Index Reduction of FM Waves by Feedback and Power-Law Nonlinearities

By V. E. BENEŠ

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Feedback systems which achieve subharmonic response by use of power-law nonlinearities and trigonometric identities are described, and the problem of "modes," i.e., multiple responses to the same excitation, in some of these systems is discussed. It is shown, by a thorough discussion of an example, that suitable choices of a trigonometric identity and a loop filter can be made which lead to locally asymptotically stable subharmonic orbits. An application to FM demodulation is suggested: the subharmonic modes can be used to reduce the index of an input wide-index wave so that a narrower IF filter than for conventional FM suffices, as in the FM demodulator with feedback, but without a controlled oscillator or even a mixer.

I. INTRODUCTION

As is known, it is possible to design feedback systems that reduce the index of an FM wave by an explicit use of power-law nonlinearities based on certain simple trigonometric identities. Circuits which use feedback to achieve subharmonic operation, and which are stable under changes in the input frequency over at least a limited range, have been built, tested, and described in the literature.[1] They have the behavior predicted by the trigonometric identities. Some of them depend on nonlinear conversion of a signal containing harmonics of θ into one containing only the first harmonic, and others depend on the inverse process of generating harmonics.

Our purpose is to discuss the problem of "modes" (i.e., different responses to the same excitation) in these systems, and the problem of the stability of the interesting subharmonic modes. We also indicate an application to frequency modulation: by incorporating the principle on which the systems are based into a frequency modulation receiver, one obtains circuits with some of the properties and advantages of the FM demodulator with feedback[2,3] proposed by J. G. Chaffee. However, none of the circuits suggested here contains a voltage-controlled oscillator, and some do not even contain a mixer.

To illustrate the principles involved, let us consider the trigonometric identity[1]

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta.$$

If we set

$$\psi(x) = \operatorname{sgn} x \cdot | x |^{1/3}$$

the above identity can equally well be put in the form

$$\psi(3 \sin \theta - \sin 3\theta) = 4^{1/3} \sin \theta.$$

Thus if we view, as in Fig. 1, $4^{1/3} \sin \theta$ as the output of the nonlinearity $\psi(\cdot)$, and feed this back through a gain of $3 \cdot 4^{-1/3}$ to an adder whose input is

$$- \sin 3\theta,$$

we will have a feedback system which is driven by $-\sin 3\theta$, and which produces $\sin \theta$, a 3-to-1 reduction of index of modulation if the angle 3θ is taken to be of the form

$$3\theta(t) = \omega t + 3\varphi(t), \qquad \dot{\varphi} = \text{signal}.$$

It is apparent[1] that other trigonometric identities can be used in analogous fashion to get an n-to-1 index reduction, with n any positive integer $\geqq 2$. For example, with

$$\psi(x) = \operatorname{sgn} x \cdot | x |^{1/5}$$

we have

$$\psi(\cos 5\theta + 20 \cos^3 \theta - 5 \cos \theta) = 16^{1/5} \cos \theta,$$

corresponding to the system of Fig. 2.

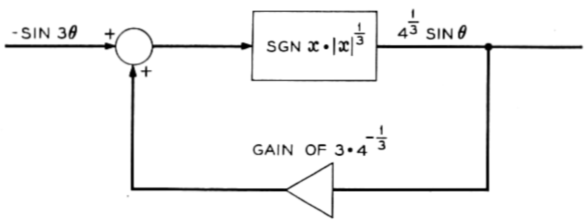

Fig. 1 — System using 1/3-power nonlinearity.

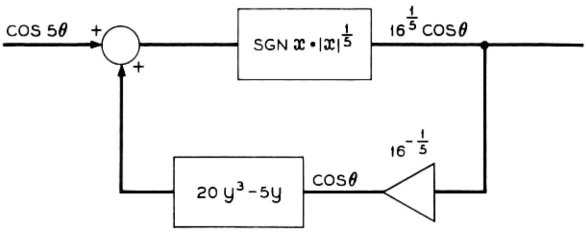

Fig. 2 — Loop based on 1/5-power nonlinearity.

