Applications of a Theorem of Dubrovskii to the Periodic Responses of Nonlinear Systems

By V. E. BENEŠ and I. W. SANDBERG

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Dubrovskii's theorem on completely continuous operators that are asymptotic to zero is applied to the study of the existence and uniqueness of periodic responses of nonlinear systems to periodic driving signals. Examples of nonexistence and nonuniqueness are given, a relationship between nonuniqueness and subharmonics is noted, and some general existence theorems are proven, giving estimates on the magnitudes of the harmonics.

I. INTRODUCTION

In 1939 V. M. Dubrovskii[1] proved the following result:

Theorem 1: If A is a completely continuous operator which maps a Banach space X into itself, with the property that

$$\lim_{\| x \| \to \infty} \frac{\| Ax \|}{\| x \|} = 0, \qquad x \, \varepsilon \, X,$$

then for each scalar λ and $y \, \varepsilon \, X$, the equation $x = y + \lambda Ax$ has at least one solution $x \, \varepsilon \, X$.

Dubrovskii's theorem was stated in the long review article of M. A. Krasnoselskii[2] on problems of nonlinear analysis, but except for a recent application,[3] it seems to have gone largely unnoticed. It is the purpose of this paper to indicate some applications of the basic idea in the theorem to integral equations (and systems thereof) that arise in the study of nonlinear electrical networks and automatic control systems.

The applications to be made all center around the existence and uniqueness of periodic responses of nonlinear systems to periodic driving signals. These properties of the equations governing nonlinear systems are frequently taken for granted. The fact is, though, that these are by no means universal properties of such equations, as simple examples (to be given) will show. Often, the nonexistence of periodic responses is related to instability of the nonlinear system, while their lack of uniqueness is closely connected with the possibility of responses with subharmonic components. Thus it is important, in control and circuit theory, to be able to distinguish nonlinear equations that have unique periodic solutions for periodic inputs from those that possess several such solutions. With the aid of the idea underlying Dubrovskii's theorem, we examine this problem in the present paper for systems described by the nonlinear integral equation

$$x(t) = y(t) + \int_{-\infty}^{\infty} k(t-u)\psi(x(u),u)\, du, \tag{1}$$

(and by a vector analog thereof,) where $y(\cdot)$ is an input, $k(\cdot)$ is an integrable (L_1) impulse response of a linear system, and $\psi(\cdot,\cdot)$ represents a periodically time-varying nonlinear element. Periodic solutions of (1) have already been considered in previous work of one of the authors;[4] almost periodic solutions of (1) have been studied in previous joint work[5] of the authors. In both these papers a basically different assumption about the growth of the element $\psi(\cdot,\cdot)$ (from that to be made here) was used.

II. SUMMARY

A discussion of the abstract Banach space setting for Dubrovskii's theorem appears in Section III. It includes a quick proof of the theorem from Schauder's fixed-point principle. There follows in Section IV an account of mathematical preliminaries, assumptions, definitions, etc., requisite for our remarks about (1). These remarks begin, in Section V, with a simple example showing that (1) may have no periodic solution and continue in Section VI with an existence theorem, for periodic solutions of (1), based on the principle of Dubrovskii's theorem. In Section VII we apply this result in discussing an example of nonuniqueness due to existence of subharmonic solutions. In Section VIII it is shown how the bound on the norm of the solutions obtained in Section VII can be improved. In Section IX, finally, a vector analog of the existence theorem of Section VII is stated and its proof sketched.

III. BACKGROUND DISCUSSION

We recall[6] that an operator A taking one Banach space into another is termed *completely continuous* if and only if it is continuous and carries every bounded set into a compact one. Dubrovskii's theorem for such operators is a straightforward consequence of Schauder's fixed-point principle:[7] Let S be a bounded, closed, convex set of a Banach space X. Let A be a continuous transformation of S into a compact subset of itself. Then there exists at least one point $x \varepsilon S$ such that $x = Ax$.

An operator A satisfying Dubrovskii's condition

$$\lim_{\| x \| \to \infty} \frac{\| Ax \|}{\| x \|} = 0$$

is said to be *asymptotically close to zero;* explicitly, the condition is that for every $\epsilon > 0$ there is an r such that $\| x \| \geqq r$ implies $\| Ax \| < \epsilon \| x \|$. To prove Dubrovskii's theorem we seek a closed ball, of radius R to be determined, that is mapped into itself by the (completely continuous) operator G defined by

$$Gx = y + \lambda Ax$$

with λ and $y \varepsilon X$ fixed. Let ϵ be a number such that $0 < | \lambda | \epsilon < 1$, and pick (by Dubrovskii's condition) an $r > 0$ such that $\| x \| \geqq r$ implies $\| Ax \| < \epsilon \| x \|$. If now s is a positive number such that

$$s \geqq \frac{\| y \|}{1 - \epsilon | \lambda |}$$

then for $r \leqq \| x \| \leqq s$

$$\begin{aligned} \| Gx \| &\leqq \| y \| + | \lambda | \cdot \| Ax \| \\ &\leqq \| y \| + | \lambda | \epsilon \| x \| \\ &\leqq s. \end{aligned}$$

