Equivalence Relations among Spherical Mirror Optical Resonators

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The frequencies, field patterns, and losses of the resonant modes of spherical mirror optical resonators can be obtained to good accuracy as the solutions of the integral equations of Fresnel diffraction theory. By a simple transformation of the variables and parameters of the integral equations, we have found certain families of resonators which have the same diffraction loss at each mirror, and whose field patterns are scaled versions of each other. In the case of the infinite strip resonator, this reduces from five to three the number of parameters necessary to specify the losses and mode patterns.

I. INTRODUCTION

The resonant frequencies, field patterns, and losses of the modes of spherical mirror optical resonators can be obtained to good accuracy as the solutions of the integral equations of Fresnel diffraction theory.[1] The equations are particularly applicable when the separation between the two mirrors forming the resonator is large compared with the dimensions of the mirrors. Unfortunately, the equations are usually not soluble analytically, and require numerical (machine) computation. There are many parameters involved: the dimensions and curvatures of the mirrors and their separation. By a simple transformation of the variables and parameters of the integral equations, we have found certain families of resonators which have the same diffraction loss at each mirror, and whose field patterns are scaled versions of each other. In the case of the infinite strip resonator, this reduces from five to three the number of parameters necessary to specify the losses and mode patterns.

II. THE TRANSFORMATION

The equations which determine the field patterns, resonant frequencies, and losses of an infinite strip resonator (see Fig. 1) are[1]

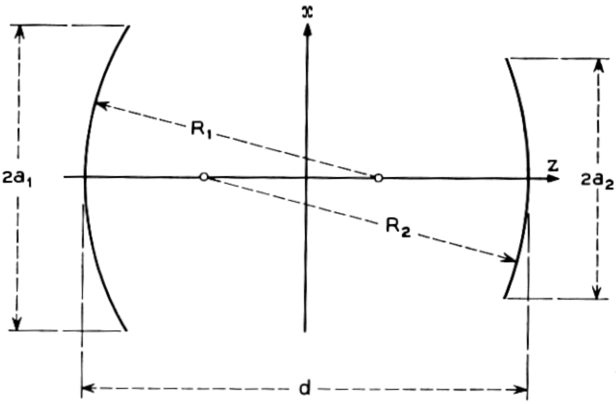

Fig. 1 — Spherical mirror resonator with mirrors of curvature radii R_1 and R_2, and of widths $2a_1$ and $2a_2$. Mirror spacing is d.

$$\gamma_1 u_1(x_1) = \frac{\sqrt{j}}{\sqrt{\lambda d}} \int_{-a_2}^{a_2} K(x_1, x_2)\, u_2(x_2) dx_2 \tag{1a}$$

$$\gamma_2 u_2(x_2) = \frac{\sqrt{j}}{\sqrt{\lambda d}} \int_{-a_1}^{a_1} K(x_2, x_1)\, u_1(x_1) dx_1 \tag{1b}$$

with the complex symmetric kernel

$$K(x_1, x_2) = K(x_2, x_1) = \exp\left[-j(\pi/\lambda d)(g_1 x_1^2 + g_2 x_2^2 - 2x_1 x_2)\right]. \tag{1c}$$

Here

$$g_i = 1 - (d/R_i), \qquad i = 1, 2,$$

the mirror separation is d, R_1 and R_2 are the radii of mirror curvature, $2a_1$, $2a_2$ are the corresponding mirrors widths, and λ is the wavelength in the resonator medium. Also, $u_1(x_1)$ is the (generally complex) normalized field distribution on the left-hand mirror of Fig. 1, while $u_2(x_2)$ is the normalized field distribution on the right-hand mirror. If the two functions are normalized so that

$$\int_{-a_1}^{a_1} |\, u_1(x_1)\, |^2\, dx_1 = \int_{-a_2}^{a_2} |\, u_2(x_2)\, |^2\, dx_2 \tag{2}$$

then one notes* that the power reflection coefficient of the left mirror

* According to (1b) a light beam with a field distribution $u_1(x_1)$ across the left mirror causes a field $\gamma_2 u_2(x_2)$ across the right mirror. Therefore, the power reflected from this latter (perfectly reflecting) mirror is proportional to

is $|\gamma_1|^2$ and the reflection coefficient of the right mirror is $|\gamma_2|^2$. Therefore the loss at the left mirror is $1 - |\gamma_1|^2$, the loss at the right mirror is $1 - |\gamma_2|^2$, and the round-trip loss is $1 - |\gamma_1\gamma_2|^2$. The condition for resonance is that $\gamma_1\gamma_2 \exp[-j2\pi(d/\lambda)]$ be real and positive.