Another system, based on the identity

$$4 \cos^3 \theta = \cos 3\theta + 3 \cos \theta,$$

is shown in Fig. 3. It generates the "compressed" signal $-3 \cos \theta$ by subtracting $4 \cos^3 \theta$ from the wide-index signal $\cos 3\theta$.

II. FILTERING

If the principles illustrated above are to be used in a communications receiver, it is probably desirable to perform some filtering to remove undesired components of noise or feedback signal. Thus in practice the feedback loop would (e.g.) include a filter which removed all components not in the (essential) band of $\sin \theta(\cdot)$. For many filters, and choices of input phase $\theta(\cdot)$, presence of the filter will of course mean that the signal in the loop is no longer so simply related to $\theta(\cdot)$ as it was in the examples above: use of the trigonometric identity to relate the loop signal to the input may be inexact. However, there exist filters and choices of $\theta(\cdot)$ for which this does not occur. (See Ref. 5, and Section V herein.) In any case, if the filter passes $\sin \theta(\cdot)$ without essential distortion the identity will remain true for practical purposes. For example, with

$$\psi(x) = \operatorname{sgn} x \cdot |x|^{1/3}$$

again, and the identity

$$\psi(\cos 3\theta + 3 \cos \theta) = 4^{1/3} \cos \theta$$

we would follow $\psi(\cdot)$ with a filter that passed $\cos \theta$ but removed out-of-band noise, and get a system like that of Fig. 4.

III. THE POSSIBILITY OF SEVERAL "MODES"

It has been pointed out in the literature[4] that certain frequency di-

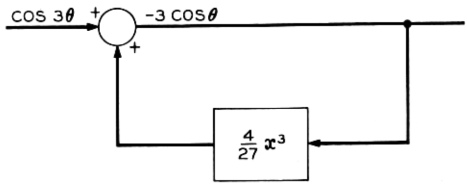

Fig. 3 — System based on cubic nonlinearity in the feedback.

viders based on regeneration and modulation are not necessarily "self-starting," and that sizeable starting voltages may be needed to set them off. Put in the language of differential equations, this means for example that there may be two entirely different kinds of steady-state response (to the same steady-state signal), one (say) oscillating with large amplitudes, and the other taking place in a region of asymptotic stability around a critical point, with small oscillations. The "starting voltage" is needed to get the system out of the region of stability.

A. J. Giger has suggested that a similar situation will obtain in realizations of the circuits described above, even though they contain an adder rather than a multiplier (modulator). Clearly, whether a realization is self-starting is going to depend on the circuit details as well as on the principles at issue here, and each case will have to be studied on its own merits. However, these general remarks are pertinent:

(*i*) The non-self-starting frequency divider is just a special case of the well-known but incompletely understood phenomenon that a system may not have a unique asymptotic response. It is a property specific to the systems described so far that they depend on a fractional-power nonlinearity, and as a result it is possible, though not necessary, that they fail to have unique periodic responses to some periodic signals.[5]

(*ii*) The desired operation of the systems proposed above depends on evoking a suitable *subharmonic* response. It is known[5] that not all solutions need contain such components of lower frequency than the input.

(*iii*) It is also known[5] that even when a subharmonic periodic solution exists, it is itself only unique up to certain phase shifts. Specifically, if we obtain a solution with a component $e^{\pi i \omega t}$ when the input contains only harmonics of $e^{\pi i n \omega t}$, $n > 2$, then any *translation* of a solution by $2k/n$ for $k = 1, \cdots, n$ is also a solution.

For example, the steady-state response of the circuit of Fig. 4 might or might not contain the desired subharmonics, and thus it might not work as planned unless care is taken to ensure that it slips into the right "mode" of operation initially, and that this mode is stable under the perturbations due to the presence of signal and noise. The mathematical analysis of such phenomena is often arduous and, in many cases, quite incomplete. From a practical point of view the mode problem (when it occurs) might best be resolved by encouraging the desired mode by pulsing the (tuned circuit) filter, by adding automatic gain control features that cut in when the amplitudes of the desired mode are low, or by designing the filter to have zeros of transmission at certain values of frequency associated with the undesired modes, such as their fundamentals.

