B.S.T.J. BRIEFS

A Note on a Signal Recovery Problem

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In a recent study[1] of the recoverability of square-integrable bandlimited signals that are distorted by a frequency-selective time-varying nonlinear system and subsequently are bandlimited to the original bands, certain assumptions were made concerning three of the four functions of frequency that characterize the linear time-invariant part of the system. These assumptions, which are stated in Section 3.4 of Ref. 1, are satisfied in most, but not all, cases of engineering interest. The purpose of this note is to report on an extension of Theorem I of Ref. 1 that covers cases in which the conditions of Section 3.4 of Ref. 1 are not met. More specifically, a proof of the following result is outlined.

Theorem: Let $\mathfrak{L}_{2R}$ and $\mathfrak{B}(\Omega)$ be as defined in Section II of Ref. 1. Let **A**, **B**, **C**, **D**, α, $\psi(\cdot,\cdot)$, and **P** be as defined in Sections 3.1, 3.2 and 3.3 of Ref. 1, and, for any $f \;\varepsilon\; \mathfrak{L}_{2R}$, let $\psi[f]$ denote the function with values $\psi(f(t),t)$ $(-\infty < t < \infty)$.

With Ω^* the complement of Ω with respect to $(-\infty, \infty)$, and

$$\begin{aligned} E(\omega) &= D(\omega) \qquad \text{for} \quad \omega \;\varepsilon\; \Omega \\ &= 1 \qquad\qquad \text{for} \quad \omega \;\varepsilon\; \Omega^*, \end{aligned}$$

let

$$\operatorname*{ess\,sup}_{-\infty<\omega<\infty} \left| [E(C-1) - PAB]^{-1} \right| < \infty,$$

and

$$(1-\alpha) \operatorname*{ess\,sup}_{-\infty<\omega<\infty} \left| E[E(C-1) - PAB]^{-1} \right| < 1.$$

Let s_3 be an arbitrary element of $\mathfrak{B}(\Omega)$. Then $\mathfrak{B}(\Omega)$ contains a unique element s_1, and $\mathfrak{L}_{2R}$ contains unique elements w, v, and s_2 such that

$$v = \mathbf{A}s_1 + \mathbf{C}w, \qquad s_2 = \mathbf{D}s_1 + \mathbf{B}w,$$

$$s_3 = \mathbf{P}s_2, \qquad \text{and} \qquad v = \psi[w]$$

[i.e., such that (1), (2), (3), and (4) of Ref. 1 are satisfied]. Furthermore, there exists a positive constant k, that depends only on **A**, **B**, **C**,

D, and α, such that if

$$\bar{v} = \mathbf{A}\bar{s}_1 + \mathbf{C}\bar{w}, \qquad \bar{s}_2 = \mathbf{D}\bar{s}_1 + \mathbf{B}\bar{w},$$

$$\bar{s}_3 = \mathbf{P}\bar{s}_2, \qquad \text{and} \qquad \bar{v} = \psi[\bar{w}]$$

where $\bar{w}$, $\bar{v}$, $\bar{s}_2 \varepsilon \mathcal{L}_{2R}$ and $\bar{s}_1$, $\bar{s}_3 \varepsilon \mathcal{B}(\Omega)$, then

$$\| s_1 - \bar{s}_1 \| \leqq k \| s_3 - \bar{s}_3 \|.$$

Outline of Proof:

Let the mapping of $\mathcal{L}_{2R}$ into itself represented in the frequency domain by multiplication by $E(\omega)$ be denoted by **E**, and let $\mathcal{K}(\Omega)$ denote the Banach space of two-vector-valued functions of t belonging to

$$\mathcal{L}_{2R} \times \mathcal{B}(\Omega),$$

with norm $\|\cdot\|'$ defined by

$$\| f \|' = \left(\int_{-\infty}^{\infty} | f_1(t) |^2 \, dt + \int_{-\infty}^{\infty} | f_2(t) |^{\frac{1}{2}} \, dt \right)^{\frac{1}{2}}, \qquad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \varepsilon \mathcal{L}_{2R} \times \mathcal{B}(\Omega).$$

