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On the Simultaneous Measurement of a Pair of Conjugate Observables

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A precise theory of the simultaneous measurement of a pair of conjugate observables is necessary for obtaining the classical limit from the quantum theory, for determining the limitations of coherent quantum mechanical amplifiers, etc. The uncertainty principle, of course, does not directly address this problem, since it is a statement about the variances of two hypothetical ideal measurements. We will adopt the approach that there exist instantaneous inexplicable ideal measurements of a single observable. Just as von Neumann[1] uses an ideal measurement together with an interaction to explain an indirect observation, we use ideal measurements together with interactions to explain the simultaneous measurement of an observable and its conjugate.

The joint measurement described below is complete in the sense that all pertinent past history is subsumed under the meter readings. A precise formula for the joint probability distribution of the results of the measurements is given. The variances given by these distributions satisfy an inequality like the uncertainty principle but with an extra factor or two. It is also shown that this inequality governs any conceivable joint measurement. The single measurement of an observable is a limiting case of the joint measurement when the variance of one of the measured variables is allowed to approach infinity.

Since we are trying to measure two observables, we will introduce two meters, that is, two one-dimensional systems which will be coupled to the object system. Since the two meter positions commute, we can make ideal simultaneous measurements of them. Our interpretation will be that these two measurements will constitute a simultaneous measurement of the two noncommuting observables of the object system. We will see that we cannot let the strength of our interaction become infinite, unlike the indirect measurement of Ref. 1, but must adjust it to a certain critical value. We will find that after the interaction and the measurement of the meter values that the system is left in a state which is completely determined by the meter readings and a certain parameter which we will call the "balance," having to do with how near our joint measurement is to either of two ideal measurements.

To effect the desired measurement, the coupling must be described by the following Hamiltonian:

$$H_{\text{int}} = K(qP_x + pP_y) \tag{1}$$

where p,q are the positions and momentum we wish to be measured and P_x, P_y are the momenta of the two single-degree-of-freedom systems we are using for meters. The positions of the two meters will be x and y. In addition, we require that the meters initially be in the states $M(x)$ and $N(y)$, where

$$\begin{aligned} M(x) &= \left(\frac{2}{\pi b}\right)^{\frac{1}{4}} e^{-x^2/b} \\ N(y) &= \left(\frac{2b}{\pi}\right)^{\frac{1}{4}} e^{-by^2} \end{aligned} \tag{2}$$

and b is the "balance." The strength of the interaction K is assumed to be sufficiently large that other terms in the Hamiltonian can be ignored. Hence the Schrödinger equation will be ($\hbar = 1$)

$$\frac{\partial \varphi}{\partial t} = -K\left(q\frac{\partial \varphi}{\partial x} - i\frac{\partial^2 \varphi}{\partial q \partial y}\right) \tag{3}$$

with the initial condition

$$\varphi(q,x,y,0) = F(q)M(x)N(y) \tag{4}$$

where $F(q)$ is the state of the system to be measured and the system and the two meters are assumed to be independent prior to the interaction. Equation (3) is solved by Fourier transforming on y. The solution is

$$\varphi(q,x,y,t) = \int_{-\infty}^{\infty} F(q - wtk)M(x - qtk + \tfrac{1}{2}wt^2k^2) \cdot \frac{\exp - (w^2/4b)}{(4\pi b)^{\frac{1}{2}}} \exp(iwy)\, dw. \tag{5}$$

To obtain the results we desire it is necessary to make ideal measurements of x and y at $t = 1/K$. In the following, t will be assumed to be equal to $1/K$ and the time will be suppressed in the notation.

The joint probability distribution for the commuting observables x and y, $P(x,y)$ is given, of course, by

$$P(x,y) = \int_{-\infty}^{\infty} |\varphi|^2\, dq. \tag{6}$$

Then, using (5), we have

$$P(x,y) = \left(\frac{1}{4\pi^3 b}\right)^{\frac{1}{2}} \left| \int_{-\infty}^{\infty} F(q) \exp - \left[\frac{1}{2b}(x-q)^2\right] \cdot \exp - (iqy)\, dq \right|^2 \tag{7}$$

or, if $G(p)$ is the momentum wave function of the system

$$P(x,y) = \left(\frac{b}{4\pi^3}\right)^{\frac{1}{2}} \left| \int_{-\infty}^{\infty} G(p) \exp - \left[\frac{b}{2}(y-p)^2\right] \cdot \exp - (ixp)\, dp \right|^2. \tag{8}$$

That is, the joint probability distribution is the Fourier spectrum of the wave function multiplied by a Gaussian window whose width is related to the balance of the measurement.