However, while it is necessary to emphasize that the problems mentioned above exist, it is also important to state that the picture is not all black: known methods of analysis and design suffice to ensure local asymptotic stability of some of the subharmonic modes described above An example is worked out in some detail in Section V.

IV. BIAS

For theoretical reasons it may be undesirable, and for practical reasons impossible, to use a nonlinear characteristic which has an infinite slope at the origin. For example the singular nature of the fractional-power nonlinearities at the origin may preclude a conventional closed-loop, open-loop stability analysis by linearization around that point. Or, if the nonlinearities are being obtained by the use of diodes, such a slope is physically unattainable. These difficulties, along with most passages of the system through or near such a singular point, can be avoided by the addition of what amounts in electrical terms to "dc bias" at various points of the system, in such a way as to (roughly) move the operating point of the system to a desired region of the nonlinear characteristic $\psi(\cdot)$. In this region $\psi(\cdot)$ might be of Lipschitz character, or it might be particularly well represented by a particular diode. Such biasing can also be used to eliminate some equilibrium points of the system, and thus to reduce the number of solutions; it can also be used to increase

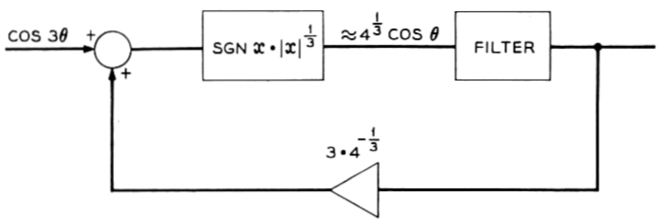

Fig. 4 — Filtering in a loop using 1/3-power nonlinearity.

the size of certain regions of asymptotic stability, and thereby enhance the dynamic stability of desired solutions. It therefore furnishes substantial latitude for design.

Thus, e.g., to use the identity

$$\sin^2 \theta = \frac{1 - \cos 2\theta}{2},$$

we can multiply by $b^2 \neq 0$ and add

$$a^2 + 2ab \sin \theta$$

to both sides, so that

$$(a + b \sin \theta)^2 = a^2 + (b^2/2) + 2ab \sin \theta - (b^2/2) \cos 2\theta.$$

Choosing $|a| > |b|$ ensures that the right-hand side is bounded away from zero. Taking the square root of both sides, we base design on the "biased" identity

$$a + b \sin \theta = (a^2 + \tfrac{1}{2}b^2 + 2ab \sin \theta - \tfrac{1}{2}b^2 \cos 2\theta)^{1/2}. \tag{1}$$

Discussion of an example of design based on (1) follows.

V. EXAMPLE, WITH ANALYSIS OF ORBITAL ASYMPTOTIC STABILITY

We now consider the circuit depicted in Fig. 5, with

$$\psi(x) = \operatorname{sgn} x \cdot |x|^{1/2},$$

with input

$$y(t) = a^2 + \tfrac{1}{2}b^2 - \tfrac{1}{2}b^2 \cos 2t,$$

and a filter whose impulse response $k(\cdot)$ is integrable and has a Fourier transform

$$K(\omega) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} k(t) e^{-i\omega t} dt$$

such that $K(0) = 0$ and $K(1) = (2\pi)^{-1/2}(2a)$. For our example we use the second-order filter

$$K(\omega) = \frac{(2\pi)^{-1/2} 2ai\omega}{c(i\omega)^2 + i\omega + c}, \tag{2}$$

with $c > 0$, so that poles of $K(\cdot)$ are in the left half-plane.