Assume that the hypotheses of the theorem are satisfied. To prove the first part of the theorem, it clearly suffices to show that $\mathcal{B}(\Omega)$ contains a unique element s_1, and $\mathcal{L}_{2R}$ contains a unique element w, such that

$$\psi[w] = \mathbf{A}s_1 + \mathbf{C}w, \tag{1}$$

and

$$s_3 = \mathbf{D}s_1 + \mathbf{PB}w, \tag{2}$$

in which **P** is defined in Section 3.2 of Ref. 1. For this purpose, we may replace **D** in (2) by **E** and write (1) and (2) as

$$\begin{bmatrix} 0 \\ s_3 \end{bmatrix} = \begin{bmatrix} (\mathbf{C} - \mathbf{I}) & \zeta\mathbf{A} \\ \mathbf{PB} & \zeta\mathbf{E} \end{bmatrix} \begin{bmatrix} w \\ \zeta^{-1}s_1 \end{bmatrix} - \begin{bmatrix} \tilde{\psi}[w] \\ 0 \end{bmatrix} \tag{3}$$

in which $\tilde{\psi}[w] = \tilde{\psi}[w] - w$, ζ is an arbitrary positive constant, and **I** is the identity operator.

The operator

$$\mathbf{L} = \begin{bmatrix} (\mathbf{C} - \mathbf{I}) & \zeta\mathbf{A} \\ \mathbf{PB} & \zeta\mathbf{E} \end{bmatrix}$$

is a bounded mapping of $\mathcal{K}(\Omega)$ into itself. In view of the first inequality of the theorem, it possesses an inverse on $\mathcal{K}(\Omega)$, and $\mathbf{L}^{-1}$ can be represented in the frequency domain by the matrix-valued function

$$L^{-1}(\omega) = \frac{1}{E(C-1) - PAB} \begin{bmatrix} E & -A \\ -\zeta^{-1}PB & \zeta^{-1}(C-1) \end{bmatrix}.$$

In particular, (3) can be written as

$$\begin{bmatrix} w \\ \zeta^{-1}s_1 \end{bmatrix} = \mathbf{L}^{-1}\mathbf{N}\begin{bmatrix} w \\ \zeta^{-1}s_1 \end{bmatrix} + \mathbf{L}^{-1}\begin{bmatrix} 0 \\ s_3 \end{bmatrix} \tag{5}$$

in which $\mathbf{N}$ is the operator defined on $\mathcal{K}(\Omega)$ by

$$\mathbf{N}f = \begin{bmatrix} \tilde{\psi}[f_1] \\ 0 \end{bmatrix}, \qquad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \varepsilon\ \mathcal{K}(\Omega).$$

The second inequality of the theorem implies that there exists a positive number ζ_0 such that $\mathbf{L}^{-1}\mathbf{N}$ is a contraction mapping of $\mathcal{K}(\Omega)$ into itself for all $\zeta > \zeta_0$. In fact, using Parseval's identity and the frequency domain representation of $\mathbf{L}^{-1}$, we find that for all f, $g\ \varepsilon\ \mathcal{K}(\Omega)$,

$$\begin{aligned} \| \mathbf{L}^{-1}\mathbf{N}f - \mathbf{L}^{-1}\mathbf{N}g \|' &\leqq \max\,(c_1, c_2)\, \| \mathbf{N}f - \mathbf{N}g \|' \\ &\leqq (1 - \alpha) \max\,(c_1, c_2)\, \| f - g \|' \end{aligned}$$

in which

$$c_1 = \operatorname*{ess\,sup}_{\omega} | E[E(C - 1) - PAB]^{-1} |,$$

and

$$c_2 = \zeta^{-1} \operatorname*{ess\,sup}_{\omega} | PB[E(C - 1) - PAB]^{-1} |.$$

In view of the contraction-mapping fixed-point theorem, this establishes the existence and uniqueness of the functions w and s_1 (as well as the important fact that these functions can be determined by an iteration procedure that converges at a geometric rate).