We presume on the weight of much experimental and theoretical[2,3] evidence that a sequence of solutions to (1) does exist. Suppose now that we have found a mode of some resonator; i.e., we have found a solution for $u_1(x_1)$ and $u_2(x_2)$ which satisfies (1) for one set of values of the five resonator parameters a_1, a_2, g_1, g_2 and d, and have found the corresponding eigenvalues γ_1 and γ_2. Our present concern is to find a family of resonators, each of which will have a similar mode; that is, a mode with the *same* values of γ_1 and γ_2 and with similar (scaled) eigenfunctions. For this purpose we rewrite (1) in terms of dimensionless variables and eigenfunctions by substituting

$$x_i = a_i\xi_i, \qquad i = 1, 2 \tag{3}$$

and

$$v_i(\xi_i) = u_i(x_i)\cdot\sqrt{a_i}, \qquad i = 1, 2. \tag{4}$$

By this transformation we obtain a generalized set of integral equations for the modes of the resonator

$$\gamma_1 v_1(\xi_1) = \sqrt{jN}\int_{-1}^{+1} d\xi_2\, v_2(\xi_2)\, K(\xi_1, \xi_2) \tag{5a}$$

$$\gamma_2 v_2(\xi_2) = \sqrt{jN}\int_{-1}^{+1} d\xi_1\, v_1(\xi_1)\, K(\xi_1, \xi_2) \tag{5b}$$

with the kernel

$$K(\xi_1, \xi_2) = \exp[-j\pi N(-2\xi_1\xi_2 + G_1\xi_1^2 + G_2\xi_2^2)]. \tag{5c}$$

In (5) only three independent resonator parameters occur

$$N \equiv a_1a_2/\lambda d, \tag{6a}$$

$$G_1 \equiv g_1(a_1/a_2), \tag{6b}$$

$$|\gamma_2|^2 \int_{-a_2}^{a_2} |u_2(x_2)|^2\, dx_2 .$$

The power of the beam as it left the left mirror was, of course, proportional to

$$\int_{-a_1}^{a_1} |u_1(x_1)|^2\, dx_1 .$$

$$G_2 \equiv g_2(a_2/a_1). \tag{6c}$$

N is the Fresnel number of the resonator, while G_1 and G_2 are generalized g factors which describe the mirror curvatures.* A geometrical interpretation of the G's is shown in Fig. 2.

Note that the above transformation maintains the normalization of the eigenfunctions

$$\int_{-1}^{+1} d\xi_1 \mid v_1(\xi_1) \mid^2 = \int_{-1}^{+1} d\xi_2 \mid v_2(\xi_2) \mid^2 \tag{7}$$

and therefore the physical meaning of the eigenvalues γ_1 and γ_2 .

III. DISCUSSION

The integral equations of any spherical mirror resonator can be transformed into the form of (5a, b, c), which describe completely the

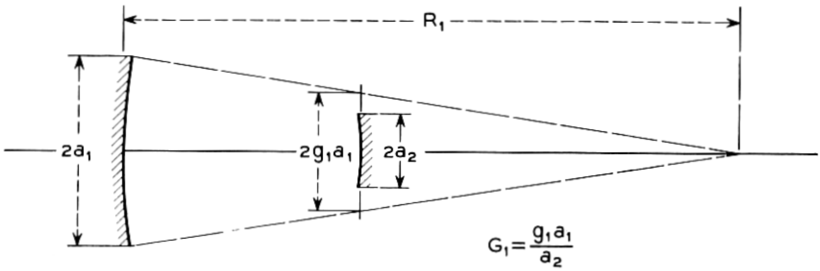

Fig. 2 — Geometrical interpretation of G_1 .

mode patterns, diffraction losses and resonant frequencies. It is clear that two resonators have the same scaled eigenfunctions, the same diffraction losses at each mirror and corresponding resonant frequencies whenever they are described by the same characteristic parameters N, G_1 , and G_2 . Two resonators are therefore equivalent if

$$a_1a_2/\lambda d = \bar{a}_1\bar{a}_2/\lambda\bar{d} = N \tag{8a}$$

$$g_1(a_1/a_2) = \bar{g}_1(\bar{a}_1/\bar{a}_2) = G_1 \tag{8b}$$

$$g_2(a_2/a_1) = \bar{g}_2(\bar{a}_2/\bar{a}_1) = G_2 \tag{8c}$$

where the overbar indicates the dimensions of a resonator equivalent to the original resonator.

