Modes in a Sequence of Thick Astigmatic Lens-Like Focusers

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Maxwell's equations are solved for a periodic sequence of lens-like focusers separated by gaps. Each focuser consists of an arbitrarily thick slab of dielectric in which the dielectric constant tapers off radially with different quadratic laws in two perpendicular directions. Since there are no limitations on the thickness of the slabs, the solutions cover the complete gamut from a sequence of infinitely thin lenses with astigmatism to a continuous dielectric waveguide, and from spherical to cylindrical lenses.

The field configurations of the modes and their propagation constants, as well as the transmission and cutoff bands, are calculated. Any arbitrary input field distribution can then be expanded in terms of the normal modes, and the expansion determines the field everywhere.

Formulas derived for sequences of weak lenses turn out to give very good results even for lenses whose thickness and separation are equal to the focal length.

I. INTRODUCTION

One possible long distance transmission medium for optical waves consists of a periodic sequence of converging lenses. In order to negotiate unwanted but unavoidable bends of the axis of the sequence it is necessary to space the lenses as closely as possible.[1] Nevertheless, ordinary dielectric lenses exhibit substantial surface scattering, and therefore the minimum spacing between lenses depends on the tolerable transmission loss.

D. W. Berreman has shown that an effective lens can be made using gas with thermal gradients,[2,3] thus avoiding the solid-to-gas transition problems. D. W. Berreman and S. E. Miller[4] proposed a gaseous lens consisting of a tube with hot walls through which a mild gas current at lower temperature is forced to flow. At any cross section the temperature increases from the center to the wall. The density and consequently the dielectric constant is then maximum on the axis and decreases radially roughly with a square law. Without the problem of scattering at the interfaces, tubular gas lenses can be closely spaced and the gaps may be comparable to the thickness of the lenses.

The advent of such a new transmission medium makes it opportune and important to generalize the theory of modes in a sequence of thin lenses by determining the normal modes in an idealized structure which consists of a periodic sequence of arbitrarily thick slabs of dielectric whose dielectric constant tapers off radially with quadratic law.

The preferential direction of gravity creates convection currents that may introduce astigmatism in the gaseous lenses. Such an aberration is included in our model by making the radial quadratic law of the dielectric different in two perpendicular directions.

We calculate the modes of propagation of the idealized structure without including the solid walls surrounding the medium. Taking them into account would perturb the modes only slightly, introducing diffraction losses. Just as in the case of a waveguide with perfect metallic walls, the idealized modes considered here are not attenuated, but their discussion is similarly expected to be useful in approximating: (a) the propagation constants; (b) the range of dimensions over which transmission is permitted or forbidden; (c) the extent of mode conversion at discontinuities or imperfections; and (d) the field at any point due to an arbitrary input such as an off-axis or tilted beam. Of these, (a) and (b) are treated in this article.

The calculations are general enough that by changing the lens parameters and the length of the gaps it is possible to cover uninterruptedly all the range from a sequence of thin lenses[5,6,7,8] to a continuous dielectric guide,[1,9,10] and from spherical to cylindrical lenses. Up to now only the extreme cases, that is, thin lenses or dielectric guide and spherical or cylindrical lenses, have been considered in the literature; this article bridges the gaps.

II. DESCRIPTION OF THE PROBLEM

Consider a periodic sequence of dielectric slabs, shown in Fig. 1. The refractive index ν of each slab is independent of z, but varies with different quadratic laws in the x and y directions as

$$\nu = \sqrt{\frac{\epsilon}{\epsilon_0}} = n\left[1 - \left(\frac{\pi x}{L_1}\right)^2 - \left(\frac{\pi y}{L_2}\right)^2\right]^{\frac{1}{2}}. \tag{1}$$

The refractive index n on the z axis and the characteristic parameters

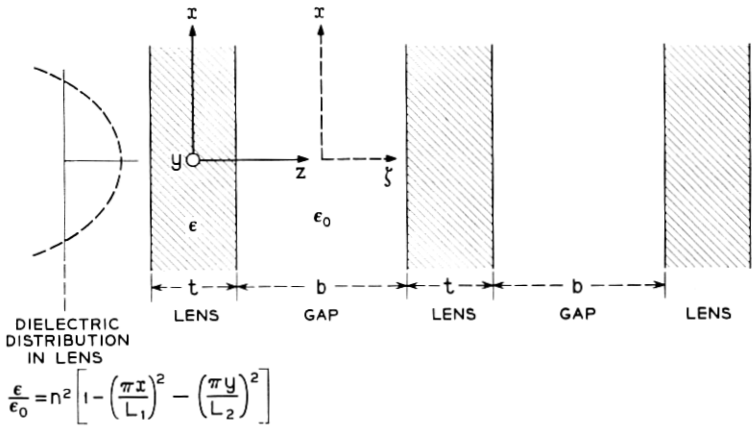

Fig. 1 — Periodic sequence of arbitrarily thick and astigmatic lenses.

of the lens-like medium, L_1 and L_2, permit adjustment of the parabolic distributions. The physical significance of L_1 and L_2 will be treated below.