Since A is completely continuous, the set

$$\{Ax: \| x \| \leqq r\}$$

is compact. Thus the continuous function $\| Ax \|$ defined on $\{ \| x \| \leqq r\}$ is bounded. If R is chosen as

$$R = \max \left\{ \frac{\| y \|}{1 - | \lambda | \epsilon}, \| y \| + | \lambda | \sup_{\| x \| \leqq r} \| Ax \| \right\}$$

then $\| x \| \leqq R$ implies $\| Gx \| \leqq R$. The closed ball of radius R is convex, and the existence of a fixed point of G in the ball follows from Schauder's fixed-point principle. To establish the result for a particular value of λ it is not necessary that A be asymptotically close to zero; clearly, it suffices that there be ϵ such that $0 < \epsilon < | \lambda |^{-1}$ and r such that $\| x \| > r$ implies $\| Ax \| < \epsilon \| x \|$.

IV. PRELIMINARIES

We shall be concerned throughout with the case in which the functions $x(\cdot)$ and $y(\cdot)$ of interest are periodic and square-integrable over a period. By $L_2(-T,T)$ we denote the Banach space of all functions $x(\cdot)$ of period $2T$ that are real-valued, measurable on $[-T,T]$, and for which the norm

$$\| x \| = \left(\frac{1}{2T} \int_{-T}^{T} | x(t) |^2 \, dt \right)^{\frac{1}{2}}$$

is finite. According to standard results in the theory of Fourier series, such a function is represented in the mean by its Fourier series

$$x(t) = \underset{N\to\infty}{\text{l.i.m.}} \sum_{m=-N}^{N} x_m e^{\pi i m t/T}$$

with Fourier coefficients

$$x_m = \frac{1}{2T} \int_{-T}^{T} x(t) e^{-\pi i m t/T} \, dt, \qquad -\infty < m < \infty .$$

The norm of $x(\cdot)$ and its Fourier coefficients are related by the Parseval identity

$$\| x \|^2 = \sum_{n=-\infty}^{\infty} | x_n |^2.$$

We shall need the following two facts from the theory of Fourier series: (1) If $z(\cdot)$, $w(\cdot) \, \varepsilon \, L_2(-T,T)$, with respective Fourier coefficients $\{z_n\}$, $\{w_n\}$, then

$$\frac{1}{2T} \int_{-T}^{T} z(t - u) w(u) \, du = \sum_{n} z_n w_n e^{i\pi n t/T},$$

the series on the right converging absolutely and uniformly; (2) the Fourier coefficients of $z(\cdot + \epsilon)$ are $\{e^{\pi i n \epsilon/T} z_n\}$.

The notation

$$\| z \|_1 = \frac{1}{2T} \int_{-T}^{T} | z(t) | \, dt$$

is used occasionally.

For a periodic function $z(\cdot) \, \varepsilon \, L_2(-T,T)$ we define the functional

$$\mu(z,\epsilon) = \frac{1}{4T} \int_{-T}^{T} | z(t + \epsilon) - z(t) | \, dt,$$

proportional to an "integral modulus of continuity," and we remark that one of the usual arguments for the Riemann-Lebesgue lemma gives, for $n \neq 0$, the inequality

$$\begin{aligned} | z_n | &= \left| \frac{1}{2T} \int_{-T}^{T} z(t) e^{-\pi i n t/T} \, dt \right| \\ &= \left| \frac{1}{4T} \int_{-T}^{T} z(t)[1 - e^{\pi i}] e^{-\pi i n t/T} \, dt \right| \\ &\leqq \frac{1}{4T} \int_{-T}^{T} | z(t) - z(t + T/n) | \, dt = \mu(z, T/n) . \end{aligned}$$

We shall make two assumptions about the nonlinear element $\psi(\cdot,\cdot)$, one about its growth and one about its continuity:

(a) there is a function $\lambda(\cdot)$ nondecreasing on $[0,\infty)$ such that for all v, t

$$| \psi(v,t) | \leqq \lambda(| v |), \tag{2}$$

(b) the function $\psi(\cdot,\cdot)$ is continuous in the first variable uniformly in both variables. Then its *modulus* of continuity $\omega(\cdot)$, defined by

$$\omega(\delta) = \sup_{u,v,t} | \psi(u,t) - \psi(v,t) | \quad \text{for} \quad | u - v | \leqq \delta, \tag{3}$$

is a continuous monotone function, and approaches zero with $\delta \to 0$. When $\psi(v,\cdot)$, considered as a function of t, has a modulus of continuity $\omega_0(\cdot)$, so that

