ing the photodetector. In the case of a photomultiplier, this is not normally necessary. Even when these terms are not completely negligible compared to $\tilde{v}_m \cos \theta$, their effect can be eliminated by varying the phase of the injected signal so that $\cos \theta$ takes on the value ± 1. The only output which depends on θ is the desired modulation signal. Thus one need only take the algebraic difference between the extreme deflections of the synchronous detector as θ is varied. The fact that there is a synchronous detector deflection which depends on the phase of the injected signal is an unambiguous indication of microwave modulation on the light.

All aspects of (2) have been verified in the course of photoelastic and electro-optic modulation experiments above 150 mc by placing variable attenuators in various parts of the circuit to see if the variation of each term had the proper dependence. Modulation depths of 10^{-6} could be easily and accurately determined with integration times following the synchronous detector of less than one second. No special shielding was required. It should also be noted that the output of the synchronous detector is proportional to the RF amplitude rather than the square of the amplitude as in most other radiometer detection schemes. Thus, the output is proportional to the amplitude of the light modulation rather than its square.

An Improved Error Bound for Gaussian Channels

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I. INTRODUCTION

The problem considered here is that of coding for the time-discrete amplitude-continuous memoryless channel with additive Gaussian noise, the code words lying on the surface of an n-dimensional hypersphere with center at the origin and radius $\sqrt{nP}$.

We define a *code* as a set of M real n-vectors $\bar{x} = (x_1, x_2, \cdots, x_n)$ satisfying the ("energy") constraint,

$$\sum_{k=1}^{n} x_k^2 = nP. \tag{1}$$

The *transmission rate* R is defined by $M = e^{nR}$, so that $R = (1/n) \ln M$.

The code words are transmitted through a channel in which they are corrupted by noise, the received word $\bar{y} = (y_1, y_2, \cdots, y_n)$ being the vector sum of the transmitted word $\bar{x}$ and a noise vector $\bar{z}$, i.e.,

$$\bar{y} = (y_1, y_2, \cdots, y_n) = (x_1 + z_1, x_2 + z_2, \cdots, x_n + z_n) = \bar{x} + \bar{z}. \quad (2)$$

The components of the noise vector $z_k (k = 1, 2, \cdots, n)$ are assumed to be statistically independent Gaussian random variables with mean zero and variance N.

The signal "energy" is $\sum_{k=1}^{n} x_k^2 = nP$, and the expected noise "energy" is $E[\sum_k z_k^2] = nN$, so that the signal-to-noise energy ratio is P/N. This quantity is also the signal-to-noise "average power."

It is the task of the decoder to examine the received vector $\bar{y}$ and decide which code word $\bar{x}$ was actually transmitted. If P_{ei} is the probability that the decoder makes an incorrect choice when code word i is transmitted $(i = 1, 2, 3, \cdots, M)$, and if each of the M code words is equally likely to be transmitted, then the over-all probability of a decoding error is

$$P_e = \frac{1}{M} \sum_{i=1}^{M} P_{ei}. \quad (3)$$

It is not hard to show that the decoding scheme which minimizes P_e for a given code is the *minimum-distance decoder*, where the decoder selects that code word which has smallest Euclidean distance from the received vector and announces that word as the one which was transmitted. Thus if $\bar{y} = (y_1, y_2, \cdots, y_n)$ is the received vector, the decoder announces that code word $\bar{x}$ which minimizes (with respect to $\bar{x}$)

$$d(\bar{x},\bar{y}) = \sum_{k=1}^{n} (x_k - y_k)^2 = \sum_k x_k^2 + \sum_k y_k^2 - 2 \sum_k x_k y_k .$$

Since $\sum_k x_k^2 = nP$, $d(\bar{x},\bar{y})$ is minimized when $\sum_k x_k y_k$ is maximized. Hence minimum-distance decoding is equivalent to selection of that code word $\bar{x}$ which minimizes the angle in n space $a(\bar{x},\bar{y})$ between $\bar{x}$ and $\bar{y}$, where

$$\cos a(\bar{x},\bar{y}) = \frac{\sum_k x_k y_k}{(\sum_k x_k^2)^{\frac{1}{2}} (\sum_k y_k^2)^{\frac{1}{2}}}. \quad (4)$$