* Note added in proof: in recently published perturbation analyses of optical resonators, Gloge[4] and Streifer and Gamo[5] have arrived at the same three resonator parameters.

Quantities which can be expressed in terms of N, G_1, and G_2 also remain invariant when we change from one resonator to an equivalent one. Some of these are listed in Table I. They are used in later computations.

The set of characteristic parameters N, G_1, and G_2 provides some insight into the behavior of resonators. The quantity

$$G^2 \equiv G_1 G_2 = g_1 g_2 \tag{9}$$

may be called the "stability number." One finds from its value whether the resonator is intrinsically of the "stable" or "unstable" type.[1,6] To be stable the resonator must satisfy

$$0 < G^2 < 1. \tag{10}$$

Table I — Some Invariants of Equivalent Resonators

Resonator Parameters (See Section III and Fig. 1)		Field Parameters (See Section X and Fig. 3)	
(1.1)	$N \equiv a_1 a_2/\lambda d$	(2.1)	$a_1 x/\lambda z$
(1.2)	$G_1 \equiv g_1(a_1/a_2)$	(2.2)	$g_{1z}(a_1/x)$
(1.3)	$G_2 \equiv g_2(a_2/a_1)$	(2.3)	$g_z(x/a_1)$
(1.4)	$G^2 \equiv g_1 g_2 = G_1 G_2$	(2.4)	$g_z g_{1z}$
(1.5)	$a_1^2 g_1/a_2^2 g_2 = G_1/G_2$	(2.5)	$a_1^2 g_{1z}/x^2 g_z$
(1.6)*	$g_1(a_1^2/\lambda d) = G_1 N$	(2.6)	$g_{1z}(a_1^2/\lambda z)$
(1.7)*	$g_2(\lambda d/a_1^2) = G_2 N$	(2.7)	$g_z(\lambda z/a_1^2)$
		(2.8)	$g_z(x^2/\lambda z)$

* Quantities like 1.6–1.7 but with subscripts 1 and 2 interchanged are also invariants.

The quantity N is the well-known "Fresnel number." For $N \gg 1$ the diffraction loss of stable resonators is typically very small indeed, and the increase of loss in crossing the boundary from a stable to an unstable type is abrupt. As N decreases toward unity, the loss of the stable resonators increases, and the boundary becomes less sharp until, as $N \ll 1$, all resonators have high loss.

Finally, at least for stable resonators with not too small Fresnel numbers, we can see that the mirror with the larger G has the smaller diffraction loss. From Ref. 6, or from Section VIII of this paper, we know that the radii* w_i of the fundamental mode "spots" on the mirrors are related by

$$w_1^2/w_2^2 = g_2/g_1\,. \tag{11}$$

* We use the word "radius" here and later to mean half the width of the mode pattern, as defined in Section VIII.

We therefore have

$$(w_1^2/a_1^2)/(w_2^2/a_2^2) = g_2a_2^2/g_1a_1^2 = G_2/G_1 . \tag{12}$$

The ratio w_i/a_i between spot radius and mirror half-width can be taken as a measure for the diffraction loss at a mirror. According to (12) the mirror with the smaller G_i has the larger ratio w_i/a_i, and thus the larger loss. In the special case $G_1 = G_2$, one sees that (5a) and (5b) become identical, and so the diffraction losses must be equal.

IV. SPECIAL ADDITIONAL EQUIVALENCES

There are two previously known[1,6] special equivalences which exist in addition to the new ones we have been discussing. These are:

(a) reversal of sign of both g_1 and g_2

(b) interchange of both g_1 and g_2, and a_1 and a_2; i.e., interchange of the mirrors.

The first of these special equivalences changes the sign of both G_1 and G_2 and does not alter N. This equivalence results because the allowed field patterns split up into those of odd and even symmetry in the x's.[1] The equivalent field patterns are complex conjugates of the old ones, but the losses are unchanged.

