The Joint Optimization of Transmitted Signal and Receiving Filter for Data Transmission Systems

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The optimum signal and receiving filters for a data transmission system with a fixed channel and detection process are found. The criteria of optimality are: (i) the minimization of mean square error between the input multilevel data signal and the output to the decision threshold and (ii) the minimization of noise power output with no intersymbol interference. Explicit frequency characteristics are obtained for the separate and joint signal and receiver optimization problems. For the joint problem it is seen that some freedom exists in assigning phases to the transmitter and receiver. In addition, explicit equations are obtained for the output signal-to-noise ratio for the problems considered. The optimization procedures are carried out in detail in some examples. In comparing binary and correlative multilevel signalling (e.g., duobinary), it is seen that there may be channels for which the optimum multilevel system is superior to the optimum binary system.

I. INTRODUCTION

In most instances a data system designer is faced with the problem of transmitting through a noisy channel over which he has no control. In other words, the designer must concern himself with transmitter and receiver terminals which reduce the disturbing effects of the channel. Nyquist's classic paper[1] considered over-all system designs which eliminate intersymbol interference. Others (see, for example, Ref. 2) have also reduced the effects of noise by designing the transmitter and receiver to minimize the noise output for a given Nyquist characteristic and an ideal channel. Tufts' recent work[3-6] has recognized that transmission without intersymbol interference may not be the most desirable. He has considered the problem of optimizing transmitted signal waveforms or receiver filters under the criterion of minimizing the mean square error (thus minimizing the joint contribution of noise and intersymbol interference). His attempt at joint optimization of the transmitter and receiver was successful only with the added condition that the transmitted waveform be time limited to one bit interval. This is an important case, but the unsolved joint optimization problem without this constraint is also important. The joint optimization solution would, in effect, provide a performance bound for a given channel.

This paper extends Tufts' results by solving the joint optimization problem (mean square error criterion) for a time-invariant transmitter and receiver subject only to an average power constraint on the transmitted waveform. Explicit equations are obtained for the signal, receiving filter, and the output signal-to-noise ratio for both the separate and joint optimization. In addition, the optimum signal and receiver are found for an arbitrary channel subject to the constraint of no intersymbol interference. The output signal-to-noise ratio for this case is obtained and is found to be only slightly less than the optimum for usual noise levels. The results show clearly the importance of the ratio of the noise spectral density to the square of the channel amplitude characteristic on the design techniques and the performance bound. The phase characteristics of the channel are irrelevant to the bound and there is some freedom in assigning phase characteristics to the transmitter and receiver in the joint optimization problems.

An important reason for the solution of the joint optimization problem is in the notation used. The frequency domain is broken into disjoint intervals and the characteristic within each interval is considered as a separate function. The equations to be minimized are easily stated and the constraints (such as no intersymbol interference) are compactly and completely represented by this notation. Finally, the joint solution is easily obtained with the notation used here whereas with the standard approaches,[6] the solution is obscured.

II. GENERAL CONSIDERATIONS

Fig. 1 illustrates the general linear, time-invariant, noisy (zero mean) multilevel transmission system considered. One may assume that the information is contained in a random sequence of impulses (of weight

$$a_l\,,\, a_l = \{-2M, \cdots, 0, \cdots 2M - 2, 2M\}$$

or

$$\{-2M - 1, \cdots, -1, 1, \cdots 2M + 1\}$$

and spaced T seconds apart) at the input of the system. Thus, a signal, $s(t)$, having an amplitude of a_l is transmitted every T seconds. The

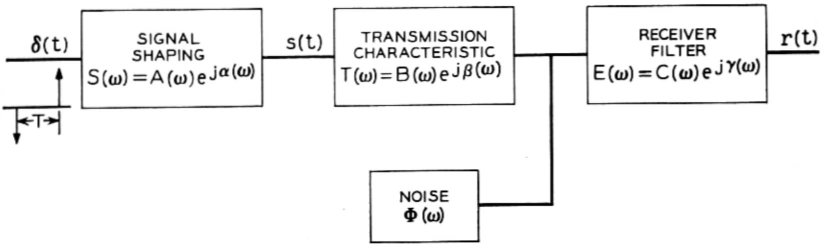

Fig. 1 — General digital transmission system.

noiseless system output, $r(t)$, has the Fourier transform

$$R(\omega) = S(\omega)T(\omega)E(\omega) \tag{1}$$

A sequence of input signals

$$\sum_l a_l s(t - lT)$$

(all sums from $-\infty$ to ∞ unless otherwise noted) will, in general, produce a sequence of overlapping output pulses

$$\sum_l a_l r(t - lT).$$

The total output at the sampling time (taken at $t = 0$ without loss of generality) is

$$v = a_0 r(0) + \sum_{l \neq 0} a_l r(-lT) + n(0) \tag{2a}$$

or

$$v = a_0 r_0 + \sum_{l \neq 0} a_l r_{-l} + n_0 \tag{2b}$$

where r_l, $l \neq 0$ represents the intersymbol interference and r_0 is the output value at the main sample point.

In order to assess the performance of such a data system, it is necessary to choose a criterion of quality. One useful measure which Tufts[3-6] has used extensively is the mean square error. This paper considers a slightly different criterion by using a normalized mean square error

$$E\{[\text{normalized input sample} - \text{output sample}]^2\}$$

or

$$E\{[a_0 r_0 - (a_0 r_0 + \sum_{l \neq 0} a_l r_{-l} + n_0)]^2\}$$

where a simple change of amplitude between input and output has been eliminated as a factor. Then

$$M.S.E. = E\{[a_0 r_0 - (a_0 r_0 + \sum_{l \neq 0} a_l r_{-l} + n_0)]^2\}$$

$$= E\{n_0^2\} + E\{2n_0 \sum_{l \neq 0} a_l r_{-l}\} + E\{(\sum_{l \neq 0} a_l r_{-l})^2\} \tag{3a}$$

$$= E\{n_0^2\} + E\{\sum_{i,j \neq 0} a_i a_j r_{-i} r_{-j}\} \tag{3b}$$

$$= \sigma^2 + \overline{a_j^2} \sum_{l \neq 0} r_l^2 + 2 \sum_{m=1}^{\infty} \overline{a_j a_{j+m}} \sum_{l \neq 0, -m} r_l r_{l+m} \tag{3c}$$

where

$$\sigma^2 = E\{n_0^2\}.$$

This expression cannot be an absolute measure of quality but should be related to the output level r_0. Much of the remainder of the paper will be concerned with the design of the transmitted signal and the receiver filter (both separately and jointly) to minimize the mean square error given by (3c) for a given value of r_0. In addition, this expression will also be minimized subject to the constraint that the intersymbol interference be zero, i.e.,

$$\sum_{l \neq 0} r_l^2 = 0.$$

It is particularly useful for our purposes to work with the frequency characteristics of the system. Appendix A gives the details of the transformation of the mean square error expression from a time domain to a frequency domain function. In terms of the system component transforms

Transmitted signal	$S(\omega)$	$= A(\omega)e^{j\alpha(w)}$
Channel characteristic	$T(\omega)$	$= B(\omega)e^{j\beta(w)}$
Receiver filter	$E(\omega)$	$= C(\omega)e^{j\gamma(w)}$
Noise spectral density		$= \Phi(\omega)$

and the shorthand notation $A[u + (2n\pi/T)] = A_n(u)$ etc., the mean square error may be written

$$M.S.E. = \int_{-\pi/T}^{\pi/T} \left\{ \sum_n C_n^2(u)\Phi_n(u) + \frac{2\pi}{T}\left[\sum_n A_n(u)B_n(u)C_n(u) \cos\left[\alpha_n(u) + \beta_n(u) + \gamma_n(u)\right] - \frac{r_0T}{2\pi}\right]^2 F(u) + \frac{2\pi}{T}\left[\sum_n A_n(u)B_n(u)C_n(u) \sin\left[\alpha_n(u) + \beta_n(u) + \gamma_n(u)\right]\right]^2 F(u) \right\} du \tag{4}$$

where

$$F(u) = \overline{a_j^2} + 2\sum_{m=1}^{\infty} \overline{a_j a_{j+m}} \cos umT. \tag{5}$$

In (4) the frequency range has been split into disjoint bands of width $2\pi/T$ and the characteristic within each band is treated as a separate function {e.g., $A_n(u) = A[u + (2n\pi/T)]$ is $A(\omega)$ for $(2n - 1)\pi/T < \omega < (2n + 1)\pi/T$}. For simplicity, the argument, u, will not be made explicit (except for the dependence of the Lagrange multipliers) throughout the rest of the development.

