}, \qquad y_b = (1 - 2\pi it)^{-1} y_a$$

$$y_d = F^{-1}\cdot[(\lambda\tilde{y}_a)*k], \qquad y_n = F^{-1}\cdot[(\lambda F\cdot y_a')*\sigma_n]$$

$$g_n = F^{-1}\cdot[\tilde{g}*\sigma_n], \qquad \sigma_n = n\sigma(nf)\,(n = 1, 2, \cdots)$$

$$\tilde{y} \equiv F\cdot y \qquad \forall y \in D$$

$$k(f) = F\cdot(1 - 2\pi it)^{-1} = \begin{cases} e^{-f} & f > 0 \\ 0 & f \leqq 0 \end{cases} \tag{29}$$

where λ and σ are any frequency functions of C_d such that

$$\lambda(f) = \begin{cases} 1 & f \in \bar{I}_c, \quad I_c = (0, f_0 + \epsilon), \quad 0 < \epsilon < \infty \\ 0 & f \notin [-\epsilon, f_0 + 2\epsilon] \end{cases}$$

$$S[\sigma(f)] \subseteq [0, \epsilon], \qquad \int_0^{\epsilon} \sigma(f)\,df = 1.$$

In addition, let $BV(\bar{I})$ and UL denote respectively the space of scalar functions of bounded variation on a closed interval $\bar{I}$ and the space of scalar functions satisfying a first-order uniform Lipschitz condition on some closed neighborhood of the origin. Finally, define the following subclass of signal space S_0:

$$S_{00} = \{x \mid x \in S_0;\ [\tilde{y}_b - \tilde{y}_b(0^+)] \in UL \cap BV(\bar{I}_c);\ \tilde{y}_b(0^+) \neq 1\}.$$

Lemma 7: Elements y_a, y_b, y_d, y_n, g_n, k, λ, and σ_n are in D.

Proof: This result follows immediately from the corresponding definitions and the test employed in Lemma 4 [cf. (22) and (23)].

Lemma 8: $S_D[\tilde{y}_b] \subseteq \bar{I}_{+\infty}$.

Proof: Clearly, by (27)

$$S_D[\tilde{y}_a] = S_D[\delta] \cup S_D[F\cdot e^{iz}] \subseteq \bar{I}_{+\infty} . \tag{30}$$

Also, taking any $\varphi \in C_d$ for which $S[\varphi] \subseteq \bar{I}_{-\infty}$, one obtains

$$F^{-1}\cdot[\,(F^{-1}\cdot k)\;(F\cdot\varphi)\,] = 0 \qquad \forall f > 0,$$

or

$$S[F^{-1}\cdot[\,(F^{-1}\cdot k)\;(F\cdot\varphi)\,]] \subseteq \bar{I}_{-\infty} .$$

Hence, for all such φ

$$\begin{aligned} F\cdot y_b\langle\varphi\rangle &= y_b\langle F\cdot\varphi\rangle = y_a\langle(F^{-1}\cdot k)\;(F\cdot\varphi)\rangle \\ &= F\cdot y_a\langle F^{-1}\cdot[(F^{-1}\cdot k)\;(F\cdot\varphi)]\rangle = 0, \end{aligned}$$

a condition implying that $\tilde{y}_b = 0$ on $I_{-\infty}$.

Lemma 9: On I_c , $\tilde{y}_b = (\lambda\tilde{y}_a)*k = \tilde{y}_d$ *and* $(F\cdot y_a')*\sigma_n = (\lambda F\cdot y_a')*\sigma_n = \tilde{y}_n$.