The second special equivalence corresponds to an interchange of the two mirrors. It leaves G^2 and N unchanged, but interchanges G_1 and G_2. It also obviously interchanges the mode patterns and the losses of the two mirrors. Combined with the equivalence relations which we have discussed before, this interchanging of the two mirrors means that two resonators are also equivalent if

$$N = \bar{N} \tag{13a}$$

$$G_1 = \bar{G}_2 \tag{13b}$$

and

$$G_2 = \bar{G}_1 . \tag{13c}$$

From these relations one deduces some rather curious equivalent resonator pairs if one postulates that the mirror curvature should be left unchanged ($g_1 = \bar{g}_1$ and $g_2 = \bar{g}_2$) and only the apertures a_1 and a_2 varied to form an equivalent resonator. With (13) one finds that

$$\bar{a}_1 = a_2(g_1/g_2)^{\frac{1}{2}} \tag{14a}$$

$$\bar{a}_2 = a_1(g_2/g_1)^{\frac{1}{2}} \tag{14b}$$

is necessary for equivalence. Note that the equivalent resonator was found simply by changing the mirror apertures. The mode pattern that appears on the left mirror of this new resonator is a scaled version of the pattern that appeared on the right mirror of the original resonator, and the pattern that was on the left is switched to the right mirror of the resonator.

Similarly, one obtains a pair of resonators equivalent in the above sense when the mirror apertures are kept constant and the curvatures are changed in accordance with (13).

V. THE CONFOCAL RESONATOR

The resonator commonly known as "the" confocal resonator is actually a very special confocal* resonator for which $R_1 = R_2 = d$, and hence $g_1 = g_2 = 0$, and $G_1 = G_2 = 0$. All of the equivalence transformations we have mentioned transform one confocal resonator into another. Our relations bring out the known fact that the losses and field patterns (apart from scale factors) of the confocal resonator depend only on the Fresnel number N and not at all on the ratio of the mirror apertures.[6]

VI. RESONATORS WITH EITHER G_1 OR G_2 EQUAL TO ZERO

When, in a system with mirrors of unequal curvature, the mirror spacing is equal to the radius of curvature of one of the mirrors, then one of the g's is zero and we have $G_1 = 0$, or $G_2 = 0$. Let $g_2 = G_2 = 0$. As a transformation to an equivalent resonator leaves G_2 invariant, we have for the equivalent resonator $\bar{g}_2 = 0$. In the stability diagram,[1,6] which shows the stable and unstable resonator regions versus g_1 and g_2, our transformation yields equivalent resonators that are represented by points on a straight line (in the general case, one has a branch of a hyperbola $g_1 g_2 = \text{const}$).

The parameters of equivalent resonators with $G_2 = 0$ are related by

$$G_1 = \bar{G}_1 = g_1(a_1/a_2) = \bar{g}_1(\bar{a}_1/\bar{a}_2). \tag{15}$$

This relation allows one to find for each resonator with $g_2 = 0$ and *unequal* apertures an equivalent resonator with $\bar{g}_2 = 0$ and *equal* apertures, which is discussed in Ref. 1. Resonators of the former type have

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* Any resonator whose mirrors have coincident foci may be termed confocal, whether or not the mirrors have equal curvature. As has been noted,[6] only "the" confocal resonator is a low-loss resonator.

been of interest for the selection of transverse modes in optical maser oscillators.[7] The mode selection properties of equivalent resonators are, of course, the same. For a resonator formed by a spherical mirror and a small plane mirror at its center of curvature[7] ($g_2 = 0$) one finds equivalent resonators of equal mirror apertures which are the closer to the confocal resonator the smaller the flat mirror [compare (15)]. As it appears that the confocal resonator has the best mode selection properties of all spherical mirror resonators, the above behavior would imply that reducing the size of the flat mirror will improve the mode selectivity of the above system. T. Li[7] has indeed found this to be so on the basis of computer calculations.

VII. RESONATORS WITH RECTANGULAR OR CIRCULAR MIRRORS

The integral equations which determine the modes of resonators with rectangular mirrors decompose into two sets of equations identical to (1), each set involving a single one of the two transverse Cartesian coordinates.[1,6] Hence all of the above applies immediately to such resonators, including resonators with astigmatic mirrors, provided the principal directions of the astigmatism are parallel to the edges of the mirrors.

Equivalent families of resonators with circular mirrors can also easily be found by a similar method, starting from the appropriate integral equations which are indicated in the Appendix. The resulting parameters are of the same form as (6), but with the a_i now redefined as the radii of the mirrors.