The last two terms on the right side of (4) represent the intersymbol distortion. By simultaneously making these terms zero, one has the necessary conditions for transmission without intersymbol interference

$$\sum_n A_nB_nC_n \cos(\alpha_n + \beta_n + \gamma_n) = \frac{r_0T}{2\pi} \tag{6}$$

$$\sum_n A_nB_nC_n \sin(\alpha_n + \beta_n + \gamma_n) = 0 \qquad -\frac{\pi}{T} \leqq u \leqq \frac{\pi}{T} \tag{7}$$

which have been discussed in a previous paper.[7]

In the remainder of the paper, the optimum signal and equalizer (receiver filter) will be determined (separately in Section III and jointly in Section IV) under the constraint that the average signal power is limited and the output level is fixed. That is, the mean square error

$$\int_{-\pi/T}^{\pi/T} \left\{ \sum_n C_n^2\Phi_n + \frac{2\pi F}{T}\left(\sum_n A_nB_nC_n \cos(\alpha_n + \beta_n + \gamma_n) - \frac{r_0T}{2\pi}\right)^2 + \frac{2\pi F}{T}\left(\sum_n A_nB_nC_n \sin(\alpha_n + \beta_n + \gamma_n)\right)^2 \right\} du$$

will be minimized subject to

$$\int_{-\pi/T}^{\pi/T} \sum_n A_n B_n C_n \cos(\alpha_n + \beta_n + \gamma_n)\, du = r_0 \tag{8}$$

and the average power constraint which may be written[8] (assuming $\overline{a_j} = 0$)

$$P = \int_{-\infty}^{\infty} F(\omega) A^2(\omega) d\omega = \int_{-\pi/T}^{\pi/T} F \sum_n A_n^2 du. \tag{9}$$

Note that this differs from Tufts' development[5] in that he considers a fixed pulse energy as the constraint.

In addition, the suboptimum signal and equalizer which eliminate intersymbol interference will be found by minimizing

$$\int_{-\pi/T}^{\pi/T} \sum_n C_n^2 \Phi_n \, du$$

subject to the constraints given by (6), (7), and (9).
Note that (6) contains the constraint on r_0, given by (8).

III. RESULTS OF THE SEPARATE OPTIMIZATION OF SIGNAL AND RECEIVER FILTER

3.1 *Simultaneous Reduction of Noise and Intersymbol Distortion*

Appendix B shows the details of the minimization of the *M.S.E.* given by (4) subject to the constraints of (8) and (9). For a specified receiver filter the optimum signal characteristics are given by

$$\sin(\alpha_k + \beta_k + \gamma_k) = 0 \tag{10}$$

and

$$A_k = \frac{T(2r_0 - \lambda_2/F)}{4\pi} \frac{B_k C_k}{\sum_k B_k^2 C_k^2 + \frac{\lambda_1 T}{2\pi}} \tag{11}$$

where it is assumed that $F \neq 0$ at any point. The constants λ_1 and λ_2 may be determined by using the constraining equations.

For a fixed signal, the optimum receiver characteristics are given by

$$\sin(\alpha_k + \beta_k + \gamma_k) = 0 \tag{10}$$

and

$$C_k = \frac{T(2r_0 - \lambda_2/F)}{4\pi} \frac{A_k B_k F/\Phi_k}{F \sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}}. \tag{12a}$$

The coefficient is found in Appendix B to be

$$\frac{T(2r_0 - \lambda_2/F)}{4\pi} = \frac{\dfrac{r_0 T}{2\pi}\left[\displaystyle\int_{-\pi/T}^{\pi/T} \frac{\sum_k \frac{A_k^2 B_k^2}{\Phi_k}\, du}{F \sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}} + \frac{1}{F}\int_{-\pi/T}^{\pi/T} \frac{\frac{T}{2\pi}\, du}{F \sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}}\right]}{\displaystyle\int_{-\pi/T}^{\pi/T} \frac{\sum_k \frac{A_k^2 B_k^2}{\Phi_k}\, du}{F \sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}}}. \tag{12b}$$

The ratio of the signal output to the square root of the mean square error is

$$\frac{r_0}{\sqrt{M.S.E.}} = \frac{2\pi}{T}\left[\frac{\displaystyle\int_{-\pi/T}^{\pi/T} \frac{F\,du}{\Delta} \int_{-\pi/T}^{\pi/T} \frac{\sum_k \frac{A_k^2 B_k^2}{\Phi_k}\, du}{\Delta} + \frac{T}{2\pi}\left(\int_{-\pi/T}^{\pi/T} \frac{du}{\Delta}\right)^2}{\displaystyle\int_{-\pi/T}^{\pi/T} \frac{\sum_k \frac{A_k^2 B_k^2}{\Phi_k}\, du}{\Delta}}\right]^{-\frac{1}{2}} \tag{13a}$$

where

$$\Delta = F \sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}.$$

This function may be approximated by

$$\frac{r_0}{\sqrt{M.S.E.}} \approx \frac{2\pi}{T}\left[\int_{-\pi/T}^{\pi/T} \frac{du}{\sum_k \frac{A_k^2 B_k^2}{\Phi_k}} - \frac{T}{2\pi}\int_{-\pi/T}^{\pi/T} \frac{du}{F\left(\sum_k \frac{A_k^2 B_k^2}{\Phi_k}\right)^2}\right]^{-\frac{1}{2}} \tag{13b}$$

for large transmitted power to noise power ratios

$$\frac{2\pi}{T} F \sum_k \frac{A_k^2 B_k^2}{\Phi_k} \gg 1.$$

In each case, the variable phase should be adjusted to produce zero phase (with any fixed delay removed) across the band

$$(\text{i.e., } \alpha_k = -\beta_k - \gamma_k).$$

The amplitude characteristics may be considered to be the product of a term which depends upon u and k and one that only depends upon u, e.g.,

$$C_k(u) = [B_k(u)A_k(u)/\Phi_k(u)]q(u) \tag{14}$$

where

$$q\left(\omega + \frac{2k\pi}{T}\right) = q\left(\omega + \frac{2l\pi}{T}\right)$$

for all k, l. The function $q(u)$ may then be considered as the output of a tapped delay line

$$\sum_p q_p e^{-j\omega pT}$$

since this function satisfies the condition on $q(u)$. Thus, the solutions may be interpreted as linear filters matched to the signal and channel followed by a tapped delay line, e.g.,

$$C(\omega) = [B(\omega)A(\omega)/\Phi(\omega)]\, q(\omega). \tag{15}$$

3.2 *Minimization of Noise Output Power with No Intersymbol Distortion*

The minimization of

$$\int_{-\pi/T}^{\pi/T} \sum_n C_n^2\Phi_n\, du + \lambda_3(u) \sum_n A_nB_nC_n \cos(\alpha_n + \beta_n + \gamma_n)$$
$$+ \lambda_4(u) \sum_n A_nB_nC_n \sin(\alpha_n + \beta_n + \gamma_n) + \lambda_1 \int_{-\pi/T}^{\pi/T} F \sum A_n^2\, du$$

subject to the constraints of (6, 7, 9) is similar to the previous case and will not be presented in detail. For a fixed receiver filter, the optimum signal characteristics are given by

$$\sin(\alpha_k + \beta_k + \gamma_k) = 0 \tag{10}$$

and

$$A_k = \frac{-\lambda_3(u)}{2\lambda_1 F} B_kC_k. \tag{16a}$$

The coefficient is easily found and

$$A_k = \frac{r_0 T}{2\pi} \frac{B_kC_k}{\sum_k B_k^2C_k^2}. \tag{16b}$$

For a fixed signal, the optimum receiver characteristics are given by

$$\sin(\alpha_k + \beta_k + \gamma_k) = 0 \tag{10}$$

and

$$C_k = \frac{-\lambda_3(u)}{2} \frac{A_k B_k}{\Phi_k}. \tag{17a}$$

Again, the coefficient is easily found and

$$C_k = \frac{r_0 T}{2\pi} \frac{\dfrac{A_k B_k}{\Phi_k}}{\sum_k \dfrac{A_k^2 B_k^2}{\Phi_k}}. \tag{17b}$$

The ratio of signal output to the noise output is

$$\frac{r_0}{\sqrt{\sigma^2}} = \frac{2\pi}{T} \left[\int_{-\pi/T}^{\pi/T} \frac{du}{\sum_k \dfrac{A_k^2 B_k^2}{\Phi_k}} \right]^{-\frac{1}{2}}. \tag{18}$$

This function is seen to be identical to the first term of (13b). The difference between (18) and (13b) is small under the condition that

$$\frac{2\pi}{T} F \sum_k \frac{A_k^2 B_k^2}{\Phi_k} \gg 1$$

and one does not sacrifice much in designing for no intersymbol interference. In this case, there is no significant advantage in using the true optimum design with large signal-to-noise ratio systems.

The interpretation of the above results is the same as for the previous minimization problem.