Proof: Take any $\varphi \in C_d$ for which $S[\varphi] \subseteq \bar{I}_c$. With regard to the first relation

$$S[F^{-1}\cdot[(F^{-1}\cdot k)\;(F\cdot\varphi)]] \subseteq (-\infty, f_0 + \epsilon],$$

and by (30)

$$\begin{aligned} F\cdot y_b\langle\varphi\rangle &= y_b\langle F\cdot\varphi\rangle = y_a\langle(F^{-1}\cdot k)\;(F\cdot\varphi)\rangle \\ &= F\cdot y_a\langle F^{-1}\cdot[(F^{-1}\cdot k)\;(F\cdot\varphi)]\rangle = F\cdot y_a\langle\lambda F^{-1}\cdot[(F^{-1}\cdot k)\;(F\cdot\varphi)]\rangle \\ &= F\cdot[(F^{-1}\cdot k)F^{-1}\cdot(\lambda F\cdot y_a)]\langle\varphi\rangle = [(\lambda\tilde{y}_a)*k]\langle\varphi\rangle. \end{aligned}$$

As to the second relation

$$\int_{-\infty}^{\infty} \sigma_n(\bar{f})\varphi(f+\bar{f})d\bar{f} = \sigma_n\langle\varphi(f+\bar{f})\rangle,$$

$$S[(2\pi if)\sigma_n\langle\varphi(f+\bar{f})\rangle] \subseteq [-\epsilon, f_0 + \epsilon],$$

and by (30) and Theorem 1

$$\begin{aligned} [(F\cdot y_a')*\sigma_n]\langle\varphi\rangle &= F\cdot y_a\langle 2\pi if\sigma_n\langle\varphi(f+\bar{f})\rangle\rangle \\ &= F\cdot y_a\langle 2\pi if\lambda(f)\sigma_n\langle\varphi(f+\bar{f})\rangle\rangle = [(\lambda F\cdot y_a')*\sigma_n]\langle\varphi\rangle. \end{aligned}$$

Lemma 10: $\tilde{y}_d \in L_2(I_\infty)$ *and* $\tilde{y}_n \in C_d$.

Proof: On the basis of the Tonelli-Hobson theorem

$$F^{-1}\cdot(\lambda F\cdot y_a)\langle\varphi\rangle = y_a\langle F\cdot(\lambda F^{-1}\cdot\varphi)\rangle$$

$$= \int_{-\infty}^{\infty} y_a(\tau) \left\{ \int_{-\infty}^{\infty} e^{-2\pi i \tau f} \lambda(f) \left[\int_{-\infty}^{\infty} e^{2\pi i f t} \varphi(t) dt \right] df \right\} d\tau$$

$$= \int_{-\infty}^{\infty} y_a(\tau) \left\{ \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} e^{2\pi i f(t-\tau)} \lambda(f) df \right] \varphi(t) dt \right\} d\tau$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} y_a(\tau) (F^{-1} \cdot \lambda)_{t-\tau} d\tau \right] \varphi(t) dt \qquad \forall \varphi \in C_d.$$

Therefore, as indicated operationally,

$$F^{-1} \cdot (\lambda F \cdot y_a) = (F^{-1} \cdot \lambda) * y_a = \int_{-\infty}^{\infty} y_a(\tau) (F^{-1} \cdot \lambda)_{t-\tau} d\tau,$$

and

$$| F^{-1} \cdot (\lambda F \cdot y_a) | \leqq (\sup_{t \in I_\infty} | y_a(t) |) \int_{-\infty}^{\infty} | (F^{-1} \cdot \lambda)_\tau | d\tau < \infty,$$

which indicates that y_d, $\tilde{y}_d \in L_2(I_\infty)$. Lastly, since

$$(\lambda F \cdot y_a') \in B([-\epsilon, f_0 + 2\epsilon]),$$
$$F^{-1} \cdot \sigma_n \in C_d,$$

by Lemma 1

$$F^{-1} \cdot (\lambda F \cdot y_a') \in C_g,$$
$$y_n = (F^{-1} \cdot \sigma_n) [F^{-1} \cdot (\lambda F \cdot y_a')] \in C_d,$$
$$\tilde{y}_n \in C_d.$$

Lemma 11: $\lim_{n \to \infty}^{(D)} g_n = g$, $g_n \in C_d$, *and* $S_D[\tilde{g}_n] \subseteq \bar{I}_c$.