$$| \psi(v,t + \epsilon) - \psi(v,t) | \leqq \omega_0(\epsilon)$$

for all v and t, we set

$$q_n = \begin{cases} 0 & n = 0 \\ \omega_0(T/n) & n \neq 0. \end{cases}$$

Jensen's inequality for a concave function $\varphi(\cdot)$ reads

$$\varphi\left(\frac{\int_a^b f(x)p(x) \, dx}{\int_a^b p(x) \, dx} \right) \geqq \frac{\int_a^b \varphi(f(x))p(x) \, dx}{\int_a^b p(x) \, dx} \tag{4}$$

where $\varphi(\cdot)$ is concave in an interval containing the range of $f(\cdot)$ over $[a,b]$, $p(x) \geqq 0$, $p \not\equiv 0$, and all the integrals in question exist.

We now return to $k(\cdot)$ in (1). Since $k(\cdot)$ belongs to L_1, it has a bounded Fourier transform

$$K(\omega) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-i\omega t} k(t)\, dt.$$

The convolution operator K on $L_2(-T,T)$ defined by

$$Kx(t) = \int_{-\infty}^{\infty} x(t-u) k(u)\, du$$

is described in terms of its effect on Fourier coefficients by the identity

$$(Kx)_m = (2\pi)^{\frac{1}{2}} K\left(\frac{\pi m}{T}\right) x_m,$$

and takes $L_2(-T,T)$ into itself continuously.

V. NONEXISTENCE OF SOLUTIONS

It is easy to see that in some very simple cases (1) has no periodic solution. An example is furnished by

Remark 1: If $\psi(v,t) = v$ for all v and t, and if, for some integer n, both the nth Fourier coefficient y_n of $y(\cdot)$ does not vanish and

$$h_n = (2\pi)^{\frac{1}{2}} K\left(\frac{n\pi}{T}\right) = 1,$$

then (1) has no periodic solution $x(\cdot)$ belonging to $L_2(-T,T)$. For if there were such a solution, the left side of (1) would have nth Fourier coefficient x_n, while the right-side would have $y_n + x_n \neq x_n$.

VI. EXISTENCE OF SOLUTIONS

Theorem 2: If $\lambda(\cdot)$ and $\omega(\cdot)$ are concave, $y(\cdot) \in L_2(-T,T)$,

$$\kappa^2 = 2\pi \sum_{m=-\infty}^{\infty} \left| K\left(\frac{\pi m}{T}\right)\right|^2 < \infty$$

and if the scalar equation

$$r = \| y \| + \kappa\lambda(r) \tag{5}$$

has a positive solution r, then there exists a solution $x(\cdot)$ of (1), with period $2T$, and such that

$$\| x \| \leqq r,$$

$$| x_m | \leqq | y_m | + (2\pi)^{\frac{1}{2}} K\left(\frac{\pi m}{T}\right) \lambda(r)$$

where x_m, y_m are the respective mth *Fourier coefficients of x, y.*

Proof: In the complex sequence space l_2 of Fourier coefficients, isometric to $L_2(-T,T)$, consider the set $\mathcal{S}$ of sequences $x = \{x_n, -\infty < n < \infty\}$ such that, with the overbar denoting the complex conjugate,

$$x_{-m} = \bar{x}_m$$

$$| x_m | \leqq | y_m | + \left| K\left(\frac{\pi m}{T}\right) \right| \lambda(r).$$

By Minkowski's inequality, $x \, \varepsilon \, \mathcal{S}$ implies

$$\| x \| = \left(\sum_m | x_m |^2\right)^{\frac{1}{2}} \leqq \| y \| + \kappa\lambda(r) = r.$$

The set $\mathcal{S}$ is compact, being an analog of the Hilbert cube or parallelotope. It is easily verified that $\mathcal{S}$ is convex.

Now let $x(\cdot) \, \varepsilon \, L_2(-T,T)$, and consider the magnitudes of the Fourier coefficients of the function $w(\cdot)$ defined by

$$w(t) = \psi(x(t),t) = \psi(x(t + 2T), t + 2T).$$

We find

$$\begin{aligned} | w_n | = \left| \frac{1}{2T} \int_{-T}^{T} \psi(x(t),t) e^{-\pi i n t/T} \, dt \right| &\leqq \frac{1}{2T} \int_{-T}^{T} | \psi(x(t),t) | \, dt \\ &\leqq \frac{1}{2T} \int_{-T}^{T} \lambda(| x(u) |) \, dt \qquad (6) \\ &\leqq \lambda\left(\frac{1}{2T} \int_{-T}^{T} | x(u) | \, du\right) \\ &\leqq \lambda(\| x \|), \end{aligned}$$

where the second inequality follows from the fact that $\lambda(\cdot)$ bounds the growth of $\psi(\cdot,t)$, the third inequality follows from the concavity of $\lambda(\cdot)$ by the Jensen inequality (4), and the fourth inequality follows from Schwarz's and the monotone nature of $\lambda(\cdot)$. Hence if $\| x \| \leqq r$, then $| y_m + (Kw)_m | \leqq | y_m | + (2\pi)^{\frac{1}{2}} K(m\pi/T) \lambda(r)$, and it follows that the operator A defined on $L_2(-T,T)$ by