In spite of the fact that it reaches negative values for large x or y, this dielectric distribution is useful because it matches the dielectric distribution of the gaseous lens, especially for small values of $\pi x/L_1$ and $\pi y/L_2$. Besides, it turns out that the field of most modes is negligible in the region where the dielectric constant is small or negative, and consequently that region does not contribute essentially to the guidance of the modes.

In the Appendix we solve approximately Maxwell's equations. The sequence of lens-like focusers supports hybrid modes EH_{pq}, characterized by the indexes p and q. These integers indicate that the intensity of each transverse field component passes through p zeros in the x direction and q zeros in the y direction.

The only approximation in the solution of Maxwell's equations consists in neglecting terms of the order of $p\lambda/L_1$ and $q\lambda/L_2$ compared to unity. λ is the free-space wavelength. Typically λ/L_1 and λ/L_2 are of the order of 10^{-5}; therefore, except for very high-order modes (p and/or q very large), the results must be satisfactorily precise.

The modes have no electric field in the y direction nor magnetic field in the x direction. The remaining components — E_x, E_z, H_y and H_z in the dielectric slabs and E_{xg}, E_{zg}, H_{yg} and H_{zg} in the gaps — are found assuming as normal modes only those field configurations that repeat themselves periodically at each lens. Therefore the equiphase surfaces of each mode are planes at $z = 0$ and $\zeta = 0$, as shown in Fig. 1. We reproduce here only E_x (55) and E_{xg} (58).

All the other components can be deduced from them with the help of (45).

$$E_x = \exp\left\{-i\left[kn\left(z - \frac{x^2}{2R_1} - \frac{y^2}{2R_2}\right) - \left(p + \frac{1}{2}\right)\tan^{-1}\left(\frac{w_1^2}{s_1^2}\tan\frac{\pi z}{L_1}\right) - \left(q + \frac{1}{2}\right)\tan^{-1}\left(\frac{w_2^2}{s_2^2}\tan\frac{\pi z}{L_2}\right)\right] - \left(\frac{x}{\rho_1}\right)^2 - \left(\frac{y}{\rho_2}\right)^2\right\} \cdot H_p\left(\sqrt{2}\,\frac{x}{\rho_1}\right) H_q\left(\sqrt{2}\,\frac{y}{\rho_2}\right); \tag{2}$$

$$E_{xg} = \exp\left\{-i\left[k\left(\zeta - \frac{x^2}{2R_{g1}} - \frac{y^2}{2R_{g2}}\right) - \left(p + \frac{1}{2}\right)\tan^{-1}\frac{2\zeta}{ks_{g1}^2} - \left(q + \frac{1}{2}\right)\tan^{-1}\frac{2\zeta}{ks_{g2}^2}\right] - \left(\frac{x}{\rho_{g1}}\right)^2 - \left(\frac{y}{\rho_{g2}}\right)^2\right\} \cdot H_p\left(\sqrt{2}\,\frac{x}{\rho_{g1}}\right) H_q\left(\sqrt{2}\,\frac{y}{\rho_{g2}}\right) \tag{3}$$

where [see (55) to (67)], $k = \omega\sqrt{\epsilon_0\mu} = 2\pi/\lambda$ is the free-space propagation constant and $H_\mu(\alpha)$ is the Hermite polynomial of order μ.

The physical significance of the symbols s_1, s_2, s_{g1}, s_{g2}, R_1, R_2, etc. will be developed below. We give first their mathematical meaning and in order to avoid repetition, from now on the letter m will stand for either the subindex 1 or 2, depending on whether the symbol under consideration refers to a dimension in the plane $y = 0$ or $x = 0$ respectively. Calling the thickness of each dielectric slab t, and the gap between them b,

$$s_m = w_m\left(\frac{1 + C_m \operatorname{ctn}\varphi_m}{1 - C_m \tan\varphi_m}\right)^{\frac{1}{4}} \tag{4}$$

$$s_{gm} = w_m(1 + C_m \operatorname{ctn}\varphi_m)^{\frac{1}{4}}(1 - C_m \tan\varphi_m)^{\frac{1}{4}} \tag{5}$$

$$w_m = \frac{1}{\pi}\sqrt{\frac{\lambda L_m}{n}} \tag{6}$$

$$C_m = n\,\frac{\pi}{2}\,\frac{b}{L_m} \tag{7}$$

$$\varphi_m = \frac{\pi}{2}\,\frac{t}{L_m} \tag{8}$$

$$R_m = \frac{L_m}{\pi}\left\{\frac{1 + \left(\frac{w_m}{s_m}\right)^4}{\left[1 - \left(\frac{w_m}{s_m}\right)^4\right]\sin\frac{2\pi z}{L_m}} + \operatorname{ctn}\frac{2\pi z}{L_m}\right\} \tag{9}$$

$$\rho_m = s_m\sqrt{\frac{1}{2}\left\{1 + \left(\frac{w_m}{s_m}\right)^4 + \left[1 - \left(\frac{w_m}{s_m}\right)^4\right]\cos\frac{2\pi z}{L_m}\right\}} \tag{10}$$

$$R_{gm} = \frac{k^2 s_{gm}^4}{4\zeta}\left[1 + \left(\frac{2\zeta}{k s_{gm}^2}\right)^2\right] \tag{11}$$

$$\rho_{gm} = s_{gm}\sqrt{1 + \left(\frac{2\zeta}{k s_{gm}^2}\right)^2} \tag{12}$$