The behavior of codes for this channel has been investigated in detail by Shannon,[1,2] who has shown the following:

Fundamental Coding Theorem: Let R be any number such that

$$R < C = \tfrac{1}{2} ln\, [1 + (P/N)].$$

For each n, there exists an n-dimensional code with rate $R(M = e^{nR})$ *such that the error probability is*

$$P_e = e^{-nE(R)+o(n)}, \tag{5}$$

where the exponent $E(R)$ *(called the "reliability") is positive when* $R < C$ *(so that* $P_e \xrightarrow{n} 0$*).*

Shannon[2] also obtained estimates of the best possible exponent

$$E(R) = \lim_{n\to\infty} - (1/n) \ln P_e .$$

In this note we establish the following upper bound on $E(R)$ (i.e., a lower bound on P_e):

$$E(R) \leqq \frac{P}{4N} e^{-2R} \tag{6}$$

For small rates R, (6) is sharper than the bounds of Ref. 2. Inequality (6) is plotted together with the estimates on $E(R)$ in Ref. 2 in Fig. 1.

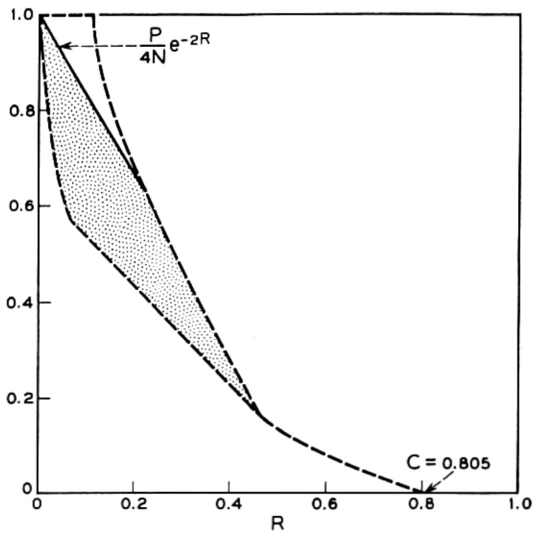

Fig. 1 — New upper bound on $E(R)$ vs R for $P/N = 4$ (solid line). The bounds on $E(R)$ of Ref. 2 are in dotted lines. $E(R)$ lies in the shaded area.

II. DERIVATION OF THE BOUND

Consider an n-dimensional code with M code words $\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_M$. Let θ be the minimum angle between pairs of code words $a(\bar{x}_i, \bar{x}_j)$ $(i \neq j)$. Denote by $\theta_n(M)$ the largest possible minimum angle θ in an n-dimensional code with M code words, and by

$$s_n(M) = 2\sqrt{nP} \sin [\theta_n(M)/2],$$

the largest possible minimum distance between pairs of code words in an n-dimensional code with M code words. Paralleling an argument of Shannon [Ref. 2, pp. 647–648] it is not hard to show that the error probability satisfies

$$P_e \geqq \tfrac{1}{2}\Phi\left(-\sqrt{\frac{nP}{N}} \sin \frac{\theta_n(M/2)}{2}\right), \tag{7}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

is the cumulative error function.

We now employ the following result of Rankin[3] to obtain an upper bound on $\theta_n(M)$:

$$M \leqq \frac{\pi^{\frac{1}{2}}\Gamma\left(\frac{n-1}{2}\right) \sin \beta \tan \beta}{2\Gamma\left(\frac{n}{2}\right) \int_0^\beta (\sin \varphi)^{n-2}(\cos \varphi - \cos \beta) d\varphi}, \tag{8}$$

where $\beta = \sin^{-1} \sqrt{2} \sin (\theta/2)$, and θ is the minimum angle in an n-dimensional code with M code words. Taking logarithms of (8) yields

$$R = \frac{1}{n} \ln M \leqq \frac{1}{n} \ln \frac{\pi^{\frac{1}{2}}}{2} \sin \beta \tan \beta + \frac{1}{n} \ln \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \tag{9}$$
$$- \frac{1}{n} \ln \int_0^\beta (\sin \varphi)^{n-2}(\cos \varphi - \cos \beta) d\varphi.$$