It is now easy to show, using the theory of Fourier series, that the system of Fig. 5 has a subharmonic response

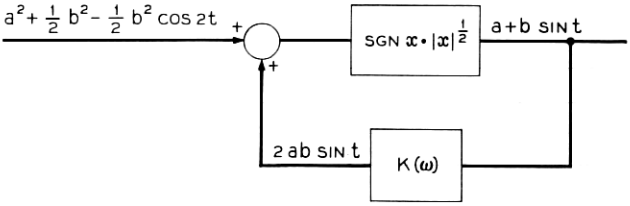

Fig. 5 — Example for orbital asymptotic stability.

$$x(t) = a^2 + \tfrac{1}{2}b^2 + 2ab \sin t - \tfrac{1}{2}b^2 \cos 2t, \tag{3}$$

where $x(\cdot)$ is the input to the nonlinearity. The loop equation for this solution (with no transients!) is simply

$$x(t) = y(t) + \int_{-\infty}^{\infty} k(t - u) \operatorname{sgn} x(u) \mid x(u) \mid^{1/2} du.$$

Substituting the expression that (3) gives for $x(\cdot)$, using the identity (1), shows that the filter removes the dc term a from the output of the nonlinearity and multiplies the amplitudes of frequencies ± 1 by $2a$. We remark that the constant $c > 0$ in (2) does not occur in the subharmonic solution (3), and that the constant b does not occur in the transfer function $K(\omega)$. Also, for the particular input we chose, the presence of the filter does not render the trigonometric identity being used inexact.

As is well known, for fixed values of a and c there are many ways of describing the circuit of Fig. 5 by differential equations so as to give rise to (2) as the transfer function of the filter. For some of these ways the solution $x(\cdot)$ found above may be orbitally asymptotically stable, for others it may not. In short, the stability of this subharmonic solution probably depends on the way in which the action of the transfer function is represented by differential equations. We shall consider representations of the form

$$\dot{z} = Az + \delta\psi(\beta' z + y(t)), \tag{4}$$

where $z(\cdot)$ is a 2-vector valued function, β is a 2-vector, δ is a 2-vector proportional to the unit 2-vector, and A is a stable 2×2 matrix. In this representation the periodic solution $x(\cdot)$ of (3) will have the form

$$x(t) = \beta' z(t) + y(t).$$

These representations and their simple properties are used merely as an illustration, because they readily admit an analysis of the asymptotic stability of the orbit corresponding to the solution $x(\cdot)$. They do not come near exhausting the possibilities for finding stable subharmonic forced oscillations in feedback systems.

Since for $t_2 \geqq t_1$, (4) gives

$$\beta' z(t_2) = \beta'[\exp A(t_2 - t_1)]z(t_1) + \int_{t_1}^{t_2} \beta'[\exp A(t_2 - u)]\delta\psi(\beta' z(u) + y(u))du$$

it is clear that (4) will represent the circuit if and only if

$$\beta'(i\omega I - A)^{-1}\delta = (2\pi)^{1/2}K(\omega) = \frac{2ai\omega}{c(i\omega)^2 + i\omega + c},$$

i.e., if and only if both

$$\det(i\omega I - A) = (i\omega)^2 + c^{-1}i\omega + 1,$$

$$\beta_1 \begin{vmatrix} \delta_1 & -a_{12} \\ \delta_2 & i\omega - a_{22} \end{vmatrix} + \beta_2 \begin{vmatrix} i\omega - a_{11} & \delta_1 \\ -a_{21} & \delta_2 \end{vmatrix} = 2ai\omega/c. \tag{5}$$

The first condition is met if

$$a_{11}a_{22} - a_{12}a_{21} (= \det A) = 1,$$

$$a_{11} + a_{22} = -c^{-1}.$$

The second condition is equivalent to $\beta'\delta = 2ac^{-1}$ and $\beta' A^{-1}\delta = 0$, taken together. Since, to facilitate stability analysis for this example, we wish to impose $\delta_1 = \delta_2 \neq 0$, the latter condition is $\beta' A^{-1}1 = 0$, $1 =$ unit 2-vector. All these conditions together can be met in many ways. A convenient choice is $\beta = (2a/c,0)'$, $\delta = 1$, $a_{22} = -a_{21}$, $(a_{11} - a_{12})a_{22} = 1$, and

$$a_{11}(-a_{11} - c^{-1}) - a_{12}(-a_{11} - c^{-1}) = 1,$$

the last imposing a rational relation between a_{11} and a_{12}, thus leaving one parameter still free.