Proof: As

$$\lim_n \int_{-\infty}^{\infty} n\sigma(nf)\varphi(f) df = \lim_n \int_{-\infty}^{\infty} \sigma(f) \varphi\left(\frac{f}{n}\right) df = \varphi(0) \qquad \forall \varphi \in C_d,$$

then

$$\lim_n^{(D)} \sigma_n = \delta,$$

and

$$\lim_n g_n\langle\varphi\rangle = \lim_n (gF^{-1} \cdot \sigma_n)\langle\varphi\rangle = \lim_n \sigma_n\langle F^{-1} \cdot (g\varphi)\rangle$$

$$= \delta\langle F^{-1} \cdot (g\varphi)\rangle = gF^{-1} \cdot \delta\langle\varphi\rangle = g\langle\varphi\rangle \qquad \forall \varphi \in C_d.$$

Furthermore, with $\sigma_n \in C_d$, all three elements $F^{-1}\cdot\sigma_n$, g_n, and $\tilde{g}_n$ are also of C_d. Finally, from Lemmas 6 and 7 it follows that

$$S_D[\tilde{g}_n] = S_D[\tilde{g}] \cup S[\sigma_n] \subseteq \bar{I}_c .$$

Theorem 5: For $x \in S_{00}$, elements $\tilde{g}_n$ satisfy the functional equations

$$\tilde{g}_n(f) = \gamma\left[\int_0^f \tilde{g}_n(f-\bar{f})\, d\tilde{y}_i(\bar{f}) - \tilde{y}_n(f)\right] \qquad f \in \bar{I}_c; \quad n = 1, 2, \cdots, \tag{31}$$

where

$$\tilde{y}_i(f) = \tilde{y}_d(f) - \tilde{y}_b(0^+) + \int_0^f \tilde{y}_d(\bar{f})\, d\bar{f} \qquad f \in I_c$$

$$\tilde{y}_i(0) = \tilde{y}_i(f_0 + \epsilon) = 0$$

$$\gamma = [1 - \tilde{y}_b(0^+)]^{-1}.$$

Proof: According to definitions (29)

$$y_a' F^{-1}\cdot\sigma_n = (1 - 2\pi i t) g_n y_b - g_n \tag{32}$$

with $y_b \in L_2(I_\infty)$, $y_a' F^{-1}\cdot\sigma_n \in C_d$, and $(1 - 2\pi i t) g_n \in C_d$. Expression (32) and Parseval's relation combine to give

$$(F\cdot y_a')*\sigma_n = \tilde{y}_b*(\tilde{g}_n + \tilde{g}_n') - \tilde{g}_n \qquad f \in I_\infty ,$$

which by Lemmas 8 and 11 reduces to

$$(F\cdot y_a')*\sigma_n = \int_0^f \tilde{y}_b(\bar{f})\left[\tilde{g}_n(f-\bar{f}) - \frac{\partial}{\partial \bar{f}}\tilde{g}_n(f-\bar{f})\right]d\bar{f} - \tilde{g}_n(f).$$

However, if considering f on I_c, one need specify $(F\cdot y_a')*\sigma_n$ and $\tilde{y}_b$ on this interval only; consequently, Lemmas 9, 10, and 11 apply, yielding

$$\tilde{y}_n(f) = \int_0^f \tilde{y}_d(\bar{f})\left[\tilde{g}_n(f-\bar{f}) - \frac{\partial}{\partial \bar{f}}\tilde{g}_n(f-\bar{f})\right]d\bar{f} - \tilde{g}_n(f) \qquad f \in I_c .$$