Let us find the physical significance of R_m, R_{gm}, ρ_m, ρ_{gm}, s_m, s_{gm}, w_m and L_m. Equating in (2) and (3) the imaginary parts of the exponents to constants we obtain two equations of equiphase surfaces (wavefronts), one applicable within a lens and the other in a gap. At the z axis, each wavefront has a radius of curvature in the plane $y = 0$ which in general is different from that in the plane $x = 0$. Within a lens those main radii of curvature are R_1 and R_2 [see (9)], while those in a gap are R_{g1} and R_{g2} [see (11)]. If $L_1 = L_2$, then $R_1 = R_2$ and $R_{g1} = R_{g2}$.

For the fundamental mode $p = q = 0$, at a given abscissa z or ζ the field amplitudes (2) and (3) decrease with different Gaussian laws in the x and y directions. The distances at which the field is $1/e$ of the maximum occurring on the z axis are the beam sizes ρ_1 and ρ_2 [see (10)] within a lens, and ρ_{g1} and ρ_{g2} [see (12)] in a gap.

For $z = 0$ and $\zeta = 0$ we find from (10) and (12) that $\rho_m = s_m$ and $\rho_{gm} = s_{gm}$. Therefore s_m and s_{gm} are the beam sizes at the planes of symmetry of each lens and each gap respectively.

The physical significance of w_m becomes obvious on reducing the gaps between lenses to zero. Then instead of a sequence of lenses we have an uninterrupted dielectric waveguide and we derive from (7), (4), (5) (10) and (12) that

$$\rho_m = \rho_{gm} = s_m = s_{gm} = w_m. \tag{13}$$

Therefore in the continuous guide the propagating normal modes do not change size along z, and for the fundamental mode w_1 and w_2 measure the beam sizes in the x and y directions.

From (10) we find that within a lens the beam sizes ρ_1 and ρ_2 in the $y = 0$ and $x = 0$ planes vary periodically along z; their periods are L_1 and L_2 respectively.

For the particular case in which $L_1 = L_2$ the field in the gap (3) coincides with that found by Boyd and Gordon[6] for the resonator made with confocal mirrors of infinite aperture.

III. TRANSMISSION AND CUTOFF CONDITIONS

Both s_m [see (4)] and s_{gm} [see (5)] must be real quantities, otherwise the fields given in (2) and (3) become infinite as x or $y \to \infty$. This establishes that a mode can propagate in the sequence of lenses either when

$$C_m \leqq \operatorname{ctn} \varphi_m \tag{14}$$

or when

$$C_m \leqq -\tan \varphi_m \,. \tag{15}$$

Their equivalents in explicit form are

$$b \leqq (2L_m/n\pi) \operatorname{ctn} (\pi t/2L_m) \tag{16}$$

and

$$b \leqq -(2L_m/n\pi) \tan (\pi t/2L_m). \tag{17}$$

Which equation must we use? Since b and L_m are positive, (14) or (16) must be used when $\varphi_m = \pi t/2L_m$ falls in an odd quadrant and (15) or (17) when it falls in an even quadrant. Naturally, if these equations are satisfied for only one of the two indexes, that sequence of lenses cannot propagate any nonattenuating mode.

If $b = 0$, the sequence of lenses is reduced to a continuous waveguide and transmission takes place, as it must, no matter what the values of φ_1 and φ_2 are. If now we increase the gap b, transmission will take place as long as (16) or (17) is satisfied.

IV. DISCUSSION OF THE FIELD INSIDE AND OUTSIDE THE LENSES

The sequence of lenses admits a complete set of modes. For each mode, the field inside (2) and outside (3) the lenses is a wave traveling in the z direction whose amplitude, period and equiphase surfaces (wavefronts) vary along z.

The amplitude depends on x as a product of a Gaussian function and a Hermite polynomial (parabolic cylinder function) whose degree depends on the mode under consideration. A similar type of variation occurs along y.

In Fig. 2 we plot qualitatively the beam sizes ρ_m and ρ_{gm} for $\varphi_m = \pi t/2L_m$ in the first, second and third quadrants. For φ_m in an odd quadrant, as in Figs. 2(a) and 2(c), the maximum and minimum beam sizes within each lens are

$$\rho_{m\,\max} = s_m = w_m \left(\frac{1 + C_m \operatorname{ctn} \varphi_m}{1 - C_m \tan \varphi_m}\right)^{\frac{1}{4}} \tag{18}$$

and

$$\rho_{m\,\min} = \frac{w_m^{\ 2}}{s_m} = w_m \left(\frac{1 - C_m \tan \varphi_m}{1 + C_m \operatorname{ctn} \varphi_m}\right)^{\frac{1}{4}}. \tag{19}$$

The period between two successive maxima is L_m. The square root of the product of the maximum and minimum beam sizes in the dielectric is a constant

$$(\rho_{m\,\max}\, \rho_{m\,\min})^{\frac{1}{2}} = w_m$$

and coincides with the beam size w_m of the lens-like medium.