IV. JOINT OPTIMIZATION OF SIGNAL AND RECEIVER FILTER

4.1 *Simultaneous Reduction of Noise and Intersymbol Interference*

The over-all optimization requires the minimization of (4) with respect to both the signal and receiver functions A_k, C_k, α_k and γ_k. From the minimization details in Appendix B, this requires the solution of

$$\sin(\alpha_k + \beta_k + \gamma_k) = 0 \qquad \text{for all } k, \tag{10}$$

$$A_k = \frac{-2\pi}{\lambda_1 T} B_k C_k \left\{ \sum_n A_n B_n C_n - \frac{T}{4\pi} \left(2r_0 - \frac{\lambda_2}{F} \right) \right\}, \tag{19}$$

and

$$C_k = \frac{-2\pi}{T}\frac{A_k B_k F}{\Phi_k}\left\{\sum_n A_n B_n C_n - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right)\right\} \qquad \text{for all } k \quad (20)$$

where it is assumed $F \neq 0$. In some correlation schemes F may be zero at certain points and it is easily seen that $A_k = C_k = 0$ at these points. For (19) and (20) to have nontrivial solutions it is necessary that the determinant

$$1 - \left(\frac{2\pi}{T}\right)^2 \frac{B_k^{\,2} F}{\lambda_1 \Phi_k}\left\{\sum_n A_n B_n C_n - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right)\right\}^2$$

vanish. This requires

$$\sum_n A_n B_n C_n - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right) = \pm \frac{T}{2\pi}\frac{\lambda_1^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}} \qquad \text{for all } k. \quad (21)$$

In general (unless $\Phi_k^{\frac{1}{2}}/B_k$ is a constant with k) for any u the equation can only be satisfied for one k. This means that for any u there is only one value of k for which there is a nontrivial solution and for the other values of k the solutions vanish.

The question now becomes one of choosing, for each u, the proper A_k and C_k which leads to the true minimization. Combining (19) and (20)

$$A_k = \frac{\Phi_k^{\frac{1}{2}}}{\lambda_1^{\frac{1}{2}} F^{\frac{1}{2}}} C_k \qquad (22)$$

and using

$$A_k B_k C_k - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right) = \frac{-T}{4\pi}\frac{2\lambda_1^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}} \qquad (21)$$

one gets

$$A_k^{\,2} = \frac{T}{4\pi}\left\{2r_0 - \frac{\lambda_2}{F} - \frac{2\lambda_1^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}}\right\}\frac{\Phi_k^{\frac{1}{2}}}{\lambda_1^{\frac{1}{2}} B_k F^{\frac{1}{2}}} \qquad (23)$$

$$C_k^{\,2} = \frac{T}{4\pi}\left\{2r_0 - \frac{\lambda_2}{F} - \frac{2\lambda_1^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}}\right\}\frac{\lambda_1^{\frac{1}{2}} F^{\frac{1}{2}}}{\Phi_k^{\frac{1}{2}} B_k}. \qquad (24)$$

Equations (23) and (24) represent the nonzero solutions. The best k (frequency region) for any u is the one which minimizes the function

$$\int_{-\pi/T}^{\pi/T}\left[\sum_k C_k^{\,2}(u)\Phi_k(u) + \frac{2\pi F}{T}\left\{\sum_k A_k(u)B_k(u)C_k(u) - \frac{r_0 T}{2\pi}\right\}^2\right] du$$

where the dependence on u has been reinserted for clarity. For every u there is at most one C_k and A_k with a nonzero value so the function may be written

$$\sum_k \int_{L_k} \left[C_k^2(u)\Phi_k(u) + \frac{2\pi}{T} F(u) \left\{ A_k(u)B_k(u)C_k(u) - \frac{r_0 T}{2\pi} \right\}^2 \right] du$$

where L_k is the subinterval within the interval

$$[(2k - 1)\ (\pi/T), \qquad (2k + 1)\ (\pi/T)]$$

over which $A_k(u)$ and $C_k(u)$ have a value. It is necessary for the true optimum that L_k be chosen to minimize the above expression. The true minimization may not require the use of the total interval $-\pi/T < u < \pi/T$, i.e., it may be possible to decrease noise output faster than intersymbol distortion is increased by using a smaller interval. Because this situation only arises when $\Phi_k^{\frac{1}{2}}/B_k$ in the optimum L_k regions have large variation, and is cumbersome to consider in detail, it will be assumed that nonzero A_k and C_k will exist over $-\pi/T < u < \pi/T$ (unless $F = 0$). Using (21, 23, 24) the above function may be written

$$\sum_k \int_{L_k} \left[\frac{T}{4\pi} \left\{ 2r_0 - \frac{\lambda_2}{F(u)} - \frac{2\lambda_1^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}(u)}{B_k(u)F^{\frac{1}{2}}(u)} \right\} \frac{\lambda_1^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}(u)F^{\frac{1}{2}}(u)}{B_k(u)} + \frac{T}{4\pi}\frac{F(u)}{2} \left\{ -\frac{\lambda_2}{F(u)} - \frac{2\lambda_1^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}(u)}{B_k(u)F^{\frac{1}{2}}(u)} \right\}^2 \right] du.$$

The constants may be evaluated by using (21, 23, 24) and the constraining equations. Thus,

$$r_0 = \int_{-\pi/T}^{\pi/T} \sum_k A_k B_k C_k \, du \tag{8}$$

$$= \sum_k \int_{L_k} \frac{T}{4\pi} \left(2r_0 - \frac{\lambda_2}{F} - \frac{2\lambda_1^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}} \right) du \tag{25a}$$

$$= r_0 - \frac{\lambda_2 T}{4\pi} \sum_k \int_{L_k} \frac{du}{F} - \frac{T\lambda_1^{\frac{1}{2}}}{2\pi} \sum_k \int_{L_k} \frac{\Phi_k^{\frac{1}{2}}}{B_k} du \tag{25b}$$

since the total range of all L_k is the interval on u $(-\pi/T, \pi/T)$ and

$$\frac{\lambda_2}{\lambda_1^{\frac{1}{2}}} = -2 \sum_k \int_{L_k} \frac{\Phi_k^{\frac{1}{2}}}{B_k} du \Big/ \sum_k \int_{L_k} \frac{du}{F}. \tag{26}$$

Using the average power condition

$$P = \int_{-\pi/T}^{\pi/T} F \sum_k A_k^2 \, du \tag{9}$$

$$= \sum_k \int_{L_k} \frac{T}{4\pi} \left\{ 2r_0 - \frac{\lambda_2}{F} - \frac{2\lambda_1^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}} \right\} \frac{\Phi_k^{\frac{1}{2}}F^{\frac{1}{2}}}{\lambda^{\frac{1}{2}}B_k} \tag{27a}$$

$$\frac{r_0 T}{2\pi\lambda_1^{\frac{1}{2}}} \sum_k \int_{L_k} \frac{F^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k}\, du = P + \frac{T}{2\pi} \sum_k \int_{L_k} \frac{\Phi_k}{B_k^2}\, du + \frac{T}{4\pi} \frac{\lambda_2}{\lambda_1^{\frac{1}{2}}} \sum_k \int_{L_k} \frac{\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}}\, du. \tag{27b}$$

Combining (27b) and (26) one gets

$$\lambda_1^{\frac{1}{2}} = \frac{r_0 T}{2\pi} \sum_k \int_{L_k} \frac{F^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k}\, du \left[P + \frac{T}{2\pi} \sum_k \int_{L_k} \frac{\Phi_k}{B_k^2}\, du - \frac{T}{2\pi} \frac{\left(\sum_k \int_{L_k} \frac{\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}}\, du \right)^2}{\sum_k \int_{L_k} \frac{du}{F}} \right]^{-1}. \tag{28}$$

Finally, combining (21–23, 25 and 27)

$$M.S.E. = \frac{\left(\frac{r_0 T}{2\pi} \sum_k \int_{L_k} \frac{F^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k}\, du \right)^2}{\left[P + \frac{T}{2\pi} \sum_k \int_{L_k} \frac{\Phi_k}{B_k^2}\, du - \frac{T}{2\pi} \frac{\left(\sum_k \int_{L_k} \frac{\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}}\, du \right)^2}{\sum_k \int_{L_k} \frac{du}{F}} \right]^2}. \tag{29}$$

Equation (29) represents the mean square error in the transmission and L_k must be chosen to minimize this expression. This is rather complicated but to a close approximation, the minimization of

$$\sum_k \int_{L_k} \frac{F^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k}\, du$$

corresponds to the minimization of (29). The minimum mean square error results by choosing for each value of u

$$\left(-\frac{\pi}{T} \leqq u \leqq \frac{\pi}{T} \right)$$

the k for which

$$\frac{\Phi_k^{\frac{1}{2}}(u) F^{\frac{1}{2}}(u)}{B_k(u)}$$

is a minimum. The interval L_k is that region of u for which

$$\frac{\Phi_k^{\frac{1}{2}}(u)F^{\frac{1}{2}}(u)}{B_k(u)}$$

is a smaller than any other

$$\frac{\Phi_j^{\frac{1}{2}}(u)F^{\frac{1}{2}}(u)}{B_j(u)}.$$

Since $F(u)$ is independent of j, L_k may also be defined as the region for which $\Phi_k^{\frac{1}{2}}(u)/B_k(u)$ is smaller than any other $\Phi_j^{\frac{1}{2}}(u)/B_j(u)$. Thus, the choice of L_k to minimize

$$\sum_k \int_{L_k} \frac{\Phi_k^{\frac{1}{2}}}{B_k}\, du$$

also corresponds to the minimization of the $M.S.E.$ and the regions in which power is transmitted are independent of the correlation properties of the data.

