Using Digit Statistics to Word-Frame PCM Signals

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Framing of PCM signals can be accomplished by statistical means. For signal samples whose probability distribution tends to be concentrated at the center of the coding range, the second digit of the Gray code generated has a probability of mostly 1's. This information can be used to frame PCM words. Three circuits are proposed that test this probability. Reliability and reframe time for each circuit are obtained either analytically or experimentally. The first circuit uses a pair of racing counters: one counts 0's in the second digit and the other 0's in the third digit of the Gray code. When the system is in-frame, the first counter seldom reaches full count before the second, whereas during out-of-frame either counter can reach full count first with equal probability. The second circuit uses a reversible counter which advances on a 0 and retards on a 1. When connected to the second digit of the Gray code, the preponderance of 1's will keep the counter at or near zero count; when connected to any other digit, where the probability of a 1 is at most 0.5, the counter will reach full count in a finite time. The third circuit uses an RC integrator in place of the reversible counter: each 0 of the second digit generates a pulse to charge the capacitor and each 1 permits the accumulated charge on the capacitor to decay. The action is similar to that of the reversible counter but is difficult to analyze. Experimental framing performance is given for this circuit.

I. INTRODUCTION

When a signal is transmitted by PCM, the receiver must be able to group the serial pulse train into code words before it can properly recover the original signal. This process is called "framing." It is also called "word synchronization," as distinguished from bit synchronization where the time base of the individual pulses is sought. When the pulse train contains several PCM signals multiplexed together, there is also the task of multiplex framing or frame synchronization whereby the individual channels must be identified. Word synchronization can be derived from frame synchronization if the words are always arranged in a definite order within a multiplex frame; otherwise, word synchronization is acquired independently. This article will consider only the problem of word synchronization, hereafter simply called "framing."

Framing is ordinarily accomplished by using supplementary framing pulses inserted among the information-bearing pulses at predetermined intervals. The receiver will then find these framing pulses by searching and testing for the unique pattern of these pulses. If the framing pulses are inserted between every word, a substantial loss of channel capacity will result; on the other hand, if framing pulses are inserted only occasionally, the PCM words will not be uniformly spaced, which is inconvenient for a sampled-data system. When the PCM signal contains known redundancies, it is possible to accomplish framing without the use of supplementary pulses. The signal is then said to be framed "statistically." The receiver now searches for the word grouping which will yield the expected statistics for the signal. A simple example of such a statistic is the intelligibility of voice. Voice transmitted by PCM is intelligible only when the PCM words are grouped correctly. Other criteria, easier to instrument than intelligibility, are available. Most signals have amplitude distributions other than the uniform distribution or have frequency spectra other than the flat spectrum. Both of these properties will be altered when framing is incorrect. One of the easiest statistics to measure is the average occurrence of 1's and 0's in the code words. Measurement of this statistic for the case of a linear coder operating on a Gaussian signal source will be the main theme of this article. The next section will elaborate on the digit probabilities, followed by descriptions and analyses of framing circuits which acquire framing by comparing the probabilities of 1's and 0's in the second digit of the Gray code.

II. PROPERTIES OF THE GRAY CODE

If the amplitude of the signal before PCM encoding is centrally distributed — Gaussian, for example — and the Gray code is used to convert this signal into PCM, then the individual digits of each code word will not have equal probability of being either a 1 or a 0. This fact can be demonstrated by observing the Gray code assignments illustrated in Fig. 1. Because the signal amplitudes are centrally distributed, the center codes will be used more frequently than the codes at the extremes; the second digit, being a 1 for the center codes, will thus be dominated by 1's. It should be noted that this redundancy is the result of a linear coder

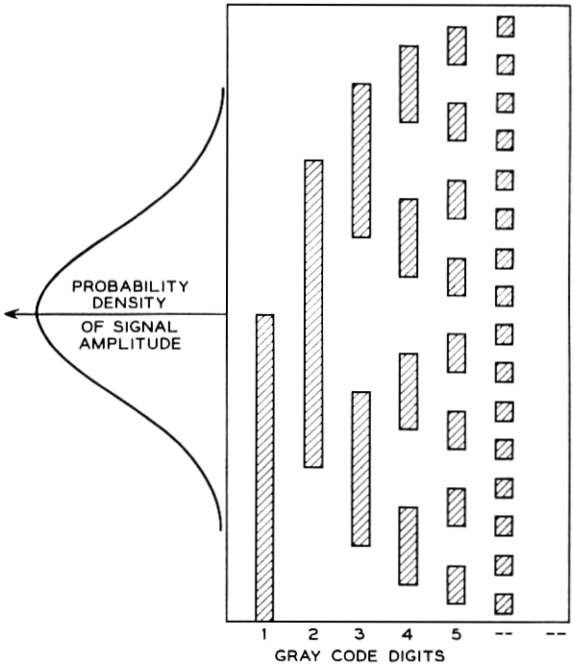

Fig. 1 — Gray code digit assignments.

operating on a Gaussian source. If a more efficient digitizer is used for this source, as for example (1) a nonlinear coder or (2) a linear coder followed by a digital processor to produce variable length codes or block codes, then this redundancy can be removed. The amount of redundancy in question is approximately one bit. Efficient coding would therefore exclude the use of statistical framing.

Fig. 2 illustrates the probabilities of 1's for all the digits; we can see that the probabilities of each digit being a 1 obey the following inequalities:

$$P(D_3 = 1) < P(D_4 = 1) < \cdots < P(D_1 = 1) < P(D_2 = 1) \quad (1)$$

or, equivalently, the probabilities of each digit being a 0 conform to

$$P(D_3 = 0) > P(D_4 = 0) > \cdots > P(D_1 = 0) > P(D_2 = 0). \quad (2)$$

Any out-of-frame condition is represented by a cyclic permutation of the digits so that one of the inequality signs in (1) will be reversed and similarly for (2). Any circuit which examines the validity of (1) or (2) is therefore a framing detector. A few such circuits will be listed here.