The new wave function for the system after the measurement is given by substituting the results of the meter readings in the wave function and renormalizing. If we denote the measured value of x by x_m and the measured value of y, y_m, the new state of the system is given by

$$F'(q) = \frac{\varphi(q, x_m, y_m)}{\int |\varphi(q, x_m, y_m)|^2 dq} = \left(\frac{1}{\pi b}\right)^{\frac{1}{4}} \cdot \exp - \left[\frac{1}{2b}(q - x_m)^2 + iqy_m\right]. \tag{9}$$

Notice that the measurement is complete, in that the state of the system after the measurement is dependent only on the meter readings and not otherwise on the state of the system before measurement. We also notice that the system is left in a minimum Gaussian packet after the measurement, with mean position x_m and mean momentum y_m, which is an intuitively satisfying result.

From (7) or (8) it is easy to verify that the expected value of x is equal to the expected value of q before the interaction and that the expected value of y is equal to the expected value of p before the interaction. The variances of x and y are related to the variances of q and p before the interaction by

$$\begin{aligned} \sigma_x^2 &= \sigma_q^2 + b/2 \\ \sigma_y^2 &= \sigma_p^2 + 1/2b. \end{aligned} \tag{10}$$

Hence the variances are individually larger than those of the wave function $F(q)$ due to the disturbances caused by the joint measurement. From

(7) and (10) it can be seen that in the limit $b \to 0$ distribution of x is the same as that of an ideal measurement of position and the system is left in an eigenstate of position. Similarly, if $b \to \infty$ we have an ideal momentum measurement and the system is left in an eigenstate of momentum.

From (10) and the uncertainty principle

$$\sigma_q \sigma_p \geqq 1/2 \tag{11}$$

we can deduce

$$\sigma_x \sigma_y \geqq 1 \tag{12}$$

which is the proper uncertainty principle for the joint measurement. The minimum can actually be met when $F(q)$ is a minimum Gaussian packet and the balance b is suitably adjusted. It is interesting that, when $F(q)$ is a minimum Gaussian packet, (8) shows that the meter readings are distributed as independent Gaussian random variables.

We will now show that the bound expressed by (12) is valid for any joint measurement that meets certain reasonable requirements.

Let us consider a joint measurement from a more general point of view. As before, we will have a meter (a system with at least two degrees of freedom) interact with our system. The initial wave function for the meter plus system will be the product of the system wave function $F(q)$ and the meter function $M(w_1 w_2, \ldots)$. After allowing the interaction to proceed for t seconds we will measure two observables, say $x(t)$ and $y(t)$, which will hopefully measure the system position and momentum. In the Heisenberg representation we may write without loss of generality

$$\begin{aligned} x(t) &= q(0) + A \\ y(t) &= p(0) + B. \end{aligned} \tag{13}$$

If we normalize with a scale factor of unity on both measurements, it is natural to require that the expectations of $x(t)$ and $y(t)$ satisfy

$$\begin{aligned} \langle x(t) \rangle &= \langle q(o) \rangle \\ \langle y(t) \rangle &= \langle p(o) \rangle \end{aligned} \tag{14}$$

uniformly for *all* initial states of the system, i.e., for all $F(q)$. This implies that

$$\begin{aligned} \langle A \rangle &= 0 \\ \langle B \rangle &= 0 \end{aligned} \tag{15}$$

identically for all $F(q)$. From this and the fact that the initial wave function for the system plus meter factors, it can be shown

$$\langle qA \rangle = \langle Aq \rangle = \langle Bq \rangle = \langle qB \rangle = 0$$
$$\langle pA \rangle = \langle Ap \rangle = \langle Bp \rangle = \langle pB \rangle = 0. \tag{16}$$

Secondly, we require that $x(t)$ and $y(t)$ commute so that they may be simultaneously measured. From this and (13) we have

$$[B,A] = [q,p] + [q,B] + [A,p]. \tag{17}$$

Squaring both sides of (17) and taking expectations, it follows from (16) that

$$\langle -[A,B]^2 \rangle \geqq 1 \tag{18}$$

which implies

$$\langle A^2 \rangle \langle B^2 \rangle \geqq 1/4. \tag{19}$$

Using (13) and (16), we obtain

$$\sigma_x{}^2 \sigma_y{}^2 = \sigma_p{}^2 \sigma_q{}^2 + \langle A^2 \rangle \langle B^2 \rangle + \sigma_q{}^2 \langle B^2 \rangle + \sigma_p{}^2 \langle A^2 \rangle \tag{20}$$

where $\sigma_x{}^2 = \langle (x - \langle x \rangle)^2 \rangle$, etc.

From (11) and (19) it follows that

$$\sigma_x \sigma_y \geqq 1 \tag{21}$$

which is the desired result.

REFERENCES

1. von Neumann, J., *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, Princeton, N. J., 1955.