Once the L_k are chosen, the resulting characteristics are

$$A_k^2 = \left[\frac{P + \dfrac{T}{2\pi}\displaystyle\sum_k \int_{L_k} \frac{\Phi_k}{B_k^2}\, du - \frac{T}{2\pi}\left(\sum_k \int_{L_k} \frac{\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}}\, du\right)^2 \Big/ \sum_k \int_{L_k} \frac{du}{F}}{\displaystyle\sum_k \int_{L_k} \frac{\Phi_k^{\frac{1}{2}} F^{\frac{1}{2}}}{B_k}\, du}\right] \times \frac{\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}} + \left[\frac{T}{2\pi}\sum_k \int_{L_k} \frac{\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}}\, du \Big/ \sum_k \int_{L_k} \frac{du}{F}\right] \frac{\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}} - \frac{T}{2\pi}\frac{\Phi_k}{B_k^2 F} \quad (30)$$

and

$$C_k^2 = \frac{\lambda_1 F}{\Phi_k} A_k^2 \quad (22)$$

where λ_1 is given by (28). For (30) to be a solution requires that $A_k^2 \geqq 0$ or

$$\frac{\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}} < K_1 + \frac{K_2}{F} \quad (31)$$

for the regions defined by the L_k. Taking into account the difference in problems considered, this equation corresponds to Tuft's condition for a solution [Ref. 5, (28)]. Equation (31) states that the solution must be limited to regions without sharp peaks in the $\Phi_k^{\frac{1}{2}}/B_k F^{\frac{1}{2}}$ characteristic. The phases, α_k and γ_k, may be arbitrarily assigned to A_k and C_k subject to the condition of (10). The ratio of the signal to the square root of the mean square distortion is

$$\frac{r_0}{\sqrt{M.S.E.}} = \frac{2\pi}{T}$$

$$\times \frac{\left[P + \frac{T}{2\pi}\sum_k \int_{L_k} \frac{\Phi_k}{B_k^2}\,du - \frac{T}{2\pi}\left(\sum_k \int_{L_k} \frac{\Phi_k^{\frac{1}{2}}}{B_k F^{\frac{1}{2}}}\,du\right)^2 \Big/ \sum_k \int_{L_k} \frac{du}{F}\right]^{\frac{1}{2}}}{\sum_k \int_{L_k} \frac{F^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k}\,du}. \qquad (32)$$

It is difficult to make a general comparison of this result and (13) because of the arbitrary signal, A_k, which appears in (13). A comparison for a specific case is found in the examples of Section V.

4.2 *Minimization of Noise Output with No Intersymbol Interference*

From the previous results, the minimization of the noise output with respect to both the signal and receiver functions, A_k, C_k, α_k and γ_k requires the simultaneous solution of

$$\sin(\alpha_k + \beta_k + \gamma_k) = 0 \qquad \text{for all } k, \qquad (10)$$

$$A_k = \frac{-\lambda_3(u)}{2\lambda_1 F} B_k C_k \qquad (16a)$$

and

$$C_k = \frac{-\lambda_3(u)}{2} \frac{A_k B_k}{\Phi_k} \qquad \text{for all } k \qquad (17a)$$

where again nonzero F is assumed (if $F = 0$, $C_k = 0$, $A_k = \infty$). A nontrivial solution of (16a) and (17a) requires that the determinant

$$1 - \frac{\lambda_3^2(u)}{4\lambda_1 F} \frac{B_k^2}{\Phi_k}$$

vanish. This requires

$$\lambda_3(u) = \pm \frac{2\lambda_1^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}F^{\frac{1}{2}}}{B_k}. \qquad (33)$$

Again, for a particular u, the right hand side of the equation depends upon k and can only be satisfied for one value of k. For the other values of k, (16a) and (17a) have trivial solutions. At each value of u there must be at least one nontrivial A_k and C_k because of the condition

$$\sum_k A_k B_k C_k = \frac{r_0 T}{2\pi} \qquad -\frac{\pi}{T} \leqq u \leqq \frac{\pi}{T}. \qquad (6)$$

Thus, for each value of u there is one and only one nonzero A_k and C_k and the question becomes one of finding which k leads to the true optimum.

Using the same techniques as shown in the previous section, the noise output is given by

$$\sum_k \int_{L_k} C_k^2(u)\Phi_k(u)du = \left(\frac{r_0 T}{2\pi}\right)^2 \frac{1}{P} \left\{\sum_k \int_{L_k} \frac{F^{\frac{1}{2}}(u)\Phi_k^{\frac{1}{2}}(u)}{B_k(u)}\, du\right\}^2. \quad (34)$$

This expression is minimized by choosing for each value of u, the k for which

$$\frac{\Phi_k^{\frac{1}{2}}(u)}{B_k(u)} \qquad \left(\text{or } \frac{\Phi_k^{\frac{1}{2}}(u)}{B_k(u)}\, F^{\frac{1}{2}}(u)\right)$$

is a minimum. The interval L_k is that region of u for which

$$\frac{\Phi_k^{\frac{1}{2}}(u)}{B_k(u)}$$

is smaller than any other

$$\frac{\Phi_j^{\frac{1}{2}}(u)}{B_j(u)}.$$

The resulting characteristics are easily found to be

$$A_k = P^{\frac{1}{2}} \left[\sum_k \int_{L_k} \frac{F^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k}\, du\right]^{-\frac{1}{2}} \frac{\Phi_k^{\frac{1}{4}}}{B_k^{\frac{1}{2}}F^{\frac{1}{4}}} \quad (35)$$

and

$$C_k = \frac{r_0 T}{2\pi}\, P^{-\frac{1}{2}} \left[\sum_k \int_{L_k} \frac{F^{\frac{1}{2}}\Phi_k^{\frac{1}{2}}}{B_k}\, du\right]^{\frac{1}{2}} \frac{F^{\frac{1}{4}}}{\Phi_k^{\frac{3}{4}}B_k^{\frac{1}{2}}}. \quad (36)$$

Note here and in the previous case, the bound depends upon the ratio $\Phi_k^{\frac{1}{2}}/B_k$ but the transmitting and receiving filters required to achieve the bound depend upon Φ_k and B_k separately.

Again, the phases, α_k and γ_k, may be arbitrarily assigned to the transmitter and receiver subject to the condition of (10). The ratio of the signal output to the noise output is

$$\frac{r_0}{\sqrt{\overline{\sigma^2}}} = \frac{2\pi}{T}\, P^{\frac{1}{2}} \left[\sum_k \int_{L_k} \frac{F^{\frac{1}{2}}\,\Phi_k^{\frac{1}{2}}}{B_k}\, du\right]^{-1}. \quad (37)$$

Comparing this equation and (32) shows the degradation in the performance bound to be expected when no intersymbol interference is allowed. This degradation is small under the usual condition of low noise. The two equations are identical if

$$\frac{\Phi_k^{\frac{1}{2}}(u)}{B_k(u)} = \frac{K}{F^{\frac{1}{2}}(u)} .$$

If this condition holds, the optimum solution is one with no intersymbol interference. This condition is especially useful with uncorrelated data where F is a constant.