It can be verified that when the conditions (5) above are met, then $\beta' z(t) = 2ab \sin t$ does define a periodic subharmonic orbit of the differential system (4). We are now in a position to make a linear local asymptotic stability analysis for the subharmonic orbit $\beta' z(t) = 2ab \sin t$, by Lyapunov's classical theorem.[6] The (periodic) linearization matrix is

$$A + \operatorname{diag} \delta f(t)$$

where

$$\begin{aligned} f(t) &= \psi'(a^2 + 2ab \sin t + \tfrac{1}{2}b^2 - \tfrac{1}{2}b^2 \cos 2t) \\ &= \psi'(x(t)) \\ &= \frac{1}{2\psi(x(t))} \\ &= \tfrac{1}{2}(a + b \sin t)^{-1}. \end{aligned}$$

If diag $\delta = dI$, d a scalar, it is easily verified that the fundamental matrix associated with this periodic matrix is

$$\Phi(t) = e^{At}u(t)$$

where $u(\cdot) > 0$ is defined by

$$u_0 \equiv 1$$

$$u_{n+1}(t) = d\int_0^t f(v)u_n(v)du$$

$$u(t) = \sum_{n=0}^{\infty} u_n(t).$$

We are therefore interested in the characteristic values of $\Phi(2\pi)$, i.e.,

$$e^{2\pi A}u(2\pi).$$

These are of the form

$$u(2\pi)e^{2\pi\mu}$$

where μ is a characteristic value of A, and so we may conclude at once[6] that if

$$-\mathrm{Re}\ \mu = \frac{1}{2c} > \frac{1}{2\pi}\log u(2\pi),$$

then the orbit determined by $\beta' z(t) = 2ab \sin t$ is locally asymptotically stable, i.e., there exists a neighborhood of it from which all solutions approach the orbit.

Clearly

$$u(t) \leqq \exp\{t \sup_{0 \leqq u \leqq t} |d| \cdot |f(u)|\}$$

whence

$$\frac{1}{2\pi}\log u(2\pi) \leqq \frac{1}{2}\frac{|d|}{a - |b|}$$

so that the inequality

$$a > c \mid d \mid + \mid b \mid$$

suffices for local asymptotic stability of the orbit.

Note that changing the sign of b corresponds to changing the sign in all the odd components in the periodic solution $x(t)$ found above; the resulting function is also a periodic solution differing from $x(t)$ only in phase, by exactly π.

VI. APPLICATION TO AN FM DEMODULATOR WITH FEEDBACK

The examples of index reduction described in the preceding sections suggest that it is possible to design FM demodulators with frequency feedback that contain no voltage-controlled oscillators and even no mixers. Several methods of realizing such a possibility will be discussed, based on the principles exemplified.

The simplest demodulation scheme of this sort is obtained by a specialization of the system of Fig. 4. We merely specify that the filter have a narrow passband centered around an intermediate frequency ω that is $\frac{1}{3}$ the carrier frequency, and that it introduce negligible amplitude variations for signals in the passband, so as not to interfere unduly with the trigonometric identity. This gives rise to the demodulator of Fig. 6, in which $\theta(\cdot)$ has the form

$$\theta(t) = \omega t + \varphi(t),$$

with $\dot{\varphi}$ a baseband signal. The feedback, made at IF, reduces the modulation index 3-to-1. Here the carrier frequency is *three* times the intermediate frequency, but this relationship can easily be changed by remodulating or, for that matter, by using a different trigonometric identity as the basis of design. We note that no mixer, and no voltage-controlled oscillator, is used. Also, the phase of the signal fed back is crucial: excessive phase shift in the filter is as intolerable here as in conventional FM with feedback.