$$Ax(t) = y(t) + \int_{-\infty}^{\infty} k(t - u)\psi(x(u),u)du, \qquad | t | \leqq T$$

maps the ball $\| x \| \leqq r$ into the compact, convex, isometric image of $\mathcal{S}$, that is, into a compact, convex subset of itself. Continuity of A *on the image* follows from that of K and from the inequality

$$\frac{1}{2T}\int_{-T}^{T} |\psi(x_1(u),u) - \psi(x_2(u),u)|\, du \leqq \omega(\| x_1 - x_2 \|),$$

provable by the same method as (6). Existence of a fixed point of A in the isometric image of $\mathcal{S}$ follows from Schauder's theorem.

We remark that if $\lambda(u) = o(u)$ as $u \to \infty$, then a solution r to the scalar equation (5) always exists. This occurs, for example, if

$$\lambda(u) = \beta u^{\alpha} \qquad \beta > 0, \qquad 0 \leqq \alpha < 1.$$

VII. NONUNIQUENESS AND SUBHARMONICS

It is known that a solution of (1) may have Fourier components, called "subharmonics," of period greater than $2T$. Our purpose is to remark that if this occurs, then (1) does not have a unique solution, and in fact, the greater the period of the subharmonic, the more distinct solutions exist. We start with a simple example: Let $T = \pi/2$, set

$$\left.\begin{aligned} \psi(u,t) &= \operatorname{sgn} u \cdot | u |^{\frac{1}{2}} \\ y(t) &= \tfrac{9}{2} - \tfrac{1}{2}\cos 2t \end{aligned}\right\}, \tag{7}$$

let

$$x(t) = \tfrac{9}{2} + 4 \sin t - \tfrac{1}{2} \cos 2t,$$

and for $K(\cdot)$ take any Fourier transform of an integrable function with $K(0) = 0$ and $K(1) = 4(2\pi)^{-\frac{1}{2}}$. For example, the fourth-order filter

$$K(\omega) = \frac{(2\pi)^{-\frac{1}{2}} 16 (i\omega)^2}{(1 + i\omega)^4}$$

will do. Actually, since we need to prescribe only the two parameters $K(0)$ and $K(1)$, the second-order filter

$$K(\omega) = \frac{(2\pi)^{-\frac{1}{2}} i\omega}{(i\omega)^2 + \frac{1}{4} i\omega + 1} \tag{8}$$

would do as well.

That $x(\cdot)$ as defined is a periodic solution of (1) of period 2π can be verified from the identity

$$2 + \sin t = (\tfrac{9}{2} + 4 \sin t - \tfrac{1}{2} \cos 2t)^{\frac{1}{2}}.$$

This example, in which the solution $x(\cdot)$ contains the subharmonic component $4 \sin t$, is adapted from Hughes,[8] and has been used earlier[4] by the authors merely to illustrate the real possibility of subharmonics in relatively simple systems.

Now since the input $y(\cdot)$ has period π, while the response $x(\cdot)$ has period 2π, it can be seen that by shifting $x(\cdot)$ by $\pm\pi$, that is, by changing the sign of (all) the odd components of $x(\cdot)$, another solution of (1) for this $\psi(\cdot)$ and $y(\cdot)$ is obtained, because

$$x(t \pm \pi) = y(t) + \int_{-\infty}^{\infty} \operatorname{sgn} x(t \pm \pi - u) \, | x(t \pm \pi - u)|^{\frac{1}{2}} k(u) du.$$

Thus, there are at least two solutions of (1) for this example; the two we have identified so far differ only in phase. As an application of Theorem 2 we show that there is at least one more solution, one that has period π. The following lemma establishes a Hölder condition for the nonlinearity of the example:

Lemma 1: If

$$\psi(v) = sgn\, v \, | v |^{\frac{1}{2}}$$

then for all v and ϵ

$$| \psi(v + \epsilon) - \psi(v) | \leqq 2^{\frac{1}{2}} | \epsilon |^{\frac{1}{2}}.$$

Proof: First suppose that $\operatorname{sgn}(v + \epsilon) \neq \operatorname{sgn} v$. Then $| \epsilon | = | v + \epsilon | + | v |$, and concavity gives, by Jensen's theorem,

$$\begin{aligned} | \psi(v + \epsilon) - \psi(v)| &= | v + \epsilon |^{\frac{1}{2}} + | v |^{\frac{1}{2}} \\ &\leqq 2 \left(\frac{| v + \epsilon | + | v |}{2} \right)^{\frac{1}{2}} = 2^{\frac{1}{2}} | \epsilon |^{\frac{1}{2}}. \end{aligned}$$