V. EXAMPLES

Consider the problem of transmitting uncorrelated binary data ($a_l = -1$ or 1, and $F = 1$) at a rate of $1/T$ bits per second over the channel whose amplitude characteristics $B(\omega)$ and noise spectral density function $\Phi(\omega)$ are shown in Fig. 2. For ease of presentation these functions are limited to the region $(-2\pi/T, 2\pi/T)$ and the phase characteristic is assumed to be zero. The preceding discussion will be illustrated by assuming a fixed signal and then finding the optimum receiver filter. The bound for the utilization of this channel will then be found by finding the joint optimum receiver and signal combination. For this purpose, the function

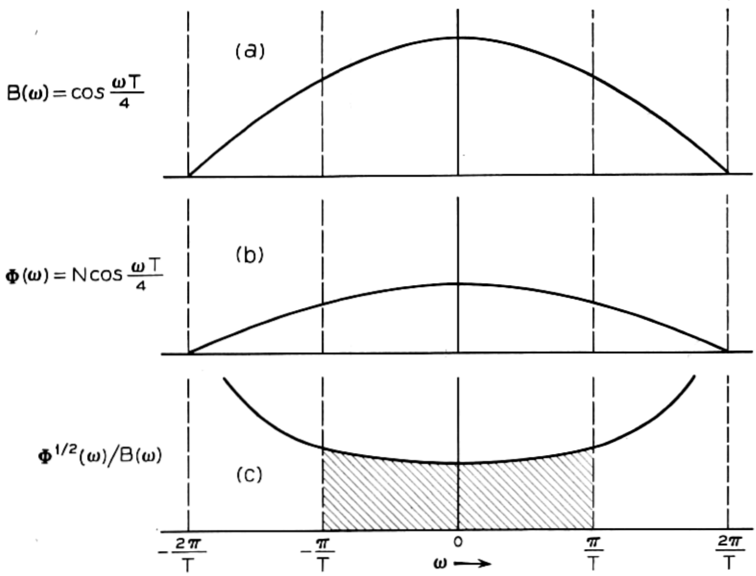

Fig. 2 — (a) Amplitude characteristic; (b) noise spectral density; (c) optimum operating region.

$$\frac{\Phi^{\frac{1}{2}}(\omega)}{B(\omega)}$$

is given in Fig. 2(c).

The various quantities of interest are given in the figure and it will be assumed that the initial choice of a signal gives a cosine roll-off and produces no intersymbol interference

$$A(\omega) = \sqrt{\frac{PT}{2\pi}} \cos \frac{\omega T}{4} \tag{38}$$

where the signal normalized to the average power restriction. The initial receiver filter is assumed to be flat. For this system

$$\frac{r_0}{\sqrt{\sigma^2}} = \sqrt{\frac{PT}{2\pi N}} \frac{\int_{-2\pi/T}^{2\pi/T} \cos^2 \frac{\omega T}{4}\, d\omega}{\left[\int_{-2\pi/T}^{2\pi/T} \cos \frac{\omega T}{4}\, d\omega\right]^{\frac{1}{2}}} = \sqrt{\frac{P\pi}{4N}} = 0.88 \sqrt{\frac{P}{N}}. \tag{39}$$

For no intersymbol interference, the optimum receiver filter is given by

$$C_k = \frac{r_0 T}{2\pi} \frac{\dfrac{A_k B_k}{\Phi_k}}{\sum_k \dfrac{A_k^2 B_k^2}{\Phi_k}} \tag{17b}$$

$$= \frac{r_0 T}{2\pi} \sqrt{\frac{2\pi}{PT}} \frac{B_k}{\cos^3 \frac{uT}{4} - \sin^3 \frac{uT}{4}} \qquad -\frac{\pi}{T} \leqq u \leqq 0 \tag{40a}$$

$$C_k = \frac{r_0 T}{2\pi} \sqrt{\frac{2\pi}{PT}} \frac{B_k}{\cos^3 \frac{uT}{4} + \sin^3 \frac{uT}{4}} \qquad 0 \leqq u \leqq \frac{\pi}{T} \tag{40b}$$

and

$$\frac{r_0}{\sqrt{\sigma^2}} = \frac{2\pi}{T} \left[\int_{-\pi/T}^{\pi/T} \frac{du}{\sum_k \dfrac{A_k^2 B_k^2}{\Phi_k}}\right]^{-\frac{1}{2}} \tag{18}$$

$$= \sqrt{\frac{2\pi P}{NT}} \left[2 \int_0^{\pi/T} \frac{du}{\cos^3 \frac{uT}{4} + \sin^3 \frac{uT}{4}}\right]^{-\frac{1}{2}} \tag{41a}$$

$$\approx \sqrt{\frac{2\pi P}{NT}} \left[\frac{2.4\pi}{T}\right]^{-\frac{1}{2}} = 0.91 \sqrt{\frac{P}{N}}. \tag{41b}$$

For the optimum filter where intersymbol interference is allowed,

$$C_k = \frac{r_0\sqrt{\dfrac{2\pi}{PT}}}{2\displaystyle\int_0^{\pi/T} \frac{\cos^3\dfrac{uT}{4}+\sin^3\dfrac{uT}{4}\,du}{\cos^3\dfrac{uT}{4}+\sin^3\dfrac{uT}{4}+\dfrac{N}{P}}} \frac{B_k}{\cos^3\dfrac{uT}{4}+\sin^3\dfrac{uT}{4}+\dfrac{N}{P}} \tag{42}$$

$$0 \leqq u \leqq \frac{\pi}{T}$$

which will not be much different from the result with no intersymbol interference for low noise. The optimum detector with no intersymbol interference (40a,b) is shown on Fig. 3(a).

The joint optimization requires constraining $C(\omega)$ and $A(\omega)$ to the region $-\pi/T \leqq u \leqq \pi/T$. From Fig. 2(c) this is seen to be the region for which

$$\frac{\Phi^{\frac{1}{2}}(\omega)}{B(\omega)}$$

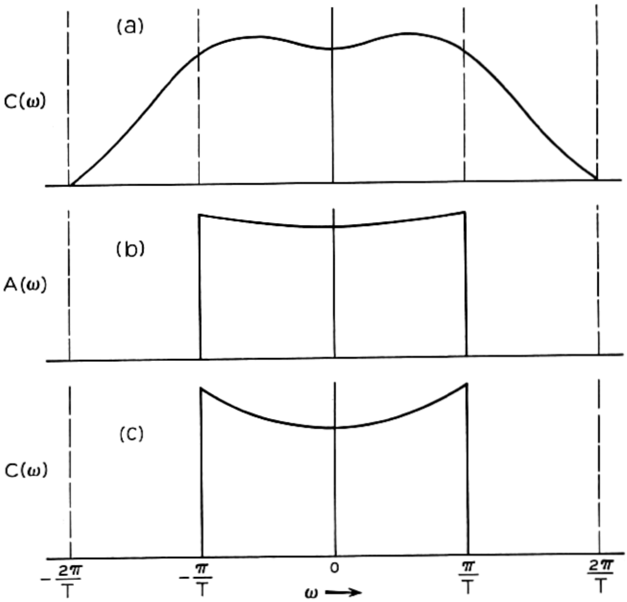

Fig. 3 — (a) Optimum receiver shaping for given $A(\omega)$; (b) signal shaping for joint optimization; (c) receiver shaping for joint optimization.

is a minimum. The optimum signal and receiver for no intersymbol interference are, using (35, 36),

$$A_0 = \frac{P^{\frac{1}{2}}}{\left\{\displaystyle\int_{-\pi/T}^{\pi/T} \frac{du}{\cos^{\frac{1}{2}} \frac{uT}{4}}\right\}^{\frac{1}{2}}} \frac{1}{\cos^{\frac{1}{4}} \frac{uT}{4}} \tag{43}$$

$$C_0 = \frac{r_0 T}{2\pi P^{\frac{1}{2}}} \frac{\left\{\displaystyle\int_{-\pi/T}^{\pi/T} \frac{du}{\cos^{\frac{1}{2}} \frac{uT}{4}}\right\}^{\frac{1}{2}}}{\cos^{\frac{3}{4}} \frac{uT}{4}} \tag{44}$$

and are shown in Figs. 3(b) and 3(c). The resulting ratio of signal-to-noise is from (37)

$$\frac{r_0}{\sigma} = \frac{2\pi}{T} \frac{P^{\frac{1}{2}}}{N^{\frac{1}{2}}} \left[\int_{-\pi/T}^{\pi/T} \frac{du}{\cos^{\frac{1}{2}} \frac{uT}{4}}\right]^{-1} \tag{45a}$$

$$\approx 0.95 \sqrt{\frac{P}{N}}. \tag{45b}$$

The true optimum filters, with intersymbol interference, will be close to the previous results for low noise. The improvement in signal-to-noise ratio may be obtained from (45b) by replacing P by

$$P + \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N}{\cos \frac{uT}{4}} \, du - \left(\frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} \frac{N^{\frac{1}{2}} \, du}{\cos^{\frac{1}{2}} \frac{uT}{4}}\right)^2$$

which is greater than P.