VIII. DETERMINATION OF SPOT RADII

If the apertures of the mirrors are sufficiently large, i.e., if $N \gg 1$, and if G^2 is not too close to 0 or 1, then the field patterns of the modes approach closely to Hermite Gaussian functions and lose their dependence on the apertures. Then one can define a "spot size", or spot radius,[6,8] where the Gaussian part of the function has dropped to e^{-1} of its maximum. In the transformations among equivalent resonators, the mode patterns scale in proportion to the apertures; hence two other invariants of equivalent resonators are obtained by replacing a_1 and a_2 in (6b) and (6c) with the spot radii w_1 and w_2. Now any quantity which is an invariant of equivalent resonators must be expressible as a function of the basic parameters N, G_1 and G_2. But since the values of N and G_1/G_2, which depend on the apertures, do not influence the spot radii, these two invariants of the equivalence transformations can be functionally dependent only on $G^2 = G_1G_2$. Hence we obtain the relations

$$w_1w_2/\lambda d = f(G^2) \tag{16a}$$

$$(w_1/w_2)^2(g_1/g_2) = 1. \tag{16b}$$

Equation (16b) follows, since we know that for mirrors of equal curvature (and hence with $g_1 = g_2$), the spot radii are also equal. The function $f(G^2)$ on the right side of (16a) may be evaluated by comparison with a known result[8] for mirrors of equal curvature

$$(g_1 = g_2 = g; \qquad w_1 = w_2 = w).$$

Equation (27) of Ref. 8 can be conveniently expressed in our present notation as

$$w^2/\lambda d = (1/\pi)(1 - g^2)^{-\frac{1}{2}} \tag{17}$$

which, on comparison with (16a), identifies $f(G^2)$ as

$$f(G^2) = (1/\pi)(1 - G^2)^{-\frac{1}{2}}. \tag{18}$$

Equations (16a) and (16b) can be rewritten with the help of (18) as

$$w_1/w_2 = (g_2/g_1)^{\frac{1}{2}} \tag{19a}$$

$$w_1w_2 = (\lambda d/\pi)(1 - g_1g_2)^{-\frac{1}{2}}. \tag{19b}$$

These last equations are identical with (39) and (40) of Ref. 6 and together determine the two spot radii. Their derivation here is included because of its relative simplicity, and as an example of the use of the invariants.

IX. FACTORS OF THE GENERAL TRANSFORMATION

Given the parameters (dimensions and curvatures) of one resonator, specification of $\bar{a}_1$ and $\bar{g}_1$ for an equivalent resonator completely determines all parameters of the equivalent resonator, apart from the special equivalences discussed in Section IV. The general transformation from the original to the equivalent resonator can be factored into a succession (product) of two simpler transformations, in the first of which a_1 is changed but g_1 is not, followed by a second for which g_1 is changed but a_1 is not.

The first of these simpler transformations effects a rather simple squeezing of all resonator dimensions, all transverse dimensions (apertures) being multiplied by the same factor ϵ, say, while all longitudinal dimensions (radii of curvature, mirror separation) are multiplied by ϵ^2. To see this we note that R_1 and d must change proportionally to leave g_1 unchanged. R_2 must change in proportion with these because of the invariance of g_1g_2, i.e., of G^2. Finally, a_1 and a_2 must change proportionally to leave G_1 invariant, and they must change as $d^{\frac{1}{2}}$ to leave N invariant.

The second simpler transformation leaves the aperture a_1 unchanged. Suppose it changes the radius of curvature R_1 in accordance with the relation

$$1/\bar{R}_1 = (1/R_1) + (1/f). \tag{20}$$

In practice a thin lens of focal length f inserted directly in front of the mirror can produce such a transformation. By using the invariants NG_1, N/G_2, and N [listed as (1.6), (1.7), and (1.1) of Table I] in succession, one can derive the following relations between the parameters of the transformed and original resonators

$$1/\bar{d} = (1/d) + (1/f) \tag{21a}$$

$$1/(\bar{d} - \bar{R}_2) = [1/(d - R_2)] + (1/f) \tag{21b}$$

$$\bar{a}_2/\bar{d} = a_2/d. \tag{21c}$$

Equations (21a), (21b) and (21c) show respectively that the position, *center* of curvature, and aperture of the original second mirror are changed to those of the new one by imaging them through the lens. In this imaging process, objects on the side of the lens toward the second mirror are taken as virtual objects, while objects on the other side of the lens are taken as real objects.