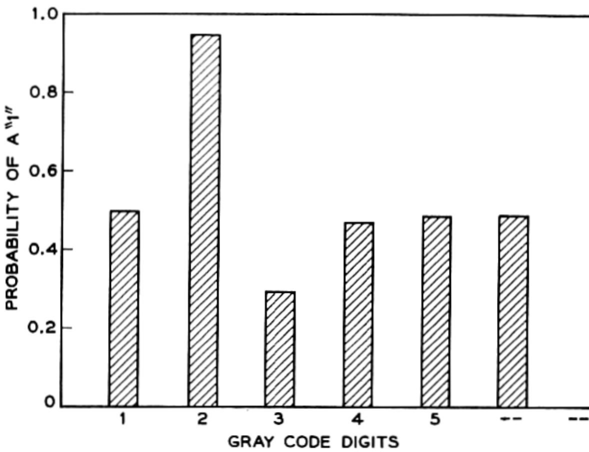

Fig. 2 — Gray code digit probabilities.

(1) Racing counters. In this scheme two counters are connected as shown in Fig. 3. When either counter reaches the full count of N, both counters are reset to zero. Now if the upper counter is connected in such a way that its count is advanced for every 0 in digit 2 and the lower counter is similarly connected for digit 3, then according to (2) the lower counter will reach full count and reset both counters most of the time. However, if the signal is out-of-frame, the counters will be actually counting the 0's of the digit pairs 3-4, 4-5, $\cdots$ or 1-2, and according to (2) the upper counter will now be able to reach full count and reset both counters much more frequently. The reset signal from the upper counter can thus be used as an out-of-frame signal. The probability of a false out-of-frame signal can be made small by increasing N, the size of the counters.

(2) Reversible counters. A single reversible counter, shown in Fig. 4,

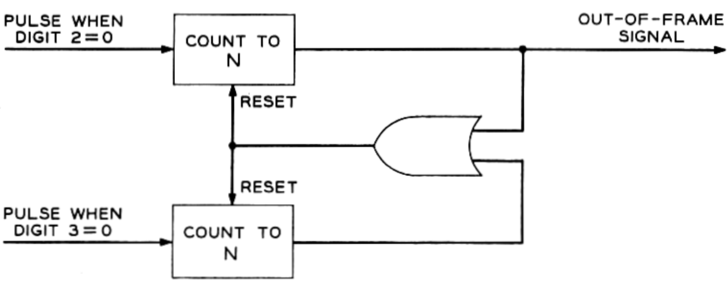

Fig. 3 — Racing counters.

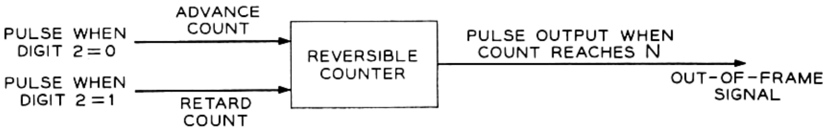

Fig. 4 — Reversible counter.

can also be used to detect the framing status. The count is increased by a 0 and reduced by a 1. When digit 2 is connected to this counter, the preponderance of 1's will keep the counter at or near the zero-count state and prevent it from reaching full count. When the receiver goes out of frame, this counter will be controlled by pulses of some other digit which, as can be seen from Fig. 2, has at least 50 per cent zeros; therefore full count will be reached within a finite time. Framing can be accomplished by searching for a word grouping such that the counter does not reach full count in a certain time interval.

(3) *RC* circuit. If a random pulse train is connected to an *RC* circuit, shown in Fig. 5, then the presence of a pulse will charge the capacitor and the absence of a pulse will permit the accumulated charge on the capacitor to discharge somewhat. The process is similar to that of the reversible counter, except that the charge and discharge rate is now a function of the accumulated charge. A threshold circuit monitoring the voltage on the capacitor can be used to indicate the framing status. A pulse train derived from the received signal such that each pulse indicates a 0 and each space indicates a 1 in the second digit of the Gray code is used as an input to the *RC* circuit. When the receiver is in frame, the pulse pattern at the input to the *RC* circuit will be sufficiently

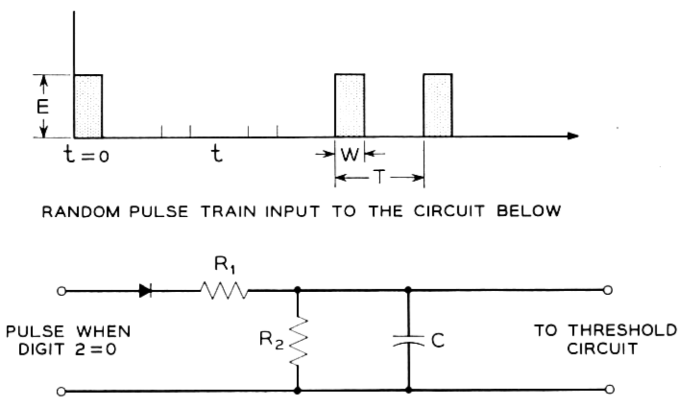

Fig. 5 — Framing with *RC* circuit.

sparse so that the accumulated charge will result in an output voltage that seldom builds up to the threshold. However, an out-of-frame condition will result in at least 50 per cent pulses present at the input, and the output of the *RC* circuit will reach threshold in a finite time.

III. FRAMING CIRCUIT CHARACTERIZATION

Two figures of merit are commonly used to characterize framing circuit performance, (1) misframe rate and (2) reframe time. Misframe rate is measured in terms of the probability that the circuit will indicate an out-of-frame condition when in fact the receiver is in frame. Reframe time is characterized by the probability distribution of the time required for the receiver to achieve correct framing; this includes the time taken to detect the out-of-frame condition. In a conventional framing circuit, wherein a known framing pulse pattern is monitored, misframe rate and reframe time are sensitive only to the error rate of the transmission medium. Performance is degraded due to masking of the framing pulses by noise. With statistical framing, performance is more dependent on signal statistics. Let the probability of a 0 in digit 2 be 0.05 at the transmitter; with an error rate of 10 per cent, the probability of a 0 will increase to about 0.14, which is still different enough from 0.5 to keep the circuit in frame. The signal itself, of course, will hardly be usable at this error rate. On the other hand, a significant change in signal statistics at the transmitter may cause a collapse of framing. Care must therefore be exercised when the performance of statistical framing circuits is to be compared with that of conventional circuits.