Clearly, in this equation, $\tilde{y}_d(0)$ and (for the development below) $\tilde{y}_d(f_0 + \epsilon)$ can be set equal to zero without affecting the associated integrals; hence, on integrating by parts, we get (31). To be noted in this theorem is that with respect to signal information, $\tilde{y}_n$, $\tilde{y}_d$, and γ derive solely from the bandlimited signal spectrum $\lambda\tilde{y} = \lambda F\cdot e^{iz}$; i.e.,

$$\tilde{y}_d = (\lambda - \lambda\tilde{y})*k,$$

$$\tilde{y}_n = -(2\pi i f \lambda\tilde{y})*\sigma_n ,$$

$$\tilde{y}_b(0^+) = \tilde{y}_d(0^+).$$

It is necessary to consider next some general properties of the integral operators in (31), viz., the mappings

$$w = T_n \cdot v = \gamma \left[\int_0^f v(f - \bar{f})\, d\tilde{y}_i(\bar{f}) - \tilde{y}_n(f) \right] \qquad f \in \bar{I}_c \tag{33}$$
$$v \in C(\bar{I}_c); \qquad n = 1, 2, \cdots.$$

Relative to the domain of T_n, define the norm

$$\| v \|_c = \sup_{f \in \bar{I}_c} | v(f) | \qquad v \in C(\bar{I}_c)$$

and metric

$$\rho_c(v,w) = \| v - w \|_c \qquad v,w \in C(\bar{I}c). \tag{34}$$

Under this scheme the pair $[C(\bar{I}_c), \rho_c] \equiv R_c$ (more precisely, the pair consisting of $C(\bar{I}_c)$ and the metric topology in $C(\bar{I}_c)$) forms a metric space which is complete.[12] We then have

Theorem 6: Corresponding to any modulating signal $x \in S_{00}$, operators T_n constitute continuous mappings of the complete metric space R_c into itself.

Proof: To show that the range as well as the domain of T_n is in $C(\bar{I}_c)$, take any $v \in C(\bar{I}_c)$ and set $w = T_n \cdot v$ for an arbitrary n. With $x \in S_{00}$, $\tilde{y}_b \in BV(\bar{I}_c)$, and by Lemma 10, $\tilde{y}_n \in C_d \subset C(\bar{I}_c)$; moreover, since $\tilde{y}_d = \tilde{y}_b$ on I_c (cf. Lemma 9), and since

$$\tilde{y}_i = \tilde{y}_d - \tilde{y}_b(0^+) + \int_0^f \tilde{y}_d\, d\bar{f} \qquad f \in I_c$$

$$\tilde{y}_i(0) = \tilde{y}_i(f_0 + \epsilon) = 0,$$

function $\tilde{y}_i \in BV(\bar{I}_o)$. Therefore, by the Lebesgue convergence theorem

$$\lim_{f_2 \to f_1} [w(f_2) - w(f_1)] = \lim \left\{ \gamma \int_0^{f_1} [v(f_2 - \bar{f}) - v(f_1 - \bar{f})]\, d\tilde{y}_i(\bar{f}) \right.$$

$$\left. + \gamma \int_{f_1}^{f_2} v(f_2 - \bar{f})\, d\tilde{y}_i(\bar{f}) - \gamma[\tilde{y}_n(f_2) - \tilde{y}_n(f_1)] \right\} = 0 \qquad \forall f_1, f_2 \in \bar{I}_c.$$

That is, $w \in C(\bar{I}_c)(T_n : C(\bar{I}_c) \to C(\bar{I}_c))$. For establishing the continuity of T_n, consider any two functions $v_1, v_2 \in C(\bar{I}_c)$, and set $w_1 = T_n \cdot v_1$, $w_2 = T_n \cdot v_2$, and $v_0 = v_2 - v_1$ for an arbitrary n. It follows from (33) and (34) that

$$\rho_c(w_2, w_1) = \sup_{f \in \bar{I}_c} \left| \gamma \int_0^f v_0(f - \bar{f}) \, d\tilde{y}_i(\bar{f}) \right|$$

$$\leqq | \gamma | V_i(f_0 + \epsilon) \sup_{f \in \bar{I}_c} | v_0(f) |$$

$$= | \gamma | V_i(f_0 + \epsilon) \rho_c(v_2, v_1)$$

where $V_i(f)$ signifies the total variation of $\tilde{y}_i$ on the interval $[0,f]$. Consequently, $\rho_c(w_2, w_1) \to 0$ if $\rho_c(v_2, v_1) \to 0$; i.e., T_n is continuous.