In the gap, the only extremum for the beam size is a single minimum which occurs at $\zeta = 0$ and, from (5) and (12), corresponds to

$$\rho_{gm\,\min} = s_{gm} = w_m(1 + C_m \operatorname{ctn} \varphi_m)^{\frac{1}{4}}\,(1 - C_m \tan \varphi_m)^{\frac{1}{4}}. \tag{20}$$

If $\varphi_m = \pi t/2L_m$ falls in an even quadrant, as in Fig. 2(b), the minimum and maximum beam sizes interchanged from the odd quadrant are (18) and (19) respectively. Again (20) corresponds to the unique minimum in each gap.

V. SPECIAL CASES

Let us consider the field in a gap assuming

$$L_1 = L_2 = L$$

and

$$t/L = \eta \quad \text{or} \quad \varphi_1 = \varphi_2 = \eta(\pi/2) \tag{21}$$

where η is an integer. Then unless the gap $b = 0$, the minimum beam size in the gap $\rho_{gm\,\min}$ (20) becomes infinitely large and the electric field (3) is reduced to a plane wave travelling in the z direction. If more generally only $\varphi_1 = \eta(\pi/2)$, but φ_2 is unrestricted, then the wave fronts are cylindrical surfaces parallel to the x axis.

Consider again

$$L_1 = L_2 = L$$

but

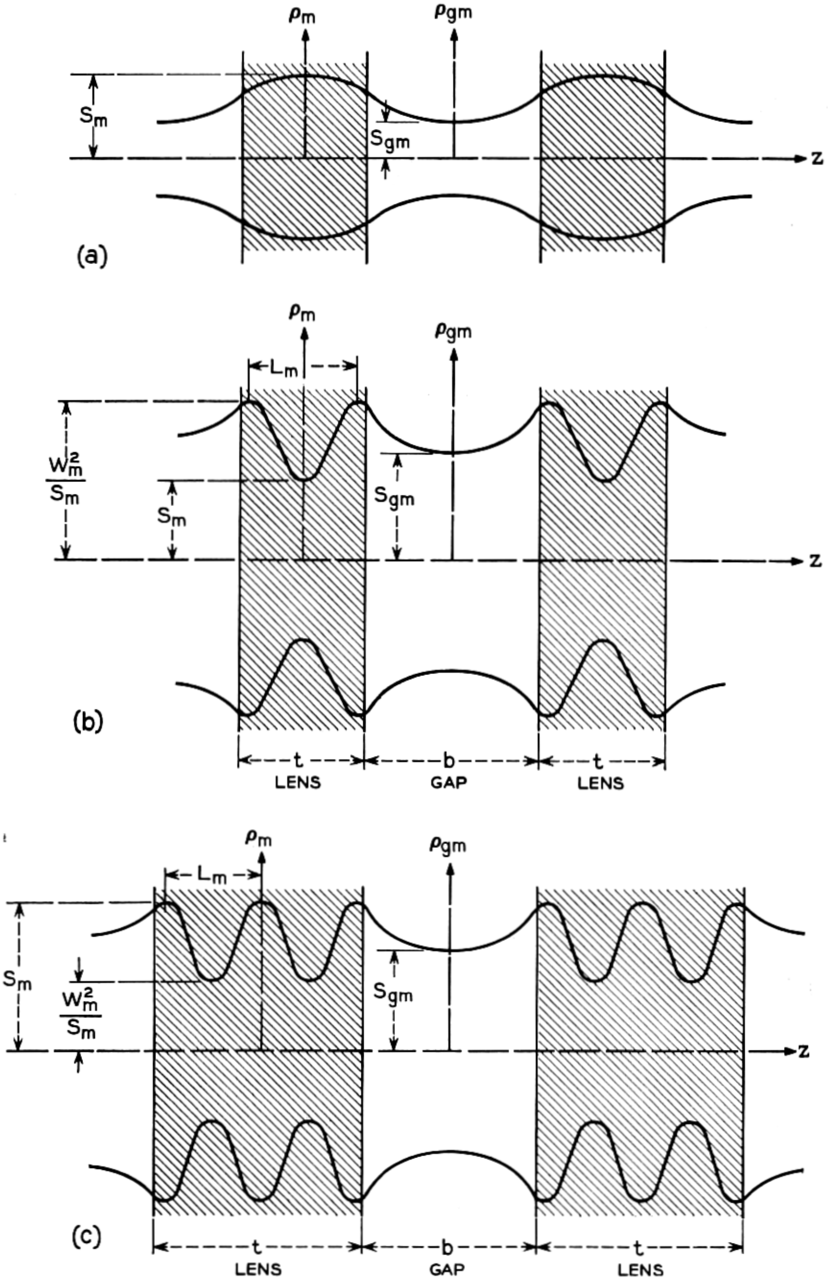

Fig. 2 — (a) Beam size for $\varphi_m = \frac{\pi}{2}\frac{t}{L_m}$ in first quadrant, $t/L_m < 1$; (b) beam size for $\varphi_m = \frac{\pi}{2}\frac{t}{L_m}$ in second quadrant, $1 < t/L_m < 2$; (c) beam size for $\varphi_m = \frac{\pi}{2}\frac{t}{L_m}$ in third quadrant, $2 < t/L_m < 3$.