It is shown in the appendix that for large n we may approximate the upper bound of (9) by $-\ln \sqrt{2} \sin (\theta/2)$, yielding

$$\sin \frac{\theta}{2} \leqq \frac{1}{\sqrt{2}} e^{-R}. \tag{10}$$

Since for large n, a code with $M/2$ points has the same rate as one with M points (10) and (7) yield (for large n)

$$P_e \geqq \tfrac{1}{2}\Phi\left(-\sqrt{\frac{nP}{N}}\,\frac{e^{-R}}{\sqrt{2}}\right). \tag{11}$$

Using the well known asymptotic form of the cumulative error function $\Phi(-x) \sim (1/x\sqrt{2\pi})e^{-x^2/2}$ (large x) we obtain

$$E(R) = \lim_{n\to\infty} -\frac{1}{n}\ln P_e \leqq \frac{P}{4N}\,e^{-2R}. \tag{12}$$

APPENDIX

We must show that the limit of the right-hand member of inequality (9) as n tends to infinity is $-\ln\sqrt{2}\sin(\theta/2)$. The first two terms of this quantity both tend to zero as n becomes large, so that we must show the following:

Let

$$I_n = \int_0^\beta \sin^{n-2}\varphi\,(\cos\varphi - \cos\beta)d\varphi,$$

then

$$E = \lim_{n\to\infty}\frac{1}{n}\ln I_n = \ln\sin\beta.$$

Proof:

(a) $I_n \leqq \int_0^\beta \sin^{n-2}\beta\,(\cos\varphi - \cos\beta)d\varphi = \sin^{n-2}\beta\,[\sin\beta - \beta\cos\beta],$

so that

$$\frac{1}{n}\ln I_n \leqq \frac{n-2}{n}\ln\sin\beta + \frac{1}{n}\ln[\sin\beta - \beta\cos\beta] \xrightarrow{n} \ln\sin\beta.$$

(b)
$$\begin{aligned} I_n &\geqq \int_{\beta-(\beta/n)}^{\beta} \sin^{n-2}\varphi\,(\cos\varphi - \cos\beta)d\varphi \\ &\geqq \sin^{n-2}\left(\beta - \frac{\beta}{n}\right)\int_{\beta-(\beta/n)}^{\beta}(\cos\varphi - \cos\beta)d\varphi. \end{aligned} \tag{13}$$

Now

$$I = \int_{\beta-(\beta/n)}^{\beta} (\cos\varphi - \cos\beta)d\varphi = \sin\beta - \sin\left(\beta - \frac{\beta}{n}\right) - \frac{\beta}{n}\cos\beta$$

$$= \sin\beta - \sin\beta\cos\frac{\beta}{n} + \cos\beta\sin\frac{\beta}{n} - \frac{\beta}{n}\cos\beta.$$

Expanding sin (β/n) and cos (β/n) into power series in (β/n), we obtain

$$I = \sin\beta\left[\frac{\beta^2}{2n^2} + o\left(\frac{1}{n^2}\right)\right] = \frac{\beta^2}{2n^2}\sin\beta[1 + o(1)].$$

Thus

$$\frac{1}{n}\ln I = \frac{1}{n}\ln\frac{\beta^2}{2n^2}\sin\beta + \frac{1}{n}\ln(1 + o(1)) \xrightarrow{n} 0.$$

From (13) we have

$$\frac{1}{n}\ln I_n \geqq \frac{n-2}{n}\ln\sin\left(\beta - \frac{\beta}{n}\right) + \frac{1}{n}\ln I \xrightarrow{n} \ln\sin\beta.$$

Therefore $E = \ln\sin\beta$, which completes the proof.

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