A basic result relating to the reconstruction of $SSB\Theta M$ signals can now be stated as

Theorem 7: Corresponding to any modulating signal $x \in S_{00}$, each of the equations

$$\tilde{g}_n = T_n \cdot \tilde{g}_n \qquad \tilde{g}_n \in C(\bar{I}_c); \qquad n = 1, 2, \cdots,$$

has a unique solution given by

$$\tilde{g}_n = \lim_{m \to \infty} \tilde{g}_{n,m} \qquad n,m = 1, 2, \cdots$$

$$\tilde{g}_{n,m+1} \equiv T_n^{\ m} \cdot \tilde{g}_{n,1} \qquad \forall n,m$$

$$\tilde{g}_{n,1} \equiv 0 \qquad f \in \bar{I}_c \qquad \forall n$$

where convergence is uniform on $\bar{I}_c$. Furthermore,

$$\frac{dx}{dt} = Im\,[F^{-1} \cdot \lim_n{}^{(D)} \cdot \lim_m \tilde{g}_{n,m}] = Im\,[\lim_n \cdot \lim_m \cdot g_{n,m}]$$

$$\tilde{g}_n, \tilde{g}_{n,m} \equiv 0 \qquad f \notin \bar{I}_c \qquad \forall n, m$$

where $Im[\cdot]$ indicates the imaginary part of the quantity in brackets.

Proof: We employ here a standard fixed-point contraction-mapping theorem (cf. Ref. 17, p. 50): If $\rho(\cdot, \cdot)$ and T represent respectively a metric in a complete metric space $R = [C,\rho]$ and a continuous mapping of R into itself, and if for some k and any two elements $v,w \in C$

$$\rho(T^k \cdot v, T^k \cdot w) \leqq \alpha\rho(v,w) \qquad \alpha < 1,$$

then there exists a unique solution to the equation $T \cdot v_0 = v_0$. Also, for an arbitrary element $v \in C$ this solution is given by

$$v_0 = \lim_{m \to \infty} T^m \cdot v,$$

where convergence is taken relative to ρ. In view of Theorem 6 we need only demonstrate in the present proof that for some k and any two elements $v,w \in C(\bar{I}_c)$ each mapping T_n satisfies the contraction condition

$$\| T_n^k \cdot v - T_n^k \cdot w \|_c \leqq \alpha \| v - w \|_c \qquad \alpha < 1.$$

First, set

$$v - w = u$$

$$M = \sup_{f \in \bar{I}_c} | u(f) | = \| u \|_c .$$

With $\tilde{y}_i \in BV(\bar{I}_c)$ (cf. Theorem 6)

$$| T_n \cdot v - T_n \cdot w | = \left| \gamma \int_0^f u(f - \bar{f}) \, d\tilde{y}_i(\bar{f}) \right|$$

$$\leqq M | \gamma | \int_0^f dV_i(\bar{f}) = M | \gamma | V_i(f) \qquad f \in \bar{I}_c,$$

where $V_i(f)$ again denotes the total variation of $\tilde{y}_i$ on $[0,f]$. However, inasmuch as

$$[\tilde{y}_b - \tilde{y}_b(0^+)] \in UL \cap BV(\bar{I}_c),$$

$$\tilde{y}_i = \tilde{y}_b - \tilde{y}_b(0^+) + \int_0^f \tilde{y}_b \, d\bar{f} \qquad f \in I_c,$$