$$C_1 = C_2 = \operatorname{ctn} \varphi_1 = \operatorname{ctn} \varphi_2 \tag{22}$$

or

$$C_1 = C_2 = -\tan \varphi_1 = -\tan \varphi_2 . \tag{23}$$

Then according to (20) the minimum beam size $\rho_{g1\,\min} = \rho_{g2\,\min} = 0$ and the field in the gap (3), for $p = q = 0$, becomes

$$E_{xg} = \exp -ik\zeta \left(1 + \frac{x^2 + y^2}{2\zeta^2}\right). \tag{24}$$

The wavefronts close to the ζ axis are concentric spheres and their centers coincide with the point $x = y = \zeta = 0$.

Therefore the two conditions indicated above correspond either to plane waves in the gap or to concentric waves (if one observes only the field in the region close to the ζ axis). They are equivalent to those in Fabry-Perot resonators with plane and concentric mirrors.[5,6,7]

The condition under which the beam is closely concentrated on the z axis is found by minimizing the maximum beam size within a lens, s_m [see (18)] or w_m^2/s_m [see (19)] depending on whether φ_m is in an odd or even quadrant.

If the gap b decreases, the value of s_m or w_m^2/s_m also decreases; for $b = 0$ the sequence of lenses becomes a dielectric waveguide, the beam size does not vary with z, and its value is $s_m = w_m^2/s_m = w_m$. On the other hand, if the thickness t of each lens is the only variable, the minimum of s_m or w_m^2/s_m is achieved by making

$$\frac{\partial s_m}{\partial t} = \frac{\partial s_m}{\partial \varphi_m} = 0 \qquad \text{if } \varphi_m \text{ is an odd quadrant}$$

or (25)

$$\frac{\partial}{\partial t}\frac{1}{s_m} = \frac{\partial}{\partial \varphi_m}\frac{1}{s_m} = 0 \qquad \text{if } \varphi_m \text{ is in an even quadrant.}$$

These conditions lead to the same requirement, namely:

$$C_m = \operatorname{ctn} 2\varphi_m \tag{26}$$

or its equivalent

$$b = (2L_m/n\pi) \operatorname{ctn} (\pi t/L_m) \tag{27}$$

which, replaced in (18) or (19), determines the minimized value of the maximum beam size within each lens

$$s_{m\,\min} = (w_m^2/s_m)_{\min} = w_m[(1 + C_m^2)^{\frac{1}{2}} + C_m]^{\frac{1}{2}}. \tag{28}$$

For the same condition (26) or (27), the beam size in the gap at any abscissa ζ is derived from (5), (12) and (26)

$$\rho_{gm} = w_m(1 + C_m^2)^{\frac{1}{4}} \left[1 + \frac{C_m^2}{1 + C_m^2} \left(\frac{2\zeta}{b}\right)^2\right]^{\frac{1}{2}}. \tag{29}$$

VI. SEQUENCE OF WEAK ASTIGMATIC LENSES

Before considering weak lenses, let us relate the characteristic lengths, L_1 and L_2 of the lens-like focusers to their focal lengths in the planes $y = 0$ and $x = 0$. To calculate the focal length in the $y = 0$ plane (see Fig. 3) the ray trajectory is determined from the equation

$$d^2x/dz^2 = (1/\nu)\,(d\nu/dx). \tag{30}$$

Taking the refractive index ν from (1)

$$\frac{d^2x}{dz^2} = \frac{1}{\sqrt{1 - (\pi x/L_1)^2}} \frac{d}{dx} \sqrt{1 - (\pi x/L_1)^2}. \tag{31}$$

For paraxial rays

$$\pi x/L_1 \ll 1 \tag{32}$$

and within a lens the trajectory of a ray entering parallel to the z axis at a distance x_0 is

$$x = x_0 \cos\,(\pi z/L_1).$$

The angle of refraction at the output surface is

$$\theta_r = (n\pi/L_1)x_0 \sin\,(\pi t/L_1). \tag{33}$$

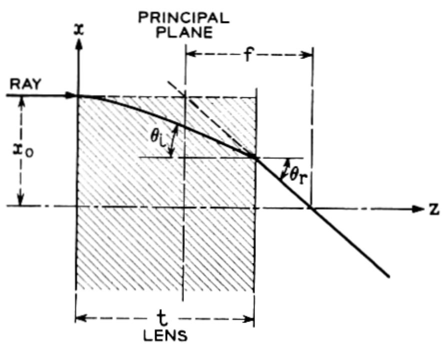

Fig. 3 — Ray trajectory in the plane $y = 0$ of a lens.

Then from simple geometric considerations in Fig. 3, the focal length f_1 results

$$f_1 = \frac{L_1}{\pi n \sin (\pi t/L_1)}. \tag{34}$$

We assume now weak lenses. They are characterized by

$$\varphi_m = (\pi/2)(t/L_m) \ll 1, \tag{35}$$

and in all previous results each circular function can be replaced by its leading term.