To evaluate the misframe rate and the reframe time of the statistical framing circuits, the response of these circuits to random inputs must be determined. Unfortunately, the statistical properties of the transient response of analog circuits such as the *RC* circuit excited by a random signal have not yet been completely solved. Therefore analytical results for framing schemes using only digital counters will be presented here; even with these circuits the results are approximate.

An experimental approach is used to determine the performance of the framing scheme using *RC* circuits. The instrumentation proves to be rather simple and some results will be given.

IV. ANALYSIS OF THE RACING COUNTERS

To lend some physical meaning to the analytical results, the analysis will be accompanied by numerical results for a typical application, namely, transmission of a mastergroup of telephone channels by PCM. A mastergroup carries 600 voice-grade channels frequency-multiplexed together, and its amplitude distribution is very close to Gaussian if the signal load is predominantly message service.[1] With normal busy hour loading the rms value of the signal is approximately $\frac{1}{4}$ of the system overload voltage. Under extreme conditions the rms may rise to $\frac{1}{3}$ of the overload voltage. These figures will be used to calculate the performances of the framing circuits. A nominal sampling rate of 6×10^6 samples per second is assumed for the mastergroup. This rate will be used to translate misframe rate into misframe interval, the mean time between misframes.

We can consider the two racing counters as a sequential machine having $(N + 1)^2$ possible states. In Fig. 6 the $(N + 1)^2$ states are depicted in a square array A; each of its elements a_{ij} represents a state where the upper counter has count i and the lower counter j. From a_{ij} transition is possible to 3 adjacent states $a_{i+1,j}$, $a_{i+1,j+1}$, or $a_{i,j+1}$ upon receiving as inputs 01, 00, or 10 respectively. In this notation the first digit represents the input to the upper counter and the second digit the input to the lower. Since the counters count only 0's, an input of 11 will not advance the counters and the state will remain at a_{ij}. Starting from the initial state a_{00}, the problems are (a) to find the probability of reaching the bottom row when digits 2 and 3 are connected to the counters (this yields the misframe rate) and (b) to find the probability distribution of the time required to reach either the bottom row or the right-hand column when other pairs of digits are connected to the counters; this leads to the distribution of reframe time when the resulting distributions are convolved.

A convenient technique for finding these probabilities is to use signal

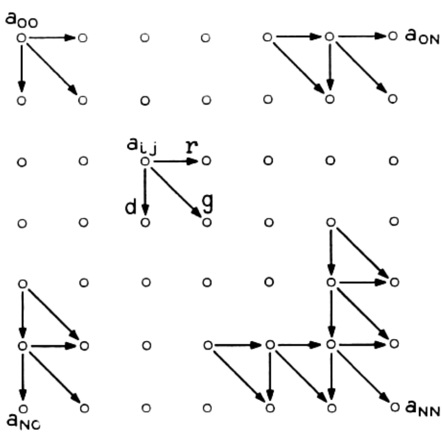

Fig. 6 — State diagram for racing counters.

flow graphs.[2] Using $x = e^{-s}$ as the time delay operator, the transitions indicated in Fig. 6 are as follows:

$$\begin{aligned} \text{down } d &= \frac{xP(01)}{1 - xP(11)} \\ \text{diagonal } g &= \frac{xP(00)}{1 - xP(11)} \qquad (3) \\ \text{and to the right } r &= \frac{xP(10)}{1 - xP(11)}. \end{aligned}$$

The denominator $[1 - xP(11)]$ is due to self-loops at each state when neither counter advances. In principle, this flow graph can be solved for the transmission from the initial state to either the bottom row or the right-hand column as rational functions of the delay operator x. From these rational functions the total probability of reaching the bottom row can be calculated by letting $x = 1$, and the probability distribution of the waiting time can be obtained by a power series expansion of the rational functions. However, in a practical situation with counters counting up to 16, the calculations become extremely involved, and even with 20 decimal digits round-off errors become excessive. Approximations are therefore used to estimate the misframe rate and the framing time.

To calculate the average misframe rate, the substitution $x = 1$ can be made before solving the flow graph of Fig. 6. This reduces complexity considerably and one can calculate the probability of reaching the bottom row before the right-hand column. Information about time delay is lost and must be estimated independently.

The flow graph can be solved by observing that

$$Q(i,j) = dQ(i - 1,j) + gQ(i - 1,j - 1) + rQ(i,j - 1) \qquad (4)$$

for

$$1 \leqq i \leqq N - 1 \qquad \text{and} \qquad 1 \leqq j \leqq N - 1$$

where $Q(i,j)$ is the probability that the state a_{ij} is reached at any time starting from a_{00}. The d, g, and r are now numerical quantities calculated from (3) with $x = 1$. The above iteration formula is valid for all states except the border states of the array A. To complete the picture we have

$$Q(00) = 1 \qquad (5)$$

since a_{00} is the initial state, and going straight down

$$Q(i,0) = dQ(i - 1,0) \qquad 1 \leqq i \leqq N. \qquad (6)$$

To the right we have

$$Q(0,j) = rQ(0,j-1) \qquad 1 \leqq j \leqq N \tag{7}$$

For the bottom row we have

$$Q(N,j) = dQ(N-1,j) + gQ(N-1,j-1) \qquad 1 \leqq j \leqq N-1 \tag{8}$$

and the rightmost column

$$Q(i,N) = gQ(i-1,N-1) + rQ(i,N-1) \qquad 1 \leqq i \leqq N-1 \tag{9}$$

and, finally, the lower right state has probability

$$Q(N,N) = gQ(N-1,N-1) \tag{10}$$

since it can be reached only by way of $a_{N-1,N-1}$. The special treatment given the bottom row and right-hand column is necessary because they are the end states; from here we start anew at a_{00}.