X. TRANSFORMATION OF THE FIELD INSIDE AND OUTSIDE THE RESONATOR

The mode patterns on the mirrors of two equivalent resonators are scaled versions of each other, and one expects also a correspondence of the fields of a mode inside and outside the equivalent systems. This correspondence is studied in this section.

With the assumptions of the diffraction theory of optical resonators the fields inside or outside the resonator structure can be expressed in terms of the field pattern on one of the mirrors via Fresnel's formula. For fields independent of y (this restriction can be removed easily; compare Appendix) we have for the field traveling to the right, say,

$$u(x,z) = \frac{\sqrt{j}}{\sqrt{\lambda z}} \int_{-a_1}^{a_1} dx_1\, u_1(x_1, 0) \cdot \exp\left[-j \frac{\pi}{\lambda z} (g_{1z} x_1^2 + g_z x^2 - 2 x x_1) \right] \tag{22}$$

where $u_1(x_1, 0)$ is the given field pattern on the left mirror and $u(x, z)$ is the field on a spherical references surface that intersects the optics axis a distance z away from the mirror (see Fig. 3). The quantities

$$g_{1z} = 1 - (z/R_1) \tag{23a}$$

$$g_z = 1 - (z/R) \tag{23b}$$

are again used to describe the curvatures of the mirror (curvature radius R_1), and that of the reference surface (curvature radius R). The mirror width is $2a_1$.

The transformation to an equivalent resonator changes the aperture and curvature of the mirror under consideration, and scales the field pattern on it accordingly, i.e., if

$$a_1 \rightarrow \bar{a}_1 \tag{24a}$$

$$g_1 \rightarrow \bar{g}_1 \tag{24b}$$

then

$$\sqrt{a_1}\, u_1(x_1, 0) = \sqrt{\bar{a}_1}\, \bar{u}_1(\bar{x}_1, 0) \tag{24c}$$

where

$$x_1/a_1 = \bar{x}_1/\bar{a}_1 . \tag{24d}$$

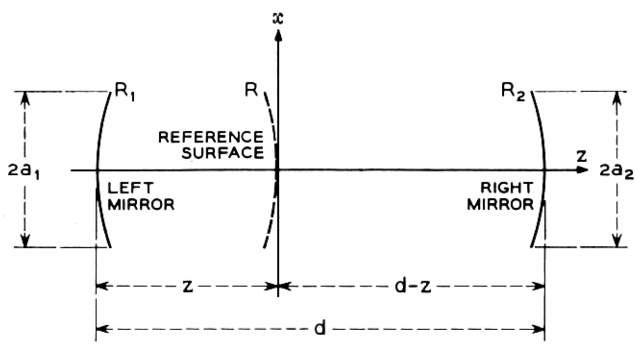

Fig. 3 — Reference surface for description of the fields inside the resonator.

We seek a new surface, described by $\bar{z}$, $\bar{g}_z$ (or $\bar{R}$), on which a scaled "image" $\bar{u}(\bar{x}, \bar{z})$ of $u(x, z)$ will be found. To do this we find, just as in Section II, a set of invariants necessary so that (22) retains its form in the transformed parameters. Essentially these invariants are the three terms in the exponential of (22), with the added fact that since x_1 transforms like a_1 we replace x_1 in those terms by a_1. By manipulation we obtain the set of invariants listed as (2.1) through (2.8) in Table I. The terms (2.6), (2.8), and (2.1) come directly from the three terms in the exponential of (22); the others may be derived from them. Finally, the transformed function is given by

$$(z/a_1)^{\frac{1}{2}}u(x, z) = (\bar{z}/\bar{a}_1)^{\frac{1}{2}}\bar{u}(\bar{x}, \bar{z}). \tag{25}$$

From this set of invariants one can find the new position ($\bar{z}$), curvature ($\bar{g}_z$) and transverse scale factor ($\bar{x}/x$) of the scaled function. First, consider the position. The invariant (2.6) may be expanded, using (23a), as

$$\frac{a_1^2}{\lambda}\left(\frac{1}{z} - \frac{1}{R_1}\right) = \frac{a_1^2}{\lambda d}\left(\frac{d}{z} - 1 + g_1\right). \tag{26}$$