$$\tilde{y}_i(0) = \tilde{y}_i(f_0 + \epsilon) = 0,$$

then $\tilde{y}_i \in UL \cap BV(\bar{I}_c)$; hence, there exists a positive constant a_0 such that

$$V_i(f) \leqq a_0 f \qquad f \in \bar{I}_c .$$

From this last condition it follows that

$$| T_n \cdot v - T_n \cdot w | \leqq M | \gamma | a_0 f$$

$$| T_n^2 \cdot v - T_n^2 \cdot w | \leqq M | \gamma |^2 a_0 \int_0^f (f - \bar{f}) \, dV_i(\bar{f}) = M | \gamma |^2 a_0 \int_0^f V_i(\bar{f}) \, d\bar{f}$$

$$\leqq M | \gamma |^2 a_0^2 \int_0^f \bar{f} \, d\bar{f} = \frac{M | \gamma |^2 a_0^2 f^2}{2} \qquad f \in \bar{I}_c,$$

and, in general,

$$| T_n^k \cdot v - T_n^k \cdot w | \leqq \frac{M | \gamma |^k a_0^{k-1}}{(k - 1)!} \int_0^f (f - \bar{f})^{(k-1)} \, dV_i(\bar{f})$$

$$= - \frac{M | \gamma |^k a_0^{k-1}}{(k - 1)!} \int_0^f V_i(\bar{f}) \, d(f - \bar{f})^{(k-1)}$$

$$\leqq -\frac{M \mid \gamma \mid^k a_0^k}{(k-1)!} \int_0^f \bar{f} d(f-\bar{f})^{(k-1)}$$

$$= \frac{M \mid \gamma \mid^k a_0^k}{(k-1)!} \int_0^f \bar{f}^{(k-1)} \, d\bar{f} = \frac{M \mid \gamma \mid^k a_0^k f^k}{k!} \qquad f \in \bar{I}_c.$$

Therefore, for k sufficiently large

$$\frac{M \mid \gamma \mid^k a_0^k (f_0 + \epsilon)^k}{k!} = \alpha < 1,$$

and

$$\| T_n^k \cdot v - T_n^k \cdot w \|_c \leqq \alpha \| v - w \|_c \qquad \alpha < 1.$$

The contraction principle as outlined then yields the main statement of the theorem, the last result being an immediate consequence of Lemma 11.

Treated next are two important classes of modulating signals which prove to be contained in S_{00} : periodic functions of S_0 and integrable $(L_1(I_\infty))$ functions of S_0 having integrable Hilbert transforms. In the following development we represent the space of periodic functions by P and the intersection $H \cdot (L_1(I_\infty)) \cap L_1(I_\infty)$ by $\hat{L}_1(I_\infty)$.

Theorem 8: $S_0 \cap P \subset S_{00}$.

Proof: Elements $x \in S_0 \cap P$ must have the form

$$x = \sum_{n=-N}^{N} b_n e^{2\pi i n f_p t}$$

$$N f_p \leqq f_0 , \qquad b_n = \bar{b}_{-n}$$

where $\bar{b}$ signifies the conjugate of b. Consequently, in accordance with Definition 2

$$z = x + i\hat{x} = \sum_{n=0}^{N} b_n e^{2\pi i n f_p t}.$$

Putting $z_p = z - b_0$, we obtain

$$y_b = (1 - 2\pi i t)^{-1} \left[(1 - e^{i b_0}) - e^{i b_0} \sum_{n=1}^{\infty} (i z_p)^n \right],$$

or

$$\tilde{y}_b = (1 - e^{i b_0}) k - e^{i b_0} \left[\sum_{n=1}^{\infty} k * F \cdot (i z_p)^n \right],$$