The probability of reaching the bottom row is the sum

$$U = \sum_{j=0}^{N} Q(N,j) \tag{11}$$

which is the probability of an output pulse from the upper counter before the lower counter reaches count N. This is the probability of a false out-of-frame signal when digits 2 and 3 are connected to the counters. The recurrence formulas are valid for signals that are independent with respect to the past, so that d, g, and r are the same for all states. Statistical dependence of the two digit inputs is considered in their joint probabilities. This iterative procedure has been carried out, and some numerical results are presented below.

Assuming a Gaussian distributed input signal the joint probabilities of digits 2 and 3 can be determined for normal loading with an rms input at ¼ of the system overload and for extreme loading with an rms input at ⅓ of the overload. The various probabilities are shown in Table I:

Table I — Probabilities of Digits 2 and 3

	01	00	10	11
RMS ¼ overload	0.0428	0.0026	0.6826	0.2720
RMS ⅓ overload	0.1092	0.0244	0.5468	0.3196

substituting these numbers into (3), we have for $x = 1$ the transition probabilities shown in Table II.

TABLE II — TRANSITION PROBABILITIES

Transition	Down d	Diagonal g	To the Right r
RMS ¼ overload	0.0588	0.0037	0.9375
RMS ⅓ overload	0.1605	0.0359	0.8036

The strong tendency to go to the right is quite evident here. The probabilities of reaching the bottom row before the right-hand column can be calculated from these data using the iteration formulas developed above. To translate these probabilities into mean time between misframes we proceed as follows. When the signal is in-frame, the lower counter almost always attains full count before the upper. For counters of size N, the lower counter resets both counters on the average of every N/p PCM words, where p is the probability of a 0 in digit 3. The mean time between misframes is then N/pU. The results are shown graphically in Fig. 7 for various counter sizes. At normal loading and $N = 16$, the mean time between misframes is 1.2×10^{12} words which, at a sampling rate of 6×10^6 per second, amounts to 2×10^5 seconds or a little more than 2 days. When the rms signal is increased to ⅓ of overload, this mean time deteriorates rapidly to fractions of a second, so that the counter size has to be more than 32 to insure adequate reliability under severe overload conditions.

To complete the picture on the racing counters, the framing time will be estimated. During search for the correct framing we observe that the

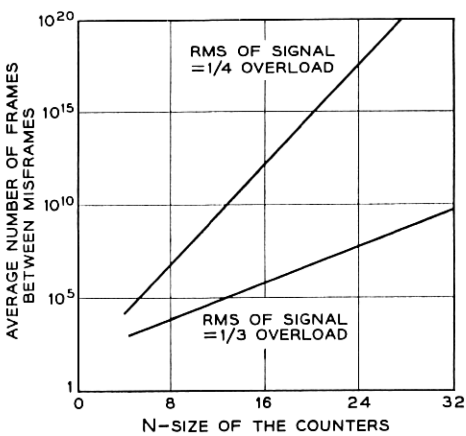

Fig. 7 — Reliability of the racing counters.

upper counter will be advanced, with 0's occurring with probability at least ½, and that it will be able to reach full count without first being reset by the lower with probability at least ½. Thus with M digits in each PCM word and assuming the worst case of searching through all $M - 1$ positions, the counters will be reset on the average of $2(M - 1)$ times. Each reset requires on the average of $2N$ words to either the upper or lower counter. A conservative estimate of the average framing time for the worst case is therefore $4N(M - 1)$ words.

As mentioned earlier, the exact distribution of the framing time is difficult to obtain; however, the variance of this distribution can be estimated. The framing time distribution can be considered as a compound distribution, where the number of times n either counter reaches full count during framing is governed by one distribution and the waiting time t for each reset is governed by another distribution. It is known that such a distribution has mean $E(n)E(t)$ and Variance $E(n)\mathrm{Var}(t) + \mathrm{Var}\ (n)E^2(t)$.[3] The distribution of the number of times either counter reaches full count before the upper counter reaches full count $M - 1$ times is governed by the negative binomial distribution.* With the upper counter having probability ½ of reaching full count, n has average $2(M - 1)$ as mentioned before and variance $2(M - 1)$. The waiting time for each reset is similarly governed by the negative binomial distribution. With probability ½ of receiving a 0, the waiting time t has mean $2N$ and variance $2N$. The variance of the framing time is therefore $2(M - 1)(2N) + 2(M - 1)(2N)^2$; for large N this is approximately $8(M - 1)N^2$.

For a 9-digit PCM system $M = 9$, and if we use $N = 32$, the average framing time for a sampling rate $F_s = 6 \times 10^6$ per second is

$$\frac{8(M-1)N}{F_s} = \frac{4 \times 8 \times 32}{6 \times 10^6} = 171\ \mu\text{sec}$$

the standard deviation is

$$\frac{[8(M-1)N^2]^{\frac{1}{2}}}{F_s} = \frac{(8 \times 8)^{\frac{1}{2}} \times 32}{6 \times 10^6} = 43\ \mu\text{sec}.$$

Since the distribution is the result of many convolutions, it can be approximated by a normal distribution; with this assumption we can use three standard deviations as the confidence limit and estimate the maximum framing time as 300 μsec. During out-of-frame conditions the upper

* See Ref. 3, p. 253. Actually the negative binomial distribution governs the number of times the lower counter reaches full count. This average is $M - 1$; the total average waiting time is therefore $(M - 1) + (M - 1) = 2(M - 1)$.

counter actually receives 0's with probability greater than ½, so that the estimates are conservative.