If $\operatorname{sgn}(v + \epsilon) = \operatorname{sgn} v$, there is no loss of generality in supposing that $v + \epsilon > v \geqq 0$, because $\psi(\cdot)$ is odd. Then using concavity again

$$\psi(v) \geqq \frac{v}{v + \epsilon} \psi(v + \epsilon)$$

$$\psi(\epsilon) \geqq \frac{\epsilon}{v + \epsilon} \psi(v + \epsilon).$$

Hence in this case

$$0 \leqq \psi(v + \epsilon) - \psi(v) \leqq \psi(\epsilon) = | \epsilon |^{\frac{1}{2}}.$$

A direct application of Theorem 2 shows that (1) for the example (7), (8) has a solution of period π. The scalar equation

$$\| y \| + \kappa r^{\frac{1}{2}} = r$$

is appropriate, and has a positive root r.

The example just discussed illustrates the following general principle regarding subharmonics:

Theorem 3: If (1) *has a solution* $x(\cdot)$ *with (minimal) period* $2nT$, $n > 1$, *then each of the functions*

$$x(t + 2kT) \qquad k = 1, \cdots, n$$

is a (distinct) solution of (1).

Proof: Since $y(\cdot)$, and the time-dependence of $\psi(\cdot\,,\cdot)$ have period $2T$, we have

$$\begin{aligned} x(t + 2kT) &= y(t) + \int_{-\infty}^{\infty} \psi(x(t + 2kT - u), t + 2kT - u)k(u)du \\ &= y(t) + \int_{-\infty}^{\infty} \psi(x(t + 2kT - u), t - u)k(u)du. \end{aligned}$$

VIII. CLOSER BOUNDS ON FOURIER COEFFICIENTS

By a more penetrating analysis it is possible to strengthen the bounds on the norm and on the Fourier coefficients given by Theorem 2. For example, the inequality (6) merely establishes a uniform bound $\lambda(r)$ for all Fourier coefficients of functions

$$w(t) = \psi(x(t),t)$$

for $\| x \| \leqq r$. However, since the argument for (6) shows that $w(\cdot)$ is absolutely integrable over a period, its Fourier coefficients actually go to zero at infinity, and it should be possible substantially to improve the estimate (6). This can be done with the help of the quantities $\{q_m, -\infty < m < \infty\}$, and the functional μ, defined in Section IV.

Throughout this section, it is assumed that $\psi(v,\cdot)$ has the modulus of continuity $\omega_0(\cdot)$ as a function of t, and that

$$\kappa^2 = \sum_{m=-\infty}^{\infty} \left| K\left(\frac{m\pi}{T}\right)\right|^2 < \infty. \tag{9}$$

It follows from (9) that there is a function $h(\cdot) \in L_2(-T,T)$ such that for any $x(\cdot) \in L_2(-T,T)$

$$\frac{1}{2T}\int_{-T}^{T} h(t - u)x(u)du = \int_{-\infty}^{\infty} k(t - u)x(u)du;$$

the Fourier coefficients of $h(\cdot)$ are

$$h_m = (2\pi)^{\frac{1}{2}} K\left(\frac{m\pi}{T}\right), \qquad -\infty < m < \infty.$$

For each positive number s, and $m \neq 0$, we define

$$a_m(s) = \mu(y,T/m) + \lambda(s)\mu(h,T/m).$$

Since for $x(\cdot) \varepsilon L_2(-T,T)$, $\mu(x,\epsilon) \to 0$ as $\epsilon \to 0$, the numbers $a_m(s)$ are bounded in $m \neq 0$ for each fixed s. By the Riesz-Fischer theorem there is for each $s > 0$ a function $u_s(\cdot) \varepsilon L_2(-T,T)$ whose Fourier coefficients are

$$u_m(s) = \begin{cases} \frac{1}{2} \mid h_m \mid \{\omega(a_m(s)) + q_m\}, & m \neq 0 \\ \mid h_0 \mid \lambda(s) & m = 0. \end{cases}$$

with (cf. Section IV)

$$q_n = \begin{cases} 0 & n = 0 \\ \omega_0(T/n) & n \neq 0. \end{cases}$$

Theorem 4: Let $\lambda(\cdot)$, $\omega(\cdot)$ be concave, and let r be a positive number satisfying the inequality

$$\| y \| + \| u_r \| \leqq r.$$

Then there exists a solution $x(\cdot) \varepsilon L_2(-T,T)$ of (1), such that

$$\| x \| \leqq r,$$

$$\mu(x,T/m) \leqq a_m(r), \qquad m \neq 0$$

$$| x_m | \leqq | y_m | + | u_m(r) |, \qquad \text{all } m.$$

Proof: Let the operator A be defined on the ball $\{ \| x \| \leqq r\}$ in $L_2(-T,T)$ by