As a further illustration, consider the utilization of the channel whose amplitude characteristic $B(\omega)$ and noise spectral density function $\Phi(\omega)$ are shown in Fig. 4. The main point of this example is that the solution of the joint optimization problem is not confined to the interval $(-\pi/T, \pi/T)$. It is, as a matter of fact, split up into three regions; L_1 for $-\pi/T \leqq u < -a$, L_0 for $-a < u < a$, and L_{-1} for $a < u \leqq \pi/T$ as shown in Fig. 4(c). These turn out to be the regions of minimum

$$\frac{\Phi_k^{\frac{1}{2}}(u)}{B_k(u)}$$

as can be seen by the curve Fig. 4(c). The optimum receiver and signal are given in Fig. 5 for the case of no intersymbol interference.

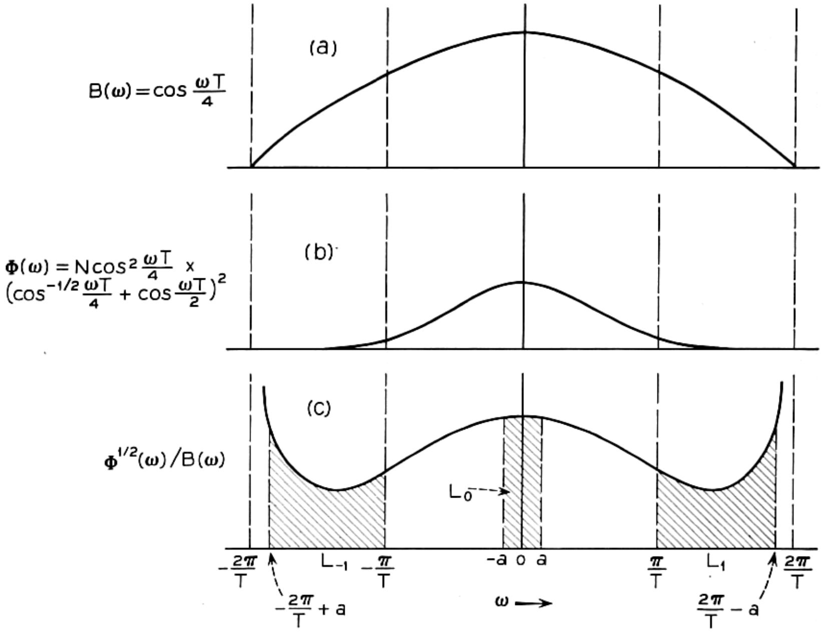

Fig. 4 — (a) Amplitude characteristic; (b) noise spectral density; (c) optimum operating region.

The previous two examples have considered the problem of finding the optimum regions, L_k, for transmitting energy. Once these regions have been determined it is of interest to examine the effect of $F(u)$ (the one parameter over which the designer may exercise some control) on the performance bound. In particular, since it is generally considered that a binary system is optimum for transmitting data in fixed time slots, it is interesting to ask if some sort of correlation scheme leads to a better output signal-to-noise ratio than binary for the same average input power. In other words, using (37) [the case for no intersymbol interference] are there cases in which

$$\sum_k \int_{L_k} \frac{F^{\frac{1}{2}}(u)\Phi_k^{\frac{1}{2}}(u)}{B_k(u)}\,du \leqq \sum_k \int_{L_k} \frac{\Phi_k^{\frac{1}{2}}(u)}{B_k(u)}\,du? \tag{46}$$

It should be emphasized that the signal-to-noise ratio is being examined and not the error rate. For correlative multilevel schemes which lead to a solution of (46) the error rate may be greater than or less than that of the binary system. In addition, there are certain error detecting

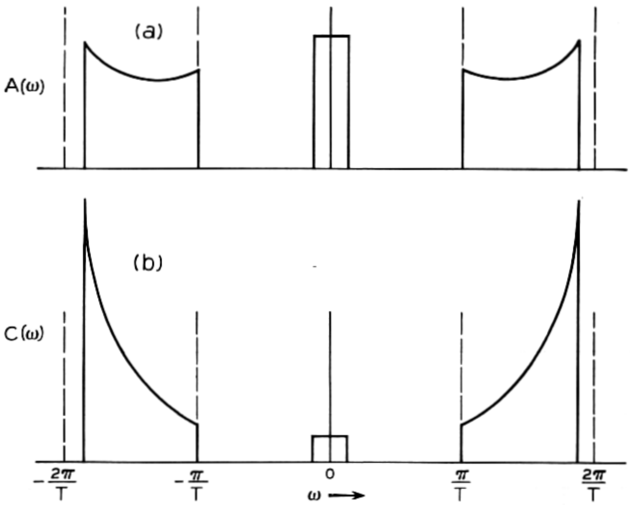

Fig. 5 — (a) Signal shaping for joint optimization; (b) receiver shaping for joint optimization.

features of the correlative multilevel systems[10] which make it difficult to compare its error rate to binary.

Equation (46) will be easier to consider if it is written

$$\int_{-\pi/T}^{\pi/T} \frac{F^{\frac{1}{2}}(u)\Phi^{\frac{1}{2}}(u)}{B(u)}\, du \leqq \int_{-\pi/T}^{\pi/T} \frac{\Phi^{\frac{1}{2}}(u)}{B(u)}\, du \tag{47}$$

where $\Phi(u)$ and $B(u)$ are the composites of $\Phi_k(u)$ and $B_k(u)$ in the (generally) disjoint regions L_k in ω. The total interval over which the L_k's extend is $-\pi/T < u < \pi/T$. For the case shown in Fig. 4 one would have

$$\frac{\Phi^{\frac{1}{2}}(u)}{B(u)} = \begin{cases} \Phi_1^{\frac{1}{2}}(u)/B_1(u) & -\pi/T < u \leqq -a, \\ \Phi_0^{\frac{1}{2}}(u)/B_0(u) & -a \leqq u \leqq a, \\ \Phi_{-1}^{\frac{1}{2}}(u)/B_{-1}(u) & a \leqq u < \pi/T. \end{cases}$$

As a specific example, consider correlated data which is formed by adding binary $(-1, 1)$ bits separated by T seconds (i.e., adding continuously the present bit and the immediately preceding bit). Such data (duobinary) has three levels $(-2, 0, 2)$ and

$$F(u) = 2 + 2 \cos uT.$$

It is easily shown that if

$$\frac{\Phi^{\frac{1}{2}}(u)}{B(u)} = d_0 - d_1 \cos \frac{uT}{2}$$

where

$$d_0 \geqq d_1 \geqq \frac{8 - 2\pi}{2\pi - 4} d_0 \approx 0.75\ d_0$$

(47) is satisfied. This means that for the above characteristic the optimum duobinary transmission system operates with greater output S/N than the best binary system. It is obvious that for

$$\frac{\Phi^{\frac{1}{2}}(u)}{B(u)}$$

constant, (the usual case considered) binary transmission is optimum (with duobinary poorer by a factor of $\pi/4$ as pointed out by Bennett[11]).

This example is not intended to exhaust the possibilities but to point out that for deviations from flat channels and white noise there is the possibility that signaling with correlated multilevel data is superior to binary.

VI. CONCLUDING REMARKS

The joint receiver and signal optimization has been carried out for the general data transmission system. The optimization considers simultaneously the noise and intersymbol distortion. The results may be applied to bandpass (baseband transmission, not carrier) or gradual cutoff systems as well as the sharp cutoff systems illustrated in the examples. Unfortunately, the joint solution is unrealizable, but it does provide a bound on the transmission performance with a fixed channel. Thus, with a given channel, one can do no better than (32) for the ratio of signal mean square error, and no better than (37) for the output signal-to-noise ratio with no intersymbol interference. From these equations, the importance of the ratio

$$\frac{\Phi^{\frac{1}{2}}(\omega)}{B(\omega)}$$

is clearly seen. It is the only contribution to the bound other than the signal correlation function F, and the average signal power. Notice that the phase characteristic of the channel is irrelevant to the bound.

In the first example, it is seen that the original choice of signal and receiver was close to the optimum bound. Hence, there is little improvement to be gained with more complex processing. The second example illustrates a case where a more complex signal and receiver would be theoretically useful. The signal and receiver of the first example would not effectively combat the noise in this case. The last example illustrates an advantage of correlative multilevel signaling in matching certain channel characteristics.

In the examples, the suboptimum (no intersymbol interference) solutions are presented in detail for three reasons:

(*i*) The solutions are of simpler form but just as illustrative as the optimum solutions.

(*ii*) The solutions depend only on the shape of $B(\omega)$ and $\Phi(\omega)$ and not upon the relative noise and signal powers.

(*iii*) The optimum and suboptimum solutions are nearly identical in the standard situation of large signal-to-noise power.

The explicit results for the suboptimum case are possible because of the explicit and complete statement of conditions for no intersymbol interference [Equations (6) and (7)].