V. ANALYSIS OF THE REVERSIBLE COUNTER

The use of a reversible counter allows greater reliability without resorting to large-capacity counters as is necessary for the racing counters. The analysis is also simpler, since only one counter is involved. The flow graph for a reversible counter is shown in Fig. 8. The probability of a 0 which increases the count is p, and $q = 1 - p$ is the probability of a 1 which decreases the count. The count cannot go below zero. The gain of the graph for any counter size N can be obtained by standard tech-

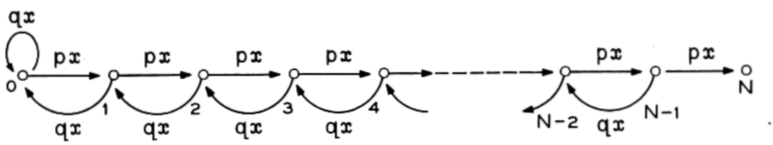

Fig. 8 — Flow graph for reversible counter.

niques. The result can be expressed conveniently in the form of a recursion formula for the denominator polynomial

$$D_N(x) = D_{N-1}(x) - pqx^2 D_{N-2}(x)$$

where

$$D_0(x) = 1 \qquad \text{and} \qquad D_1(x) = 1 - qx.$$

The numerator is simply $N_N(x) = p^N x^N$. Some representative results are

$$Q_4(x) = \frac{p^4 x^4}{1 - qx - 3pqx^2 + 2pq^2x^3 + p^2q^2x^4} \tag{12}$$

and

$$Q_8(x) = \frac{p^8 x^8}{\begin{array}{l}1 - qx - 7pqx^2 + 6pq^2x^3 + 15p^2q^2x^4 \\ \qquad - 10p^2q^3x^5 - 10p^3q^3x^6 + 4p^3q^4x^7 + p^4q^4x^8\end{array}}. \tag{13}$$

The average time between misframes can be determined from the above by differentiation. Thus*

$$T_{av} = Q_N'(1) \tag{14}$$

* See Ref. 4.

when p and q are for the second digit of the Gray code. The results for various counter sizes and for system overload at 4 and 3 times rms are shown in Fig. 9. It is seen that with a counter of size 16 and worst-case loading the misframe interval is still sufficiently long, 1000 hours at a 6-mc sampling rate.

One disadvantage of using the reversible counter is the slow reframing process. When the receiver is out of frame the counter can be assumed to receive 1's and 0's with equal probability. Using formulas developed above for $N = 16$ but substituting 0.5 for p and q, one obtains an average of 272 words to reach full count. For a 9-digit PCM system sampled at 6 mc, this amounts to 360 μsec for the average framing time. To shorten the framing time a dual-mode scheme applied frequently in conventional framing circuits can be used. The scheme is described in more detail below.

The framing circuit is designed to have two modes of operation. In the in-frame mode, the counter size is set at 16 for maximum reliability; once the out-of-frame signal is received the counter size is reduced to 8 to secure fast framing. The logic is depicted in Fig. 10.

The flip-flop determines the mode of operation. When in frame, the flip-flop is reset and the counter must reach count 16 excess 0's over 1's of the second-digit Gray code. When the system goes out of frame, the probabilities of 1's and 0's are equal, and an output from the $N = 16$ lead of the binary counter chain sets the flip-flop to the out-of-frame mode. In this mode the output from the $N = 8$ lead of the binary chain is used. At the same time a timer is turned on to reset the flip-flop after

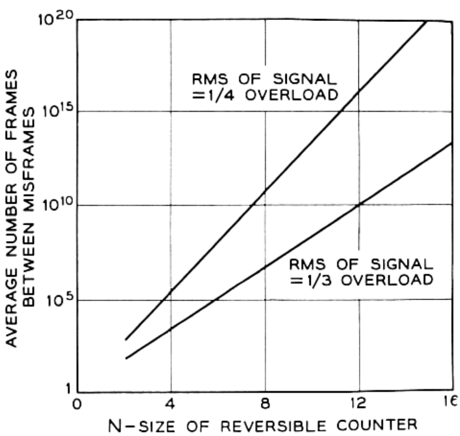

Fig. 9 — Reliability of the reversible counter.

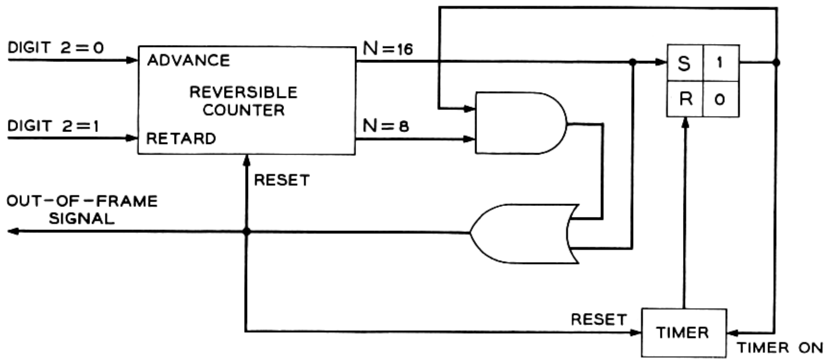

Fig. 10 — Dual-mode reversible counter framer.

a certain elapsed time. This elapsed time is selected such that it is longer than the maximum time required to get an output from the $N = 8$ lead when the system is searching but shorter than the minimum time required to get an output from this same stage when the correct frame is found. "Maximum" and "minimum" are used here in a probabilistic sense to be defined later. Thus during recovery the timer is reset before it reaches the preset time, thereby preventing the flip-flop from resetting back to the in-frame mode. When the system cycles back into frame, the timer will return the system to the in-frame mode. Each time a signal appears at the counter output, the framing counter is inhibited one time slot in order to examine the next bit position; the reversible counter is also reset automatically to zero. With the proper preset time, the system is almost always prevented from cycling past the true in-frame position.

To estimate the framing time for this scheme, we again use the example of a 9-digit PCM system sampled at 6×10^6 per second. For the worst case of searching through all 9 digits the average framing time is given by

$$T_F = Q_{16}'(1) + 7Q_8'(1). \tag{15}$$

The effect of incorrect decisions by the timer which cause recycling is ignored here. The first term corresponds to detection and the second term corresponds to the search through the next 7 positions. The time spent in verifying that the last position is the correct one is not included, because the system will already be in frame. The above equation is evaluated for $p = q = \frac{1}{2}$ and yields the worst-case average framing time of 130 μsec. This estimate is again conservative, since 0's occur with probability greater than $\frac{1}{2}$ in some digits of the Gray code.