The second part of the theorem follows directly from (5), the relation

$$\begin{bmatrix} \bar{w} \\ \zeta^{-1}\bar{s}_1 \end{bmatrix} = \mathbf{L}^{-1}\mathbf{N}\begin{bmatrix} \bar{w} \\ \zeta^{-1}\bar{s}_1 \end{bmatrix} + \mathbf{L}^{-1}\begin{bmatrix} 0 \\ \bar{s}_3 \end{bmatrix},$$

and the fact that $\mathbf{L}^{-1}\mathbf{N}$ is a contraction for $\zeta > \zeta_0$.

REFERENCE

1. Sandberg, I. W., On the Properties of Some Systems that Distort Signals — II, BSTJ, **43**, Jan., 1964, p. 91.

Detection of Weakly Modulated Light at Microwave Frequencies

By M. G. COHEN and E. I. GORDON

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Studies of the photoelastic, electro-optic or magnetic-optic properties of materials at high frequencies often require the detection of microwave modulated light. In many cases, the modulation depth is sufficiently small that quantitative measurement becomes difficult if not impossible. The purpose of this brief is to describe a homodyne-superheterodyne technique which allows measurement of modulation depths of considerably less than 10^{-6}.

At such small modulation depths, the light-associated shot noise can be large compared to the modulation signal. Under these circumstances, it is customary to use synchronous detection techniques following a sensitive superheterodyne receiver and to chop the modulation signal at some low frequency. This requires extremely good RF shielding between the modulation source and the receiver to avoid pickup. The variations in amplitude and phase of the pickup produce an unsteady output signal. Alternately, one can chop both the light and the modulation and perform the synchronous detection at a sum or difference frequency. In any case, the limiting sensitivity is determined by noise originating in the photodetector and the receiver.

The technique described here is considerably simpler and more sensitive. Fig. 1 indicates the usual synchronous detection scheme using a photodetector feeding a microwave receiver with a 30-mcs IF strip. A reference signal for the synchronous detector is derived from the light chopping wheel. The added feature is the injection into the line incident on the receiver of a small fraction of the CW modulating signal, taken from the input line to the interaction region or modulator and passed through a variable attenuator and phase shifter. The amplitude of the injected signal is kept at least 30 db larger than that of the pickup. Thus the amplitude and phase of the total injected signal, including pickup, are essentially independent of fluctuations in the phase and amplitude of the pickup. The mixer and IF strip are operated in their linear regions. Thus the signal output v_o from the final stage of the IF amplifier, which is an envelope detector, can be written

$$v_o = \left| v_i(1 + \tilde{\Gamma}) + \tilde{v}_m + \tilde{v}_s + v_r \right|_{\text{time average}} . \quad (1)$$

Here v_i is the injected signal and $\tilde{\Gamma} <<< 1$ represents that part of the