APPENDIX A

Frequency Domain Representation of Mean Square Error

By writing the time domain samples, r_l, in terms of the Fourier transform of $r(t)$ one gets

$$r(t) = \int_{-\infty}^{\infty} R(\omega)e^{j\omega t}\,d\omega \tag{48a}$$

$$r_l = \int_{-\infty}^{\infty} R(\omega)e^{j\omega lT}\,d\omega \tag{48b}$$

$$r_l = \sum_{n=-\infty}^{\infty} \int_{(\pi/T)(2n-1)}^{(\pi/T)(2n+1)} R(\omega)e^{j\omega lT}\,d\omega \tag{48c}$$

$$= \sum_{n=-\infty}^{\infty} \int_{-\pi/T}^{\pi/T} R\left(u + \frac{2n\pi}{T}\right) e^{julT}\,du. \tag{48d}$$

Assuming that

$$\sum_n R\left(u + \frac{2n\pi}{T}\right) e^{julT}$$

is a uniformly convergent series

$$\left(| R(\omega) | \to \frac{1}{\omega^q}, \qquad q \geqq 2, \qquad \text{as } \omega \to \infty \right)$$

$$r_l = \int_{-\pi/T}^{\pi/T} \sum_{n=-\infty}^{\infty} R\left(u + \frac{2n\pi}{T}\right) e^{julT}\,du. \tag{48e}$$

Noting that r_l is just the lth coefficient of an exponential Fourier series expansion of

$$\frac{2\pi}{T} \sum_{n=-\infty}^{\infty} R\left(u + \frac{2n\pi}{T}\right) \qquad -\frac{\pi}{T} \leqq u \leqq \frac{\pi}{T},$$

one may write

$$\sum_{n=-\infty}^{\infty} R\left(u + \frac{2n\pi}{T}\right) = \frac{T}{2\pi} \sum_{l=-\infty}^{\infty} r_l e^{-julT} \tag{49a}$$

or

$$\sum_{n=-\infty}^{\infty} R\left(u + \frac{2n\pi}{T}\right) - \frac{Tr_0}{2\pi} = \frac{T}{2\pi} \sum_{l\neq 0} r_l e^{-julT} \tag{49b}$$

and

$$\left[\sum_{n=-\infty}^{\infty} R\left(u + \frac{2n\pi}{T}\right) - \frac{Tr_0}{2\pi}\right]^* e^{jumT} = \frac{T}{2\pi} \sum_{l\neq 0} r_l e^{julT} e^{jumT}. \tag{49c}$$

Multiplying (49b) and (49c) together and integrating one obtains, using the shorthand notation

$$R_n(u) = R\left(u + \frac{2n\pi}{T}\right),$$

$$\sum_{l\neq 0,-m} r_l r_{l+m} = \frac{2\pi}{T} \int_{-\pi/T}^{\pi/T} \left| \sum_n R_n(u) - \frac{r_0 T}{2\pi} \right|^2 \cos umT \, du. \tag{50}$$

Further,

$$\overline{a_j^2} \sum_{l\neq 0} r_l^2 + 2 \sum_{m=1}^{\infty} \overline{a_j a_{j+m}} \sum_{l\neq 0,-m} r_l r_{l+m} = \frac{2\pi}{T} \int_{-\pi/T}^{\pi/T} \left| R_n(u) - \frac{r_0 T}{2\pi} \right|^2 F(u) \, du \tag{51}$$

where

$$F(u) = \overline{a_j^2} + 2 \sum_{m=1}^{\infty} \overline{a_j a_{j+m}} \cos umT. \tag{5}$$

The noise variance may be written

$$E\{n_0^2\} = \int_{-\infty}^{\infty} | E(\omega)|^2 \Phi(\omega) \, d\omega \tag{52a}$$

where $\Phi(\omega)$ is the channel noise power spectral density

$$E\{n_0^2\} = \sum_n \int_{\pi/T(2n-1)}^{\pi/T(2n+1)} | E(\omega)|^2 \Phi(\omega) \, d\omega \tag{52b}$$

$$= \sum_n \int_{-\pi/T}^{\pi/T} \left| E\left(u + \frac{2n\pi}{T}\right) \right|^2 \Phi\left(u + \frac{2n\pi}{T}\right) du \qquad (52c)$$

$$= \int_{-\pi/T}^{\pi/T} \sum_n | E_n(u) |^2 \Phi_n(u) \, du \qquad (52d)$$

if $\sum_n | E_n(u)|^2 \Phi_n(u)$ is a uniformly convergent series. The mean square error expression may then be written

$$M.S.E. = \int_{-\pi/T}^{\pi/T} \left\{ \sum_n | E_n(u) |^2 \Phi_n(u) + \frac{2\pi}{T}\left[\operatorname{Re} \sum_n R_n(u) - \frac{r_0 T}{2\pi} \right]^2 F(u) + \frac{2\pi}{T}\left[\operatorname{Im} \sum_n R_n(u) \right]^2 F(u) \right\} du. \qquad (53)$$

In terms of the system component functions

$$S(\omega) = A(\omega)e^{j\alpha(\omega)}$$
$$T(\omega) = B(\omega)e^{j\beta(\omega)}$$
$$E(\omega) = C(\omega)e^{j\gamma(\omega)}$$

(53) may be written

$$M.S.E. = \int_{-\pi/T}^{\pi/T} \left\{ \sum_n C_n^2(u)\Phi_n(u) + \frac{2\pi}{T}\left[\sum_n A(u)B_n(u)C_n(u) \cos\left[\alpha_n(u) + \beta_n(u) + \gamma_n(u)\right] - \frac{r_0 T}{2\pi} \right]^2 F(u) + \frac{2\pi}{T}\left[\sum_n A_n(u)B_n(u)C_n(u) \sin\left[\alpha_n(u) + \beta_n(u) + \gamma_n(u)\right] \right]^2 F(u) \right\} du. \qquad (4)$$

APPENDIX B

Simultaneous Reduction of Noise and Intersymbol Interference

The function

$$\int_{-\pi/T}^{\pi/T} \left[\sum_n C_n^2 \Phi_n + \frac{2\pi F}{T} \left\{ \sum_n A_n B_n C_n \cos\left(\alpha_n + \beta_n + \gamma_n\right) - \frac{r_0 T}{2\pi} \right\}^2 + \frac{2\pi F}{T} \left\{ \sum_n A_n B_n C_n \sin\left(\alpha_n + \beta_n + \gamma_n\right) \right\}^2 \right] du$$

$$+ \lambda_1 \int_{-\pi/T}^{\pi/T} F \sum_n A_n^2 \, du$$

$$+ \lambda_2 \int_{-\pi/T}^{\pi/T} \sum_n A_n B_n C_n \cos (\alpha_n + \beta_n + \gamma_n) du$$

is to be minimized subject to the constraints

$$\int_{-\pi/T}^{\pi/T} F \sum A_n^2 \, du = P \tag{9}$$

and

$$\int_{-\pi/T}^{\pi/T} \sum_n A_n B_n C_n \cos (\alpha_n + \beta_n + \gamma_n) du = r_0 . \tag{8}$$

For the optimum signal, this minimization is carried out with respect to A_k and α_k. Using the standard techniques of the calculus of variations[9] the minimization with respect to α_k yields

$$\begin{aligned} A_k B_k C_k \sin (\alpha_k + \beta_k + \gamma_k) \Big[\sum_n A_n B_n C_n \cos (\alpha_n + \beta_n + \gamma_n) \\ - \frac{T}{4\pi} \left(2r_0 - \frac{\lambda_2}{F} \right) \Big] - A_k B_k C_k \cos (\alpha_k + \beta_k + \gamma_k) \\ \cdot \Big[\sum_n A_n B_n C_n \sin (\alpha_n + \beta_n + \gamma_n) \Big] = 0, \end{aligned} \tag{54}$$

assuming that $F \neq 0$. The special case when $F = 0$ will be considered later. Summing over all k leads to

$$\left(2r_0 - \frac{\lambda_2}{F} \right) \sum_k A_k B_k C_k \sin (\alpha_k + \beta_k + \gamma_k) = 0 \tag{55}$$

since λ_2 depends upon the constraints

$$\sum_k A_k B_k C_k \sin (\alpha_k + \beta_k + \gamma_k) = 0 \tag{56}$$

and

$$\begin{aligned} A_k B_k C_k \sin (\alpha_k + \beta_k + \gamma_k) \Big[\sum_n A_n B_n C_n \cos (\alpha_n + \beta_n + \gamma_n) \\ - \frac{T}{4\pi} \left((2r_0 - \frac{\lambda_2}{F} \right) \Big] = 0 \end{aligned} \tag{57}$$