which, since

$$S_D[F\cdot(iz_p)^n] \subseteq [nf_p\,,\ \infty)$$

$$S_D\left[F\cdot\sum_{n=1}^{\infty}(iz_p)^n\right] \subseteq [f_p\,,\ \infty),$$

gives

$$S_D[\tilde{y}_b - (1 - e^{ib_0})k] \subseteq [f_p\,,\ \infty),$$

$$\tilde{y}_b(0^+) = (1 - e^{ib_0}) \neq 1,$$

and

$$[\tilde{y}_b - \tilde{y}_b(0^+)] \in UL \cap BV(\bar{I}_c).$$

Theorem 9: $S_0 \cap \hat{L}_1(I_\infty) \subset S_{00}$, *and for all* x *in this intersection*

$$\frac{dx}{dt} = Im\,[\lim_{m\to\infty} h_m] \qquad m = 1, 2, \cdots,$$

where

$$h_{m+1} = (F^{-1}\cdot\lambda_0)*[(y_a*F^{-1}\cdot\lambda_0)h_m] - \frac{d}{dt}(y_a*F^{-1}\cdot\lambda_0) \qquad t \in I_\infty$$

$$h_1 \equiv 0 \qquad t \in I_\infty$$

$$\lambda_0(f) = \begin{cases} 1 & f \in \bar{I}c \\ 0 & f \notin \bar{I}_c\,. \end{cases}$$

Proof: Considering that $x \in S_0 \cap \hat{L}_1(I_\infty)$, z is a bounded element of $L_1(I_\infty)$; hence, by Darboux's formula

$$|\,y_a(t)\,| = |\,1 - e^{iz}\,| \leqq |\,z\,| \int_0^1 e^{\lambda|\hat{x}|}\,d\lambda \leqq |\,z(t)\,| \exp\,(\sup_{t\in I_\infty}|\,\hat{x}\,|), \quad (35)$$

and

$$|\,\tilde{y}_b(f_2) - \tilde{y}_b(f_1)\,| \leqq |\,f_2 - f_1\,| \exp\,(\sup_t |\,\hat{x}\,|) \int_{-\infty}^{\infty} \left|\, z(t)\left(\frac{2\pi it}{1 - 2\pi it}\right) \cdot \frac{\sin \pi(f_2 - f_1)t}{\pi(f_2 - f_1)t} \right| dt$$

$$\leqq |\,f_2 - f_1\,| \exp\,(\sup_t |\,\hat{x}\,|) \int_{-\infty}^{\infty} |\,z(t)\,|\,dt$$

$$\forall f_1, f_2 \in I_\infty,$$

the latter condition indicating that $\tilde{y}_b(0^+) = \tilde{y}_b(0^-) = 0$ (cf. Lemma 8) and that $\tilde{y}_b \in UL \cap BV(\bar{I}_c)$.* In order to prove the second part of the theorem, we note first that with $x \in S_0 \cap \hat{L}_1(I_\infty)$, $z \in L_2(I_\infty)$ and $\tilde{z} \in L_2 \cap C(I_\infty)$; moreover, as $S_D[\tilde{z}] \subseteq \bar{I}_c$, $\tilde{z} \in L_1(I_\infty)$ and $\tilde{g} = 2\pi i f(i\tilde{z}) \in L_1 \cap L_2 \cap C(I_\infty)$. Similarly, by (35), $y_a \in L_1 \cap L_2(I_\infty)$ and $\tilde{y}_a \in L_2 \cap C(I_\infty)$. As a result,

$$\begin{aligned} |\tilde{g}_n| &= |\tilde{g}*\sigma_n| \\ &= \left| \int_0^f \tilde{g}(f-\bar{f})\sigma(n\bar{f})n\, d\bar{f} \right| \\ &\leqq \sup_{f \in \bar{I}_c} |\tilde{g}(f)| \int_0^\epsilon |\sigma(\bar{f})|\, d\bar{f} \leqq \text{constant} \qquad \forall n, \end{aligned}$$