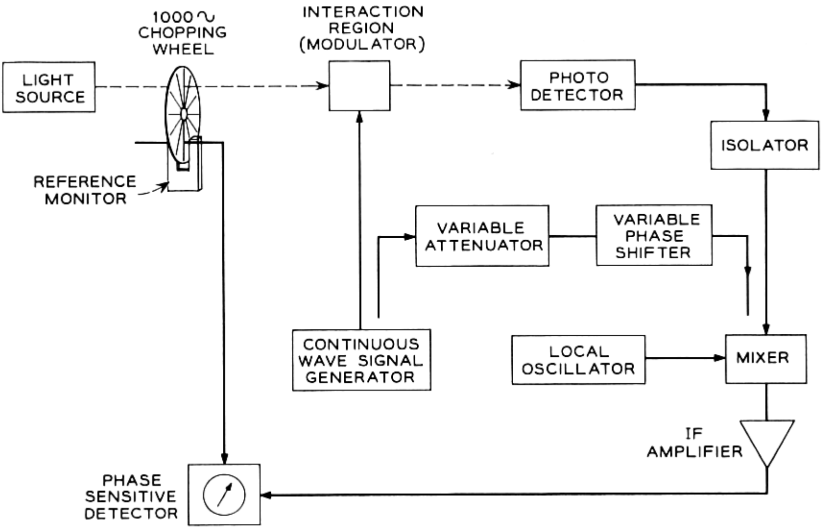

Fig. 1 — Block diagram of the homodyne-superheterodyne detection scheme.

injected signal which, because of imperfect isolation, is incident on the photodetector and is reflected with a component at the chopping frequency (because the photodetector RF impedance is dependent on the light intensity as, for example, in a photodiode) $\tilde{v}_m$ is the modulation signal from the light, $\tilde{v}_s$ is the light-associated shot noise from the photodetector and v_r is the receiver equivalent input noise. The tilde over some of the quantities indicates that they are chopped at the reference frequency of the synchronous detector. By far the largest signal is v_i, so v_o can be written to a very good approximation

$$\begin{aligned} v_o &= |v_i| [1 + 2 \operatorname{Re} \tilde{\Gamma} + 2(|\tilde{v}_m|/|v_i|) \cos \theta + \text{terms of order} \\ &\qquad |\tilde{v}_s|^2/|v_i|^2, |v_r|^2/|v_i|^2 \text{ and higher}] \\ &= |v_i| (1 + 2 \operatorname{Re} \tilde{\Gamma}) + 2|\tilde{v}_m| \cos \theta + \text{terms of order} \\ &\qquad |\tilde{v}_s|^2/|v_i|, |v_r|^2/|v_i| \text{ etc.} \end{aligned} \tag{2}$$

in which θ is the phase angle between $\tilde{v}_m$ and v_i, and Re indicates the real part. All other cross terms have a time average of zero; the only signals which are coherently related are v_i and $\tilde{v}_m$. Thus all noise terms can be made arbitrarily small compared to $|\tilde{v}_m|$ by making $|v_i|$ large. The contribution from the chopped term containing Γ can be made arbitrarily small by using more isolation in the line immediately following the photodetector. In the case of a photomultiplier, this is not normally necessary. Even when these terms are not completely negligible compared to $\bar{v}_m \cos \theta$, their effect can be eliminated by varying the phase of the injected signal so that $\cos \theta$ takes on the value ± 1. The only output which depends on θ is the desired modulation signal. Thus one need only take the algebraic difference between the extreme deflections of the synchronous detector as θ is varied. The fact that there is a synchronous detector deflection which depends on the phase of the injected signal is an unambiguous indication of microwave modulation on the light.

All aspects of (2) have been verified in the course of photoelastic and electro-optic modulation experiments above 150 mc by placing variable attenuators in various parts of the circuit to see if the variation of each term had the proper dependence. Modulation depths of 10^{-6} could be easily and accurately determined with integration times following the synchronous detector of less than one second. No special shielding was required. It should also be noted that the output of the synchronous detector is proportional to the RF amplitude rather than the square of the amplitude as in most other radiometer detection schemes. Thus, the output is proportional to the amplitude of the light modulation rather than its square.

An Improved Error Bound for Gaussian Channels

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I. INTRODUCTION

The problem considered here is that of coding for the time-discrete amplitude-continuous memoryless channel with additive Gaussian noise, the code words lying on the surface of an n-dimensional hypersphere with center at the origin and radius $\sqrt{nP}$.

We define a *code* as a set of M real n-vectors $\bar{x} = (x_1, x_2, \cdots, x_n)$ satisfying the ("energy") constraint,

$$\sum_{k=1}^{n} x_k^2 = nP. \tag{1}$$

The *transmission rate* R is defined by $M = e^{nR}$, so that $R = (1/n) \ln M$.