Because of the inequality (35) the characteristic length of the focusing medium L_1 in (34) can be calculated explicitly by

$$L_1 = \pi\sqrt{ntf_1}. \tag{36}$$

Similarly, for the plane $x = 0$

$$L_2 = \pi\sqrt{ntf_2}. \tag{37}$$

The weak lens requirement (35) then becomes

$$\varphi_m = \frac{1}{2}\sqrt{\frac{t}{nf_m}} \ll 1. \tag{38}$$

Using (36) and (37) together with the simplifying assumption (38) we re-evaluate the maximum and minimum beam sizes (18), (20) for weak lenses (φ_m in first quadrant),

$$s_m = \left(\frac{tf_m\lambda^2}{n\pi^2}\right)^{\frac{1}{4}}\left(\frac{1 + (nb/t)}{1 - (b/4f_m)}\right)^{\frac{1}{4}} \tag{39}$$

$$s_{gm} = \left(\frac{tf_m\lambda^2}{n\pi^2}\right)^{\frac{1}{4}}\left(1 + \frac{nb}{t}\right)^{\frac{1}{4}}\left(1 - \frac{b}{4f}\right)^{\frac{1}{4}}. \tag{40}$$

The distance h between the principal planes may be of interest. Using (33) and (34), this distance turns out to be

$$h_m = \frac{2L_m}{\pi n}\tan\frac{\pi t}{2L_m} - t. \tag{41}$$

Expanding the circular function in series, keeping only the first two terms and substituting L_m by their equivalents (36) and (37) we obtain,[14]

$$h_m = t\left(\frac{1}{n} - 1\right) + \frac{t^2}{12n^2f_m}. \tag{42}$$

6.1 *Example*

Let us assume a sequence of gaseous lenses such that

$$b = t = f_1 = f_2 = 0.25 \text{ m}$$

$$\lambda = 0.6328\ 10^{-6} \text{ m}$$

$$n \approx 1.$$

For these dimensions $\varphi_1 = \varphi_2 = 0.5$, and therefore the weak lens inequality (38) is hardly satisfied. Nevertheless, let us go ahead and calculate extreme beam sizes $s_1 = s_2$ and $s_{g1} = s_{g2}$ as well as the characteristic length $L_1 = L_2$ of the lens using (39), (40) and (36)

$$\begin{aligned} s_1 = s_2 &= 0.286 \text{ mm} \\ s_{g1} = s_{g2} &= 0.248 \text{ mm} \\ L_1 = L_2 &= 0.785 \text{ m}. \end{aligned} \tag{43}$$

Let us calculate again the extreme beam sizes using the exact expressions (18), (20), deriving L from (34)

$$\begin{aligned} s_1 = s_2 &= 0.276 \text{ mm} \\ s_{g1} = s_{g2} &= 0.224 \text{ mm} \\ L_1 = L_2 &= 0.704 \text{ m}. \end{aligned} \tag{44}$$

The two sets of results (43) and (44) are reasonably similar and show the usefulness of weak lens formulas even for lenses with comparable values of t and f.

VII. CONCLUSIONS

The properties of the modes in a sequence of thick, astigmatic and unbounded lens-like focusers are similar to those in a sequence of thin infinitely large lenses.

The modes are hybrid and described by parabolic cylinder functions (product of Gaussians times Hermite polynomials). Transmission takes place as long as the gap between lenses is smaller than a value given in (16) or (17).

The maximum beam size can be reduced by decreasing the distance between dielectric slabs. Nevertheless, if the gap is fixed, the minimization of the maximum beam size can be obtained by selecting the dielectric properties or the thickness of each focuser according to (27).

Simplified formulas derived for sequences of weak lenses yield good approximations even for lenses whose thickness, separation and focal length are comparable.

VIII. ACKNOWLEDGMENT

I am grateful to J. W. Tukey for his constructive criticism of this article.

APPENDIX

Solution of Maxwell's Equations in a Sequence of Thick Astigmatic Lenses

We will obtain, first, a general enough solution of Maxwell's equations for one of the lenses; see Fig. 1. Then by making $n = 1$ and $L_1 = L_2 \to \infty$, we will deduce a general solution for the uniform gap between lenses, and finally we will match the tangential fields to satisfy the boundary conditions. For modes with only four field components, E_x, E_z, H_y and H_z, Maxwell's equations become

$$\begin{aligned}\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= j\omega\mu \mathrm{H}_y\\ \frac{\partial E_x}{\partial y} &= -j\omega\mu \mathrm{H}_z\\ \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} &= -j\omega\epsilon \mathrm{E}_x\\ \frac{\partial H_y}{\partial x} &= -j\omega\epsilon \mathrm{E}_z\end{aligned} \tag{45}$$

where μ, the magnetic permeability, is a constant;

$$\frac{\epsilon}{\epsilon_0} = n^2\left[1 - \left(\frac{\pi x}{L_1}\right)^2 - \left(\frac{\pi y}{L_2}\right)^2\right] \tag{46}$$

ϵ, L_1 and L_2 are arbitrary constants; and $\omega\sqrt{\epsilon_0\mu} = 2\pi/\lambda = k$ is the free-space propagation constant.