But now the term $(a_1^2 g_1/\lambda d)$ is itself an invariant (1.6, Table I) of the resonator transformation, and hence the remaining part of (26), i.e.,

$$\frac{a_1^2}{d\lambda}\left(\frac{d - z}{z}\right)$$

also forms an invariant. Finally we can simplify this a bit by dividing by N(1.1, Table I) to yield the invariant

$$\frac{a_1}{a_2}\left(\frac{d - z}{z}\right). \tag{27}$$

We see that the ratio $z/(d - z)$ transforms like a_1/a_2. From (27), we obtain the equation from which the new position $\bar{z}$ may be derived

$$\frac{a_1}{a_2}\left(\frac{d - z}{z}\right) = \frac{\bar{a}_1}{\bar{a}_2}\left(\frac{\bar{d} - \bar{z}}{\bar{z}}\right). \tag{28}$$

Once we have found the new position, the new transverse scale factor and curvature may be found most easily using the invariants (2.1) and (2.4), respectively, of Table I; i.e.,

$$(\bar{x}/x) = (\bar{z}/z)(a_1/\bar{a}_1) \tag{29}$$

and

$$\bar{g}_z = g_z g_{1z} / \bar{g}_{1z} \tag{30}$$

and the scaled function is as given in (25).

To provide a more physical picture of the field transformation, it is interesting to note that the simple squeezing and imaging transformations discussed in Section IX apply to the arbitrary reference surface and its field as well as to the second mirror and its field.

Finally we note that the transformations of Fresnel's formula we have been discussing do not depend on the fact that $u_1(x_1, 0)$ is an eigenfunction of a resonator. The preceding discussion, with the exception of the derivation of (26), all applies equally well to the fields generated by *any* prescribed field distribution over an aperture.

XI. ACKNOWLEDGMENT

The authors would like to thank T. Li for valuable comments.

APPENDIX

Resonators with Spherical Mirrors of General Shape

Within the assumptions of the theory of optical resonators[1–9] the modes of a resonator formed by two spherical mirrors of quite general shape are governed by the integral equations

$$\sigma_1 E_1(x_1, y_1) = \frac{j}{\lambda d} \int_{A_2} dA_2 \cdot K(x_1, x_2; y_1, y_2) \cdot E_2(x_2, y_2) \tag{31}$$

and

$$\sigma_2 E_2(x_2, y_2) = \frac{j}{\lambda d} \int_{A_1} dA_1 \cdot K(x_1, x_2; y_1, y_2) \cdot E_1(x_1, y_1) \tag{32}$$

with the kernel

$$K(x_1, x_2; y_1, y_2) = \exp\left\{-j\frac{\pi}{\lambda d}[g_1(x_1^2 + y_1^2) + g_2(x_2^2 + y_2^2) - 2(x_1x_2 + y_1y_2)]\right\}. \tag{33}$$

Here (x_1, y_1) and (x_2, y_2) are coordinates in planes perpendicular to the optic axis, d is the mirror separation, and g_1 and g_2 describe the mirror curvatures as in Section II. Subscript "1" indicates quantities associated with the mirror on the left-hand side, and "2" refers to the mirror on the right. σ_1 and σ_2 are the eigenvalues corresponding to γ_1 and γ_2 discussed in Section II. The integration has to be performed over the reflecting areas A_1 and A_2 of the mirrors where dA_1 and dA_2 are the area elements. No assumptions on the curves bounding the reflecting areas have been made, and the formulation (31) and (32) includes mirrors of quite general shape. Special cases are, of course, strip mirrors, square mirrors, rectangular mirrors, and mirrors of circular shape. These are of main practical interest.

Let us compare (31), (32) and (33) with (1a), (1b), and (1c) of Section II. It is clear that the discussion of two-dimensional resonators systems given in Section II can be extended to the three-dimensional case in which we are interested now. The only difference is that we now have two transverse coordinates (x, y). If they are subjected to the transformation

$$x_i = \epsilon_1 \bar{x}_i \, ; \qquad y_i = \epsilon_i \bar{y}_i \, ; \qquad i = 1, 2 \tag{34}$$

and the mirror areas and area elements are scaled like

$$A_i = \epsilon_i^2 \bar{A}_i \, ; \qquad dA_i = \epsilon_i^2 dA_i \tag{35}$$

then the mirror curvatures and the mirror separation of two equivalent resonators are related by the same invariants as before. All we have to do is to replace a_i^2 by A_i in the table of invariants. For the special case of circular mirrors, a_i can be redefined as the mirror radius and retained in the invariance relations.

Note that we have used the same scaling factors ϵ_i for the x and y coordinates. If different scaling factors are used one obtains, of couse, equivalent resonators with mirrors that are not spherical but astigmatic.

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