$$Ax(t) = y(t) + \int_{-\infty}^{\infty} k(t-u)\psi(x(u),u)du$$

$$= y(t) + \frac{1}{2T}\int_{-T}^{T} h(t-u)\psi(x(u),u)du.$$

The argument of Theorem 2 shows that A maps $\{ \| x \| \leqq r\}$ continuously into $L_2(-T,T)$. Further, by Fubini's theorem and the concavity of $\lambda(\cdot)$,

$$\frac{1}{8T^2}\int_{-T}^{T} dt \int_{-T}^{T} | h(t+\epsilon-u) - h(t-u)| \cdot |\psi(x(u),u)| \, du$$

$$= \mu(h,\epsilon) \cdot \frac{1}{2T}\int_{-T}^{T} | \psi(x(u),u)| \, du$$

$$\leqq \mu(h,\epsilon)\lambda(\| x \|),$$

and we find that $\| x \| \leqq r$ implies $\mu(Ax,T/m) \leqq a_m(r)$ for $m \neq 0$. Moreover

$$| \psi(x(u+\epsilon),u+\epsilon) - \psi(x(u),u) | \leqq \omega(| x(u+\epsilon) - x(u) |) + \omega_0(\epsilon),$$

so that the concavity of $\omega(\cdot)$ implies

$$2\mu(\psi(x),\epsilon) \leqq \omega(\mu(x,\epsilon)) + \omega_0(\epsilon).$$

It follows that $\| x \| \leqq r$ implies

$$| (Ax)_m | \leqq | y_m | + \tfrac{1}{2} | h_m | \{\omega(a_m(r)) + q_m\}, \qquad m \neq 0$$

and also

$$| (Ax)_0 | \leqq | y_0 | + | h_0 | \lambda(r).$$

Let $\mathcal{S}$ be the compact set of l_2 sequences

$$x = \{x_n , -\infty < n < \infty\}$$

such that

$$x_{-m} = \bar{x}_m ,$$
$$| x_m | \leqq | y_m | + | u_m(r) |.$$

It can be seen that A maps the ball $\{ \| x \| \leqq r\}$ into the isometric image in $L_2(-T,T)$ of $\mathcal{S}$. This image is compact and convex, and Theorem 4 follows from Schauder's fixed-point principle, as did Theorem 2.

Theorem 5: Let $\lambda(\cdot),\omega(\cdot)$ *be concave, let* $y(\cdot)\ \varepsilon\ L_2(-T,T)$, *and let there exist a positive number* r *and a real bounded sequence* $b = \{b_m , m \neq 0\}$ *satisfying the inequalities*

$$\| y \|_1 + \| h \|_1\lambda(r) \leqq r$$

$$\mu(y,T/m) + \sum_{n\neq 0} \left| \sin \frac{n\pi}{2m} \right| | h_n | \{\omega(b_n) + q_n\} \leqq b_m , \qquad m \neq 0.$$

Then there exists a solution $x(\cdot)\ \varepsilon\ L_2(-T,T)$ of (1) such that

$$| x_m | \leqq | y_m | + \tfrac{1}{2} | h_m | \{\omega(b_n) + q_n\}, \qquad m \neq 0.$$
$$\| x \|_1 \leqq r$$

Proof: Let R be the compact, convex subset of $L_2(-T,T)$ consisting of functions $z(\cdot)$ such that

$$\| z \|_1 \leqq r$$
$$2\mu(z,T/m) \leqq b_m , \qquad m \neq 0.$$

Let $x(\cdot) \in R$, and let $\{\psi_n, -\infty < n < \infty\}$ be the Fourier coefficients of the function $w(\cdot)$ defined by

$$w(t) = \psi(x(t),t), \qquad \text{all } t.$$

Then the concavity of $\lambda(\cdot)$ implies that

$$|\psi_0| \leqq \lambda\left(\frac{1}{2T}\int_{-T}^{T} |x(t)|\,dt\right) \leqq \lambda(r)$$

and that of $\omega(\cdot)$ implies that for $m \neq 0$

$$\begin{aligned} |\psi_m| &\leqq \frac{1}{2T}\int_{-T}^{T} \omega(|x(t+T/m) - x(t)|)dt + q_m \\ &\leqq \omega(2\mu(x,T/m)) + q_m . \end{aligned}$$

Now

$$\begin{aligned} &Ax(t+\epsilon) - Ax(t) \\ &\quad = y(t+\epsilon) - y(t) + \int_{-T}^{T} \{h(t+\epsilon-u) - h(t-u)\}\psi(x(u),u)du \end{aligned}$$

and the second term on the right is

$$\sum_{n\neq 0} h_n(e^{\pi i n\epsilon/T} - 1)\psi_n e^{\pi i n t},$$

the series converging absolutely and uniformly to a quantity of modulus at most

$$2\sum_{n\neq 0} |h_n| \left|\sin\frac{\pi n\epsilon}{2T}\right| |\psi_n| .$$

Hence, with $\epsilon = T/m$, $m \neq 0$,

$$2\mu(Ax,T/m) \leqq b_m .$$

At the same time

$$\begin{aligned} \frac{1}{2T}\int_{-T}^{T} |Ax(t)|\,dt &\leqq \frac{1}{2T}\int_{-T}^{T} |y(t)|\,dt + \frac{1}{4T^2}\int_{-T}^{T} |h(t-u)|\,|\psi(x(u),u)|du\,dt \\ &\leqq \|y\|_1 + \|h\|_1\,\lambda(r) \\ &\leqq r. \end{aligned}$$

Thus $Ax(\cdot)$ belongs to R. The result follows by Schauder's theorem, as before.