By eliminating variables and by neglecting terms of the order of λ/L_1 and λ/L_2* as compared to unity we obtain identical equations for E_x and H_y. For E_x,

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} + (kn)^2\left[1 - \left(\frac{\pi x}{L_1}\right)^2 - \left(\frac{\pi x}{L_2}\right)^2\right]\mathrm{E}_x = 0. \tag{47}$$

* In practice λ/L_1 and λ/L_2 are of the order of 10^{-5}.

This equation is separable and a general solution is

$$E_x = \exp\left[-\left(\frac{x}{w_1}\right)^2 - \left(\frac{y}{w_2}\right)^2\right]$$
$$\times \sum_{\nu=0}^{\infty}\sum_{\mu=0}^{\infty} A_{\nu\mu} \exp\left[-iknz\left(1 - \frac{2\nu+1}{2n}\frac{\lambda}{L_1} - \frac{2\mu+1}{2n}\frac{\lambda}{L_2}\right)^{\frac{1}{2}}\right] \quad (48)$$
$$\times H_\nu\left(\frac{\sqrt{2}x}{w_1}\right) H_\mu\left(\frac{\sqrt{2}y}{w_2}\right).$$

where ν and μ are integers, and $A_{\nu\mu}$ is an arbitrary constant. Using m to indicate either subscript 1 or 2,

$$w_m = \frac{1}{\pi}\sqrt{\frac{\lambda L_m}{n}}. \quad (49)$$

The function

$$H_\nu(\xi) = (-1)^\nu e^{\xi^2}\,(d^\nu/d\xi^\nu)e^{-\xi^2}$$

is the Hermite polynomial[12] of order ν. Hermite polynomials of lowest degree are $H_0(\xi) = 1$; $H_1(\xi) = 2\xi$; $H_2(\xi) = 4\xi^2 - 2$; and $H_3(\xi) = 8\xi^3 - 12\xi$.

Expression (48) can be simplified provided that the important terms of the summation are those for which

$$\nu\lambda/L_1 \ll 1$$

and (50)

$$\mu\lambda/L_2 \ll 1.$$

Then the square root in the exponent can be replaced by the first two terms of a power series expansion and

$$E_x \cong \exp\left(-i\theta - \frac{x^2}{w_1^2} - \frac{y^2}{w_2^2}\right)\left[\sum_{\nu=0}^{\infty} A_\nu \exp\,[i(\pi\nu z/L_1)]H_\nu\left(\frac{\sqrt{2}x}{w_1}\right)\right]$$
$$\times\left[\sum_{\mu=0}^{\infty} B_\mu \exp\,[i(\pi\nu z/L_1)]H_\mu\left(\frac{\sqrt{2}y}{w_2}\right)\right] \quad (51)$$

where

$$\theta = knz\left[1 - \frac{\lambda}{4n}\left(\frac{1}{L_1} + \frac{1}{L_2}\right)\right].$$

We will look for a periodic field configuration that reproduces itself at each lens. For reasons of symmetry, then, the planes of symmetry of the lenses ($z = 0$) and gaps ($\zeta = 0$) must be equiphase surfaces.

We choose the field at the plane $z = 0$ to be

$$\exp\left(-\frac{x^2}{s_1^2}-\frac{y^2}{s_2^2}\right)H_p\left(\frac{\sqrt{2}x}{s_1}\right)H_q\left(\frac{\sqrt{2}y}{s_2}\right)$$

where p and q are integers and s_1 and s_2 are arbitrary parameters for the time being. Therefore for $z = 0$, we obtain from (51)

$$\sum_{\mu=0}^{\infty} A_\nu \exp\left(-\frac{x^2}{w_1^2}\right)H_\nu\left(\frac{\sqrt{2}x}{w_1}\right) = \exp\left(-\frac{x^2}{s_1^2}\right)H_p\left(\frac{\sqrt{2}x}{s_2}\right) \tag{52}$$

and

$$\sum_{\mu=0}^{\infty} B_\mu \exp\left(-\frac{y^2}{w_2^2}\right)H_\mu\left(\frac{\sqrt{2}y}{w_2}\right) = \exp\left(-\frac{y^2}{s_2^2}\right)H_q\left(\frac{\sqrt{2}y}{s_2}\right). \tag{53}$$