The exact distribution of the framing time for the worst case may be determined by expanding $Q_{16}(x)Q_8^7(x)$ in a power series. Again, this is difficult to do accurately. To get around this problem an approximation to the inverse transform of $Q_8(x)$ is determined by noting that the decay in the tail of the distribution is dependent mainly on the singularity of $Q_8(x)$ closest to the unit circle (1.01728 in this case). On this basis the inverse transform is approximately

$$q_8(k) = \frac{1.6986 \times 10^{-2}}{(1.01728)^{k-8}} \qquad \text{for} \qquad k \geqq 8$$

and

$$q_8(k) = 0 \qquad 0 \leqq k < 8$$

where 1.6986×10^{-2} is selected so that

$$\sum_{k=8}^{\infty} q_8(k) = 1.$$

Using the result and returning to the x domain

$$Q_8(x) \approx \frac{1.6986 \times 10^{-2} x^8}{\left(1 - \frac{x}{1.01728}\right)}. \tag{16}$$

We now make the further approximation of replacing $Q_{16}(x)$ by $Q_8(x)$ in the product mentioned above. We can therefore deal with the simple result given by (16) raised to the 8th power. On this basis a somewhat optimistic expression for the distribution of the framing time can be readily obtained:

$$\begin{aligned} p(n) &= \frac{(1.6986 \times 10^{-2})^8 (n - 57)!}{7!(n - 64)!(1.01728)^{n-64}} \qquad \text{for} \qquad n \geqq 64 \\ p(n) &= 0 \qquad 0 \leqq n < 64. \end{aligned} \tag{17}$$

The upper tail of $p(n)$ is shown in Fig. 11.

Taking the 10^{-3} point as the confidence limit and multiplying by the sampling period, we get 200 μsec as the maximum framing time. Since the framing process is dominated by the $Q_8^7(x)$ term, the error introduced by the substitution of $Q_8(x)$ for the $Q_{16}(x)$ term should not be significant.

Finally, we note that an optimum time must be chosen for the timer in Fig. 10 to reset the flip-flop back to the in-frame mode. Selection of this time is based on the distributions of waiting times for an output from the $N = 8$ lead of the counter, first under the out-of-frame condi-

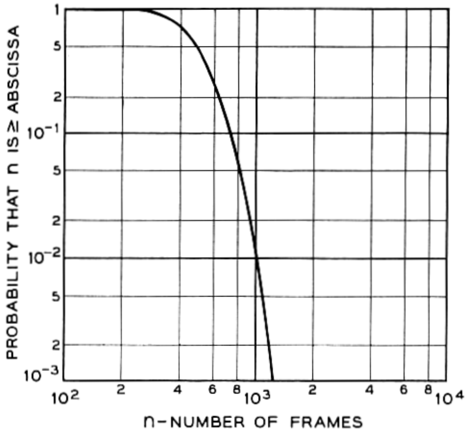

Fig. 11 — Distribution of framing time for reversible counter.

tion and second under the in-frame condition. Summing the first two columns of Table I we obtain 0.13 as the probability of a 0 for the second digit when the rms input is at ⅓ of the system overload. For the other digits a probability of 0.5 is assumed. Expanding $Q_8(x)$ in a power series when $p = 0.5$ and when $p = 0.13$ yields the desired result. This is plotted in Fig. 12. If the time is chosen to be 560 frames, the framing detector will be in the wrong operating mode only 0.01 per cent of the time,

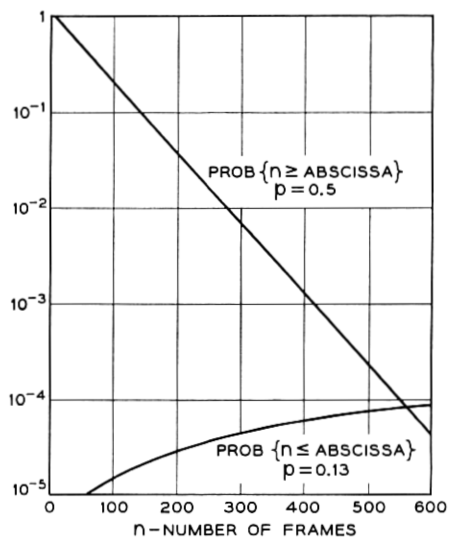

Fig. 12 — Selection of optimum time for the timer.

which means that the framer will seldom cycle past the true frame position during the framing process.

We note for future reference that the distribution of the waiting time in Fig. 12 for $p = 0.5$ is a straight line on semilog paper, which indicates that it has an exponential tail.

VI. MEASURED FRAMING PERFORMANCE FOR THE RC CIRCUIT

We introduce this section by defining the problem. Illustrated in Fig. 5 is a typical input to the RC circuit, a random pulse train

$$x(t) = \sum_{n=0}^{\infty} a_n g(t - nT) \tag{18}$$

where a_n is a sequence of independent random variables assuming values 1 or 0 with probabilities p and $(1 - p)$, and $g(t)$ is a rectangular pulse of height E and width w. When this pulse train is applied to the circuit of Fig. 5, the capacitor will charge when a pulse is present and discharge otherwise. The charging time constant is

$$\tau_c = \frac{R_1 R_2}{R_1 + R_2} C \tag{19}$$

and the discharge time constant is

$$\tau_d = R_2 C. \tag{20}$$

It is also convenient to refer to the attenuation constant

$$K = \frac{R_2}{R_1 + R_2} = 1 - \frac{\tau_c}{\tau_d}. \tag{21}$$

We are interested in the transient response of the circuit $y(t)$, particularly at times $t = w$, $t = T + w$, $\cdots$, $t = MT + w$ because they are the local maxima. We can proceed step by step:

$$y(w) = a_0 KE[1 - \exp(-w/\tau_c)] \tag{22}$$

$$y(T) = a_0 KE[1 - \exp(-w/\tau_c)] \exp[-(T - w)/\tau_d]; \tag{23}$$

at $t = T + w$, the charge due to a_1 is added, the charge due to a_0 decays further with a time constant of either τ_c or τ_d depending on the value of a_1

$$\begin{aligned} y(T + w) = {} & a_1 KE[1 - \exp(-w/\tau_c)] \\ & + \{a_0 KE[1 - \exp(-w/\tau_c)] \exp[-(T - w)\tau_d]\} \quad (24) \\ & [a_1 \exp(-w/\tau_c) + (1 - a_1) \exp(-w/\tau_d)]; \end{aligned}$$

in general

$$y(MT + w) = KE[1 - \exp(-w/\tau_c)] \sum_{n=0}^{M} a_n \exp[-(M - n)(T - w)/\tau_d] \prod_{m=n+1}^{M} [a_m \exp(-w/\tau_c) + (1 - a_m) \exp(-w/\tau_d)]. \quad (25)$$