Minimizing with respect to A_k one gets

$$\frac{4\pi}{T} F[B_k C_k \cos(\alpha_k + \beta_k + \gamma_k)] \left[\sum_n A_n B_n C_n \cos(\alpha_n + \beta_n + \gamma_n) - \frac{r_0 T}{2\pi} \right] + \frac{4\pi}{T} F[B_k C_k \sin(\alpha_k + \beta_k + \gamma_k)] \cdot \left[\sum_n A_n B_n C_n \sin(\alpha_n + \beta_n + \gamma_n) \right] + 2\lambda_1 F A_k + \lambda_2 B_k C_k \cos(\alpha_k + \beta_k + \gamma_k) = 0 \tag{58}$$

or using (56)

$$[B_k C_k \cos(\alpha_k + \beta_k + \gamma_k)] \left[\sum_n A_n B_n C_n \cos(\alpha_n + \beta_n + \gamma_n) - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right) \right] + \frac{2\lambda_1 T}{4\pi} A_k = 0. \tag{59}$$

Comparing (57) and (59) it is seen that a nontrivial solution requires

$$\sin(\alpha_k + \beta_k + \gamma_k) = 0 \tag{10}$$

and

$$A_k = -\frac{2\pi}{\lambda_1 T} B_k C_k \left\{ \sum_n A_n B_n C_n - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right) \right\}. \tag{19}$$

Multiplying each side of the equation by $B_k C_k$ and summing one gets

$$\sum_k A_k B_k C_k = -\frac{2\pi}{\lambda_1 T} \sum_k B_k^2 C_k^2 \left\{ \sum_n A_n B_n C_n - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right) \right\}. \tag{60}$$

$$\sum_k A_k B_k C_k = \frac{T}{4\pi} \frac{\left(2r_0 - \frac{\lambda_2}{F}\right) \sum_k B_k^2 C_k^2}{\sum_k B_k^2 C_k^2 + \frac{\lambda_1 T}{2\pi}} \tag{61}$$

and

$$\sum_k A_k B_k C_k - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right) = -\frac{T}{4\pi}\left(\frac{\lambda_1 T}{2\pi}\right) \frac{(2r_0 - \lambda_2/F)}{\sum_k B_k^2 C_k^2 + \frac{\lambda_1 T}{2\pi}}. \tag{62}$$

Finally,

$$A_k = \frac{T}{4\pi} \frac{(2r_0 - \lambda_2/F)}{\sum_k B_k^2 C_k^2 + \frac{\lambda_1 T}{2\pi}} B_k C_k . \tag{11}$$

For the optimum receiver filter, the minimization with respect to γ_k again yields (56) and (57) and with respect to C_k leads to

$$\frac{4\pi}{T} F[A_k B_k \cos(\alpha_k + \beta_k + \gamma_k)] \left[\sum_n A_n B_n C_n \cos(\alpha_n + \beta_n + \gamma_n) - \frac{r_0 T}{2\pi}\right] + \frac{4\pi}{T} F[A_k B_k \sin(\alpha_k + \beta_k + \gamma_k)] \left[\sum_n A_n B_n C_n \sin(\alpha_n + \beta_n + \gamma_n)\right] + 2C_k\Phi_k + \lambda_2 A_k B_k \cos(\alpha_k + \beta_k + \gamma_k) = 0 \tag{63}$$

or using (56)

$$[A_k B_k \cos(\alpha_k + \beta_k + \gamma_k)] \left[\sum_n A_n B_n C_n \cos(\alpha_n + \beta_n + \gamma_n) - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right)\right] + \frac{2T}{4\pi}\frac{C_k \Phi_k}{F} = 0. \tag{64}$$

Again, comparing this equation and (57) reveals that a nontrivial solution requires

$$\sin(\alpha_k + \beta_k + \gamma_k) = 0. \tag{10}$$

Then

$$C_k = -\frac{2\pi}{T}\frac{A_k B_k F}{\Phi_k}\left[\sum_n A_n B_n C_n - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right)\right]. \tag{20}$$

Multiplying each side of the equation by $A_k B_k$ and summing one gets

$$\sum_k A_k B_k C_k = -\frac{2\pi}{T} F \sum_k \frac{A_k^2 B_k^2}{\Phi_k}\left[\sum_n A_n B_n C_n - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right)\right] \tag{65}$$

$$\sum_k A_k B_k C_k = \frac{T}{4\pi}\frac{\left(2r_0 - \frac{\lambda_2}{F}\right) F \sum_k \frac{A_k^2 B_k^2}{\Phi_k}}{F \sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}} \tag{66}$$

and

$$\sum_k A_k B_k C_k - \frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right) = -\frac{T}{4\pi}\left(\frac{T}{2\pi}\right)\frac{\left(2r_0 - \frac{\lambda_2}{F}\right)}{F \sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}}. \tag{67}$$

Finally,

$$C_k = \frac{T}{4\pi} \frac{(2r_0 - \lambda_2/F)}{F\sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}} \frac{A_k B_k F}{\Phi_k}. \tag{12a}$$

The coefficient, $T/4\pi(2r_0 - \lambda_2/F)$ may be determined from the constraint

$$\int_{-\pi/T}^{\pi/T} \sum_k A_k B_k C_k \, du = r_0 . \tag{8}$$

Multiplying both sides of (12a) by $A_k B_k$, summing, and integrating one obtains

$$r_0 = \frac{T}{4\pi} \int_{-\pi/T}^{\pi/T} \left(2r_0 - \frac{\lambda_2}{F}\right) \frac{F\sum_k \frac{A_k^2 B_k^2}{\Phi_k}}{F\sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}} \, du \tag{68}$$

or

$$\frac{T}{4\pi}\left(2r_0 - \frac{\lambda_2}{F}\right) = \frac{\frac{r_0 T}{2\pi}\left[\int_{-\pi/T}^{\pi/T} \frac{\sum_k \frac{A_k^2 B_k^2}{\Phi_k} du}{F\sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}} + \frac{1}{F}\int_{-\pi/T}^{\pi/T} \frac{\frac{T}{2\pi} du}{F\sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}}\right]}{\int_{-\pi/T}^{\pi/T} \frac{\sum_k \frac{A_k^2 B_k^2}{\Phi_k}}{F\sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}} du}. \tag{12b}$$

The mean square error

$$\int_{-\pi/T}^{\pi/T} \left[\sum_k C_k^2 \Phi_k + \frac{2\pi F}{T}\left\{\sum_k A_k B_k C_k - \frac{r_0 T}{2\pi}\right\}^2\right] du$$

may be found using (12a,b) to be

$$\left(\frac{r_0 T}{2\pi}\right)^2 \left(\frac{\int_{-\pi/T}^{\pi/T} \frac{F du}{\Delta} \int_{-\pi/T}^{\pi/T} \frac{\sum_k \frac{A_k^2 B_k^2}{\Phi_k} du}{\Delta} + \frac{T}{2\pi}\left(\int_{-\pi/T}^{\pi/T} \frac{du}{\Delta}\right)^2}{\int_{-\pi/T}^{\pi/T} \frac{\sum_k \frac{A_k^2 B_k^2}{\Phi_k} du}{\Delta}}\right)$$

where

$$\Delta = F \sum_k \frac{A_k^2 B_k^2}{\Phi_k} + \frac{T}{2\pi}.$$

REFERENCES

1. Nyquist, H., Certain Topics in Telegraph Transmission Theory, AIEE Trans., *47,* April, 1928, pp. 617–644.
2. Bennett, W. R. and Davey, J. R., *Data Transmission,* McGraw-Hill Book Co., New York, 1965.
3. Tufts, D. W., Matched Filters and Intersymbol Interference, Cruft Laboratory, Harvard University, Cambridge, Massachusetts, Tech. Rept. No. 345, July, 1961.
4. Tufts, D. W., Nyquist's Problem in Pulse Transmission Theory, Cruft Laboratory, Harvard University, Cambridge, Massachusetts, Tech. Rept. No. 425, Sept., 1963.
5. Tufts, D. W., Certain Results in Pulse Transmission Theory, Cruft Laboratory, Harvard University, Cambridge, Massachusetts, Tech. Rept. No. 335, Feb., 1962.
6. Tufts, D. W., Nyquist's Problem — The Joint Optimization of Transmitter and Receiver in Pulse Amplitude Modulation, Proc. IEEE, *53,* March, 1965, pp. 248–259.
7. Gibby, R. A. and Smith, J. W., Some Extensions of Nyquist's Telegraph Transmission Theory, B.S.T.J., *44,* Sept., 1965, pp. 1487–1510.
8. Bennett, W. R., Statistics of Regenerative Digital Transmission, B.S.T.J., *37,* Nov., 1958, pp. 1501–1542.
9. Courant, R. and Hilbert, D., *Methods of Mathematical Physics, 1,* Interscience Publishers, New York, 1953.
10. Lender, A., The Duobinary Technique For High Speed Data Transmission, IEEE Trans. on Communication and Electronics, *82,* May, 1963, pp. 214–218.
11. Bennett, W. R., unpublished memorandum.