Using the orthogonality properties of the Gaussian-Hermitian product,[12] we obtain

$$A_\nu = \frac{\sqrt{2}}{\sqrt{\pi 2^\nu \nu!}\,w_1}\int_{-\infty}^{\infty}\exp\left[-\xi^2\left(\frac{1}{s_1^2}+\frac{1}{w_1^2}\right)\right] \cdot H_p\left(\frac{\sqrt{2}\xi}{s_1}\right)H_\nu\left(\frac{\sqrt{2}\xi}{w_1}\right)d\xi \tag{54}$$

and a similar expression for B_μ. Replacing the result in (51) and performing, as in Ref. 13, first the summation in ν and μ and then the integration in ξ, the transverse field component inside a lens expressed in closed form results

$$\begin{aligned} \mathrm{E}_x = \exp\Bigg\{-i\Bigg[kn\left(z-\frac{x^2}{2R_1^2}-\frac{y^2}{2R_2^2}\right)-\left(p+\frac{1}{2}\right) \\ \cdot\tan^{-1}\left(\frac{w_1^2}{s_1^2}\tan\frac{\pi z}{L_1}\right)-\left(q+\frac{1}{2}\right)\tan^{-1}\left(\frac{w_2^2}{s_2^2}\tan\frac{\pi z}{L_2}\right)\Bigg] \\ \times \exp\left[-\left(\frac{x}{\rho_1}\right)^2-\left(\frac{y}{\rho_2}\right)^2\right]H_p\left(\sqrt{2}\frac{x}{\rho_1}\right)H_q\left(\sqrt{2}\frac{y}{\rho_2}\right) \end{aligned} \tag{55}$$

where, for $m = 1$ or 2

$$R_m = \frac{L_m}{\pi}\left\{\frac{1+\left(\frac{w_m}{s_m}\right)^4}{\left[1-\left(\frac{w_m}{s_m}\right)^4\right]\sin\frac{2\pi z}{L_m}}+\operatorname{ctn}\frac{2\pi z}{L_m}\right\} \tag{56}$$

and

$$\rho_m = s_m \sqrt{\frac{1}{2}\left\{1 + \left(\frac{w_m}{s_m}\right)^4 + \left[1 - \left(\frac{w_m}{s_m}\right)^4\right] \cos \frac{2\pi z}{L_m}\right\}}. \tag{57}$$

The electric field in the uniform dielectric gap between two lenses can be derived from the previous expression by making

$$n = 1$$

and

$$1/L_m = 0$$

and by substituting another symbol, s_{gm}, for s_m. Again we demand the plane of symmetry of the gap, $z = (b + t)/2$ (see Fig. 1), to be an equiphase surface. This is achieved by substituting $\zeta = z - (b + t)/2$ for z.

The electric field in the gap is then

$$E_{xg} = \exp\left\{-i\left[k\left(\zeta - \frac{x^2}{2R_{g1}} - \frac{y^2}{2R_{g2}}\right) - \left(p + \frac{1}{2}\right)\tan^{-1}\frac{2\zeta}{ks_{g1}^2} - \left(q + \frac{1}{2}\right)\tan^{-1}\frac{2\zeta}{ks_{g2}^2}\right] - \left(\frac{x}{\rho_{g1}}\right)^2 - \left(\frac{y}{\rho_{g2}}\right)^2\right\} \cdot H_p\left(\sqrt{2}\,\frac{x}{\rho_{g1}}\right) H_q\left(\sqrt{2}\,\frac{y}{\rho_{g2}}\right) \tag{58}$$

where

$$R_{gm} = \frac{k^2 s_{gm}^4}{4\zeta}\left[1 + \left(\frac{2\zeta}{ks_{gm}^2}\right)^2\right] \tag{59}$$

and

$$\rho_{gm} = s_{gm}\sqrt{1 + \left(\frac{2\zeta}{ks_{gm}^2}\right)^2}. \tag{60}$$

To match the fields (55) and (58) at the interfaces, the x and y dependences of the field at both sides must coincide. The fact that it can be matched guarantees that Maxwell's equations are satisfied simultaneously in lenses and gaps. It can be verified that if the tangential electric field continuity is satisfied, the tangential magnetic field continuity is also guaranteed. By considering waves propagating in both directions, it could be possible to take into account reflections at the interfaces, but we shall instead assume that at each interface there is a matching mechanism that prevents reflections. Notice that in the case of gaseous lenses the small changes of dielectric constants automatically insure negligible reflection at the interfaces.

The exact matching of the fields at the interfaces is achieved by making equal the coefficients of x, y, x^2 and y^2 in both expressions (55) and (58) at the boundary $z = t/2$ of the lens and $\zeta = -b/2$ of the gap. Then

$$R_{m(z = t/2)} = R_{gm(\zeta = -b/2)} \tag{61}$$

$$\rho_{m(z = t/2)} = \rho_{gm[\zeta = -(b/2)]} . \tag{62}$$

From them, together with (56), (57), (59) and (60), we deduce the values of s_m and s_{gm} that guarantee the matching at the interfaces. They are:

$$s_m = w_m \left[\frac{1 + C_m \operatorname{ctn} \varphi_m}{1 - C_m \tan \varphi_m}\right]^{\frac{1}{4}} \tag{63}$$

and

$$s_{gm} = w_m(1 + C_m \operatorname{ctn} \varphi_m)^{\frac{1}{4}} (1 - C_m \tan \varphi_m)^{\frac{1}{4}} \tag{64}$$

where

$$C_m = n(\pi/2)\ (b/L_m) \tag{65}$$

$$\varphi_m = (\pi/2)\ (t/L_m) \tag{66}$$

$$w_m = (1/\pi) \sqrt{\lambda L_m/n}. \tag{67}$$

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