The framing performance of this circuit is related to the probability distribution of the first time that the output of the circuit exceeds a certain threshold. It is the distribution of the smallest M such that

$$y(MT + w) > \text{threshold}. \quad (26)$$

To find the distribution analytically from (25) appears difficult. Some simplification can be obtained by assuming that the widths of the pulses are small or by assuming that the charge and discharge time constants are the same. Under either of these conditions the product in (25) disappears and the output is essentially of the form

$$z = \sum_{n=0}^{M} a_n \beta^{(M-n)} \qquad 0 < \beta < 1. \quad (27)$$

The behavior of the random variable z when $M \rightarrow \infty$ has received some attention,[4] but the distribution of the first passage time of z with respect to some threshold is still difficult to obtain.

Here the experimental approach is taken; the circuit used is depicted in Fig. 13. The input is derived from an analog-to-digital converter with a Gaussian signal as input. The output of this converter is in Gray code. By adjusting the level of the input signal and by selecting the various digits of the Gray code, a pulse train with any desired pulse density may be obtained. The digital timer measures the waiting time; it is started at the closing of the input switch and stopped by the threshold circuit. The threshold circuit also opens the input switch and signals the recorder to write the timer output on tape. A delay circuit resets the digital timer and initiates the next cycle of measurement after the RC circuit has returned to the rest condition. Each timing and recording operation takes about one msec; about a million measurements were made and recorded in a matter of minutes. A simple computer program reads the data and compiles the cumulative distribution of these data as well as the mean and standard deviation.

Some qualitative results concerning the effects of the various parameters will be given below. First, for all of the combinations of the parame-

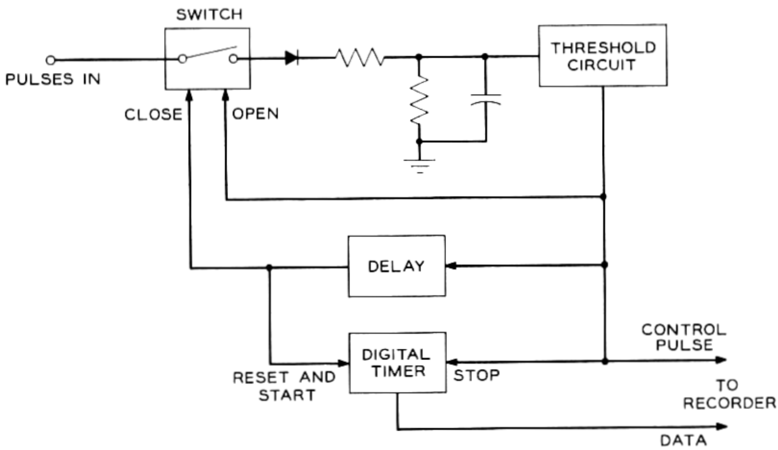

Fig. 13 — Measuring distribution of first passage time.

ters chosen, the measured distributions tend to have an exponential tail; they plot as straight lines on semilog paper (see, for example, Fig. 15 below). An intuitive argument can be given for this result. If we suppose that the threshold is set very low compared to the average output of the circuit, at voltages below this threshold the circuit acts more like an integrator than an *RC* circuit because it charges almost linearly and discharges very little between pulses. The distribution should therefore be similar to the distribution of the waiting times for the nth success in a sequence of Bernoulli trials, which has an exponential tail. Now we suppose that the threshold is set high compared to the average output of the circuit. Near this threshold, the circuit decays rapidly between pulses, so that a succession of many pulses in a row is necessary to drive the circuit over the threshold. The problem is now similar to the first occurrence of n consecutive successes in a sequence of Bernoulli trials, which again has an exponential tail. Finally, we can suppose that the threshold is set about equal to the average output of the circuit when the probability of a pulse at the input is 0.5. Near this threshold the decay due to an absence of a pulse is about equal to the charge contributed by a presence of an input pulse. The circuit therefore behaves much like a reversible counter in this region. In the previous section this has been shown to have an exponential tail. All of these arguments are of course approximate, but, lacking a complete theory, they serve to provide some insight. Knowledge that the distribution of the waiting time has an exponential tail enables us to use the techniques developed for the reversible counter to estimate the framing time distribution of this circuit.

The second qualitative result is that the measured distributions of the first passage time for the various circuits are very much the same as long as their composite time constants and relative threshold settings are the same. By "composite time constant" is meant the time required for the output to reach $(1 - e^{-1})$ of the maximum output when all pulses are present at the input. By "relative threshold" is meant the threshold as a fraction of the aforementioned maximum output. The different situations are illustrated in Fig. 14.

The composite time constant and the maximum output can be computed from (25), setting all a_n's to 1.

$$y(MT + w)$$
$$= KE[1 - \exp(-w/\tau_c)] \sum_{n=0}^{M} \exp - \left[(M - n) \left(\frac{T - w}{\tau_d} + \frac{w}{\tau_c} \right) \right]. \quad (28)$$

Letting M approach infinity we obtain the maximum output

$$y_{\max} = KE \frac{[1 - \exp(-w/\tau_c)]}{\left[1 - \exp - \left(\frac{T - w}{\tau_d} + \frac{w}{\tau_c} \right) \right]}. \quad (29)$$