Remark: Let $\omega(\cdot)$ be concave, with $\omega(u) = o(u)$ as $u \to \infty$. Let $\{z_n, n \neq 0\}$ belong to l_2, and let $\{h_n, n \neq 0\}$ belong to $l_1 \cap l_2$. Then there exists a *minimal* bounded sequence $\{b_n, n \neq 0\}$ satisfying

$$| z_n | + \sum_{m \neq 0} \left| \sin \frac{\pi m}{2n} \right| | h_m | \omega(b_m) \leqq b_n, \qquad n \neq 0. \tag{10}$$

The sequence $\{b_n\}$ is minimal in the sense that its components are less than or equal to the corresponding components of any other sequence satisfying (10).

Let B be the set of sequences v satisfying (10). To prove B is non-empty, let

$$u_n = \begin{cases} 0 & n = 0 \\ \sum_{m \neq 0} \left| \sin \frac{\pi m}{2r} \right| | h_m |, & n \neq 0, \end{cases}$$

and let r satisfy $\| z \| + \| u \| \omega(r) \leqq r$. Define $w = \{w_n, n \neq 0\}$ by

$$w_n = | z_n | + u_n \omega(r) \leqq r.$$

Then

$$| z_n | + \sum_{m \neq 0} \left| \sin \frac{\pi m}{2n} \right| | h_m | \omega(w_m) \leqq w_n,$$

so that $w \,\varepsilon\, B$ and is bounded. Now set $b_n = \inf_{v \varepsilon B} v_n$. For any $v \,\varepsilon\, B$

$$| z_n | + \sum_{m \neq 0} \left| \sin \frac{\pi m}{2n} \right| | h_m | \omega(b_m) \leqq | z_n | + \sum_{m \neq 0} \left| \sin \frac{\pi m}{2n} \right| | h_m | \omega(v_m)$$
$$\leqq v_n .$$

Thus $b \,\varepsilon\, B$ and is minimal.

IX. THE VECTOR EQUATION

In this final section, we consider a vector form of the integral equation (1). Let $k(\cdot)$ be an $N \times N$ *matrix* of real functions of L_1, and for each t, let $\psi(\cdot, t)$ be a real N-*vector* valued function of a real N-vector. Let $y(\cdot)$ be a real N-vector valued function of time t. With these re-interpretations of the notations in mind, we can leave (1) unchanged.

With M a complex matrix, we let $M', \bar{M}$, and M^* denote the transpose, the complex-conjugate, and the complex-conjugate-transpose, respectively, of M. The positive square-root of the largest eigenvalue of M^*M is denoted by $\Lambda\{M\}$.

If v is a real or complex N-vector, its norm is defined as the "Euclidean" norm

$$\| v \| = \left(\sum_{i=1}^{N} | v_i |^2 \right)^{\frac{1}{2}} = (v^* v)^{\frac{1}{2}}.$$

It is well-known that

$$\Lambda^2\{M\} = \sup_{\| v \| = 1} v^* M^* M v \tag{11}$$

and hence that $\| Mv \| \leqq \Lambda\{M\} \, \| v \|$ for complex N-vectors v.

As previously, $L_2(-T,T)$ is the space of real-valued, measurable, functions $x(\cdot)$ of the real variable t which satisfy

(i) $x(t + 2T) = x(t)$,

(ii) $\dfrac{1}{2T} \displaystyle\int_{-T}^{T} | x(t) |^2 \, dt < \infty$.

We take as our basic space the Nth power of $L_2(-T,T)$, i.e.,

$$L_2^N(-T,T),$$

and think of it as composed of column N-vector valued functions of time. A norm for $L_2^N(-T,T)$ can be defined by the formula

$$\| x \|^2 = \frac{1}{2T} \int_{-T}^{T} x' x \, dt$$

$$= \frac{1}{2T} \int_{-T}^{T} \sum_{i=1}^{N} | x_i(t) |^2 dt$$

where $x = (x_1, \cdots, x_N)' \, \varepsilon \, L_2^N(-T,T)$. This norm makes $L_2^N(-T,T)$ a Banach space. Further, an element $x(\cdot)$ of $L_2^N(-T,T)$ has the Fourier representation

$$x(t) = \underset{n \to \infty}{\text{l.i.m.}} \sum_{m=-n}^{n} x_m e^{\pi i m t / T}$$

where the N-vector x_m of mth Fourier coefficients is given by

$$x_m = \frac{1}{2T} \int_{-T}^{T} x(t) e^{-\pi i m t / T} \, dt,$$

and the Parseval identity

$$\sum_{m=-\infty}^{\infty} x_m^* x_m = \sum_{m=-\infty}^{\infty} \| x_m \|^2 = \| x \|^2$$

for $x \, \varepsilon \, L_2^N(-T,T)$ holds.