As a demodulator, the circuit of Fig. 6 shares with Chaffee's circuit the advantage that the wide band of noise which must be passed by the initial amplifier along with the wide-index signal is not admitted to the detector. This circumstance is important, because a principal object of feedback (in FM with feedback) is to reduce the noise level at the detector by filtering all but a small part of the noise. However, it remains to be seen how well the nonlinearity sgn $x \cdot \mid x \mid^{1/3}$ performs its function in the presence of the wideband noise that enters it, since the resulting amplitude modulation $a(\cdot)$ at the input renders the trigonometric identity being used here inexact. This AM due to noise might be removed

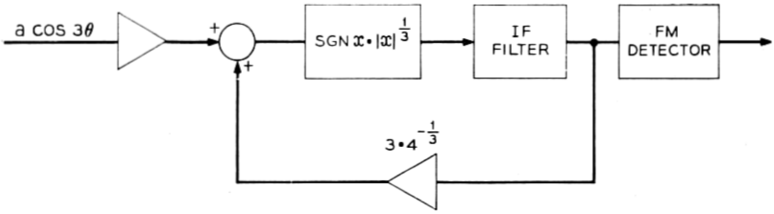

Fig. 6 — Possible utilization of 1/3-power nonlinearity in FM demodulator.

by inserting a limiter and filter to follow the IF filter, as in conventional systems, at the expense of incurring additional phase shift.

A final point, due to A. J. Giger, is that, unlike Chaffee's circuit, the present one *retains* the carrier phase instead of discarding it and operating independently of it.

VII. CIRCUITS WITHOUT FRACTIONAL-POWER NONLINEARITIES

It is straightforward to generate other, quite different designs based on the same identity,

$$\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta,$$

designs which do not depend on a *fractional*-power nonlinearity, and so do not incur the problems above. For example, Fig. 7 shows a design very much like that of the conventional FM with feedback demodulator, except that the detector-controlled oscillator is replaced by the nonlinearity

$$4y^3 - 3y.$$

Applied to $\cos \theta$, this gives $\cos 3\theta$, to be used as the feedback input to the mixer. If the other mixer input, i.e., the incoming signal, is the

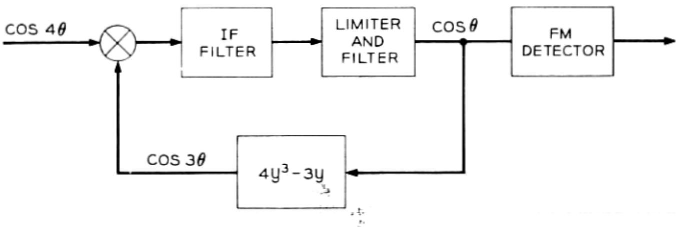

Fig. 7 — Cubic nonlinearity used in FM feedback system with modulator.

wide-index signal cos 4θ, the filter can be made to select the difference frequency component cos θ to complete the loop and provide the feedback signal. In this system the carrier frequency is four times the intermediate frequency.

A particularly simple circuit, based on Fig. 3 and using only an adder, is depicted by Fig. 8. In this design the feedback is through the simple cubic nonlinearity

$$\frac{4}{27}x^3.$$

If $-3 \cos \theta$ is applied to this, and the output is combined in the adder with an incoming wide-index signal cos 3θ, the adder output is -3 cos θ. This passes substantially unchanged through the narrow IF filter suitable for the low-index wave cos θ, while of the wide band of noise accompanying cos 3θ at the input only a narrow band can pass the IF filter. For practical purposes, the cubic characteristic would only be required over the range $|x| \leqq 3$, and standard stability analyses can be used. Again, carrier phase is retained.

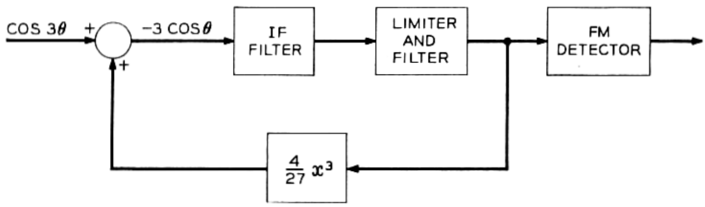

Fig. 8 — Cubic nonlinearity used in FM feedback system with adder.

VIII. ACKNOWLEDGMENTS

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