$$\lim_n{}^{(\mathrm{D})}\, \tilde{g}_n = \lim_n \tilde{g}_n = \tilde{g},$$

$$\lim_n{}^{(\mathrm{D})}\, \tilde{y}_n = \lim_n [(2\pi i f \tilde{y}_a \lambda)*\sigma_n] = (2\pi i f \tilde{y}_a \lambda),$$

and by the Lebesgue convergence theorem

$$\tilde{g} = \lim_n{}^{(\mathrm{D})}\, T_n \cdot \tilde{g}_n = \int_0^f \tilde{g}(f-\bar{f})\, d\tilde{y}_i(\bar{f}) - 2\pi i f \tilde{y}_a \lambda \equiv A \cdot \tilde{g} \qquad f \in \bar{I}_c .$$

This expression asserts that $\tilde{g}$ is a fixed point of the mapping $A: C(\bar{I}_c) \to C(\bar{I}_c)$. Precisely the same arguments as were used in Theorems 6 and 7 apply here to show that A is continuous with respect to norm $\|\cdot\|_c$, and that

$$\begin{aligned} \frac{dx}{dt} &= \mathrm{Im}\,[\lim_{m\to\infty} \cdot h_m] & & m = 1, 2, \cdots \\ \tilde{h}_{m+1} &= A^m \cdot \tilde{h}_1 & & \forall m \\ \tilde{h}_1 &\equiv 0 & & f \in \bar{I}_c \\ \tilde{h}_m &\equiv 0 \qquad f \notin \bar{I}_c & & \forall m \end{aligned}$$

where convergence is uniform on $\bar{I}_c$. On writing $\tilde{y}_i$ as

$$\tilde{y}_i = (\tilde{y}_a\lambda)*k + \int_0^f [(\tilde{y}_a\lambda)*k]\, d\bar{f}$$

* A similar calculation employing the Schwarz inequality shows that $\tilde{y}_b(0^+) = 0$ for $x \in L_2(I_\infty)$ also. Most square-integrable signals of practical interest satisfy the appropriate Lipschitz and bounded variation conditions, and are therefore contained in S_{00}.

$$= e^{-f} \int_0^f \tilde{y}_a \lambda e^{\bar{f}} \, d\bar{f} + \int_0^f e^{-\bar{f}} \left[\int_0^{\bar{f}} \tilde{y}_a \lambda e^{f'} \, df' \right] d\bar{f}$$

$$= \int_0^f \tilde{y}_a(\bar{f}) \lambda(\bar{f}) \, d\bar{f} \qquad f \in \bar{I}_c ,$$

we get

$$\tilde{h}_{m+1} = \int_0^f \tilde{h}_m(f - \bar{f}) \tilde{y}_a(\bar{f}) \lambda(\bar{f}) \, d\bar{f} - 2\pi i f \tilde{y}_a \lambda \qquad f \in \bar{I}_c$$

or, more compactly,

$$\tilde{h}_{m+1} = \lambda_0(f) \int_0^f \tilde{h}_m(f - \bar{f}) \tilde{y}_a(\bar{f}) \lambda_0(\bar{f}) \, d\bar{f} - 2\pi i f \tilde{y}_a \lambda_0 \qquad f \in I_\infty . \quad (36)$$

(Since $S_D[\tilde{g}] \subseteq [0, f_0]$, λ_0 could be defined to have the same support.) Taking the inverse Fourier transform of both sides of (36) yields the second part of the theorem.

V. SUMMARY

Definition 2 and Theorems 3 through 9, which constitute the principal results of the preceding sections, provide both a distribution-theoretic basis for the spectral representation of single-sideband angle-modulated carriers and a recurrence formulation for reconstructing most of the associated modulating signals of practical interest. It is important to emphasize again that the approach employed in this development applies also to other modulation schemes.

VI. ACKNOWLEDGMENTS

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APPENDIX

Index of Symbols

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