The matrix convolution operator K is defined on $L_2^N(-T,T)$ by

$$Kx(t) = \int_{-\infty}^{\infty} k(t - u)x(u)du$$

and the operator ψ by

$$\psi x(t) = \psi(x(t),t), \qquad \text{all } t.$$

Equation (1) assumes the concise form

$$x = y + K\psi x.$$

The matrix K_m, $m = 0, \pm 1, \cdots$ is defined by the condition

$$K_m = (k_{ij}^m) = \int_{-\infty}^{\infty} e^{-\pi imt/T}\, k(t)dt.$$

It is assumed that $\sum_m \Lambda^2\{K_m\} < \infty$. This condition is met, e.g., if

$$\sum_m | k_{ij}^m |^2 < \infty.$$

for $1 \leqq i,j \leqq N$. The matrix convolution operator K takes a function $x(\cdot) \varepsilon L_2^N(-T,T)$ with (vector) Fourier coefficients x_m into the function $z(\cdot)$ whose coefficients are

$$z_m = K_m x_m, \qquad m = 0, \pm 1, \cdots,$$

and the Riesz-Fischer theorem guarantees that $z(\cdot) \varepsilon L_2^N(-T,T)$. Further, by formula (11) we have

$$\| z_m \| \leqq \Lambda\{K_m\} \| x_m \|.$$

An analog of Hilbert's cube in $L_2^N(-T,T)$ is described by

Lemma 2: Let $\{c_n, -\infty < n < \infty\}$ *be nonnegative real numbers with*

$$\sum_n c_n^2 < \infty.$$

Then the set

$$\{x \,\varepsilon\, L_2^N(-T,T) : \| x_n \| \leqq c_n, \quad \text{all } n\}$$

is compact.

This result is a consequence of a known theorem. (See p. 136 of Ref. 5.)

Analogs of the growth condition (2) and of the uniform continuity condition (3) will be used. These are (*i*) $\| \psi(u,t \| \leqq \lambda(\| u \|)$, for all t and all real N-vectors u where $\lambda(\cdot)$ is a monotone function.

(*ii*) $\psi(\cdot,\cdot)$ is continuous in the first variable uniformly in both variables; its *modulus* of continuity $\omega(\cdot)$, defined by

$$\omega(\delta) = \sup_{u,v,t} \| \psi(u,t) - \psi(v,t) \| \quad \text{for} \quad \| u - v \| \leqq \delta,$$

is a continuous monotone function that approaches zero with δ.

Theorem 6: If $\lambda(\cdot)$ and $\omega(\cdot)$ are concave, y belongs to $L_2^N(-T,T)$,

$$\kappa^2 = \sum_m \Lambda^2\{K_m\} < \infty,$$

and if the scalar equation

$$r = \| y \| + \kappa N \lambda(r)$$

has a positive solution r, then there exists an element $x \varepsilon L_2^N(-T,T)$ satisfying

$$x = y + K\psi x$$

$$\| x \| \leqq r$$

$$\| x_m \| \leqq \| y_m \| + \Lambda\{K_m\} N \lambda(r),$$

with x_m, y_m the respective mth *(vector) Fourier coefficients of x,y.*

The proof of Theorem 6 is an exact analog of that of Theorem 2, using the compact set

$$\{x \varepsilon L_2^N(-T,T) : \| x_m \| \leqq \| y_m \| + \Lambda\{K_m\}\lambda(r), \quad \text{all } m\}$$

and with $w(t) = \psi(x(t),t)$, the inequality, (analogous to (6),)

$$\| w_m \| \leqq N\lambda(\| x \|),$$

provable by observing first that for all t

$$\sum_{j=1}^{N} | w_j(t) | \leqq N^{\frac{1}{2}} \| w(t) \|$$

$$\leqq N^{\frac{1}{2}} \lambda(\| x(t) \|),$$

so that trivially

$$| w_j(t) | \leqq N^{\frac{1}{2}} \lambda(\| x(t) \|)$$

and by concavity of $\lambda(\cdot)$,

$$\frac{1}{2T} \int_{-T}^{T} | w_j(t) | \, dt \leqq N^{\frac{1}{2}} \lambda(\| x \|).$$

Squaring both sides and summing over $j = 1, \cdots, N$ we obtain

$$\| w_m \|^2 \leqq N^2\lambda^2(\| x \|).$$

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