The expression inside the summation in (28) can be rewritten as

$$\exp - \left[T(M - n) \left(\frac{1 - w'}{\tau_d} + \frac{w'}{\tau_c} \right) \right] \quad (30)$$

where $w' = T/w$, the duty cycle of the pulses. From this we can see that the composite time constant is

$$\left(\frac{1 - w'}{\tau_d} + \frac{w'}{\tau_c} \right)^{-1}. \quad (31)$$

The third qualitative result is the following. For circuits and threshold settings such that with equal probability of pulses and spaces at the input the distributions of the first passage time are the same, the average first passage time for low probability of input pulses is longer when the relative threshold is higher. Relative threshold is defined as above. This result can be explained by using arguments similar to the first result. At low threshold settings, the circuit acts as an accumulator so that the average first passage time is inversely proportional to the average pulse density. On the other hand, for high threshold settings, the first passage time depends on the occurrence of many consecutive pulses; the probability of this occurrence decreases exponentially with the average pulse density. This result is directly applicable to the framing

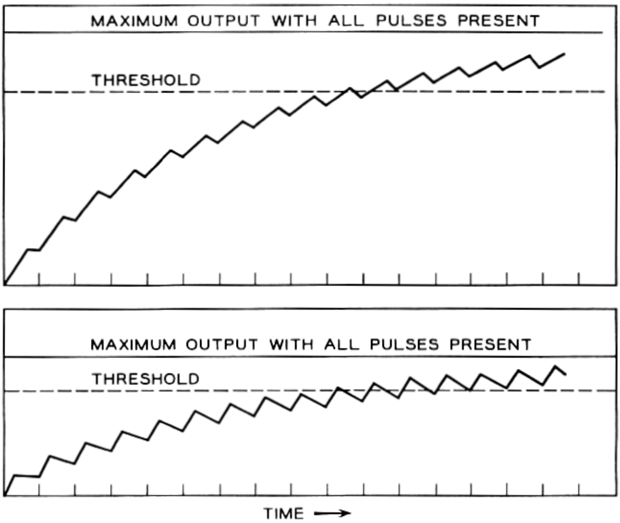

Fig. 14 — Two situations depicting different parameter settings but with substantially the same distribution of the waiting time to first passage of the threshold voltage.

problem. For the *RC* circuit, the dual-mode operation controlled by a timer used for the reversible counter is not necessary. With appropriate choice of circuit parameters and threshold, one can achieve fast framing and low misframe rate at the same time. To what extent the threshold can be adjusted to improve framing performance depends on the stability of the circuit. When the threshold is set near the level corresponding to all pulses present, a small drift in any of the parameters will cause a large change in reliability.

The framing performance of a typical *RC* circuit will be given here. Again we assume a 9-digit PCM system with 6-mc sampling rate. The parameters are as follows:

$$\begin{aligned} \text{pulse width} &= 50 \text{ per cent duty cycle} \\ \text{charging time constant} &= 0.44\ \mu\text{sec} \\ \text{discharge time constant} &= 1.2\ \mu\text{sec} \\ \text{composite time constant} &= 0.64\ \mu\text{sec}. \end{aligned}$$

With the probability of a pulse set at $\frac{1}{2}$, the variation of the distribution of the waiting time with threshold setting is illustrated in Fig. 15. The variation of the misframe interval and average framing time with threshold setting is illustrated in Fig. 16. If the threshold is chosen such that the misframe interval is 10^5 seconds (about one day), the average

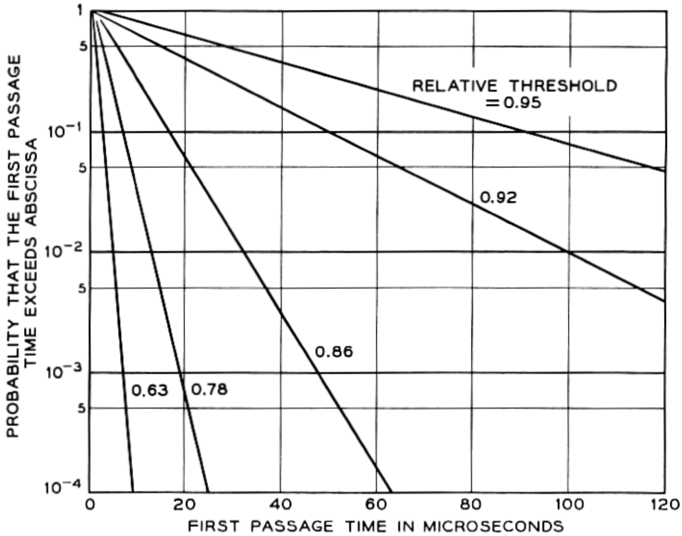

Fig. 15 — Distribution of the first passage time.

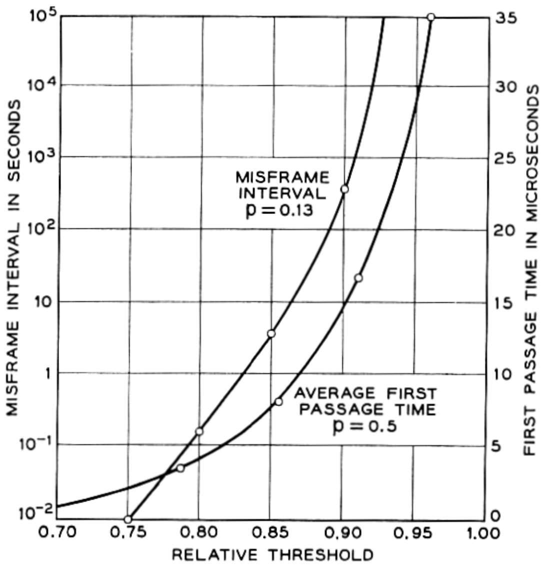

Fig. 16 — Performance variation of the *RC* framer with threshold settings.

first passage time is about 20 μsec. This yields an average framing time of 160 μsec if 8 positions are to be cycled through. Using the results of the reversible counter as a guide, the maximum framing time with 99.9 per cent confidence is about 250 μsec.

VII. SUMMARY

This paper has considered the possibility of framing a PCM signal by utilizing the statistics of the code digits. Three schemes for testing digit statistics have been proposed and their performances analyzed or measured. Statistical framing is shown to be feasible and effective whenever the signal statistics satisfies certain weak conditions.

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