On the Reception of Binary Signals in the Presence of a Small Random Delay*

By M. I. SCHWARTZ

(Manuscript received March 22, 1965)

Receiver design for a binary communication system which operates over a linear channel with a random delay is considered. It is assumed that the variance of the random delay is very small and that the rate of growth of its moments is restricted. Under certain smoothness requirements on the received signal an approximation to the test statistic, which is optimum in the Neyman-Pearson sense, is derived for the case of gaussian receiver noise with covariance $R(\tau) = R(0)e^{-\beta|\tau|}$. *It is found that the test statistic, which in general is nonlinear, assumes the linear form of a crosscorrelator when phase reversal signaling is employed.*

The case where the noise is white and phase reversal signaling is used is investigated. The correlation waveform in this case is found to consist of the expected value of the received signal plus a term dependent on the slope of the signal when the delay is equal to its mean value.

I. INTRODUCTION AND SUMMARY

In any practical communication system the signal arrival time is never exactly known. This results in a degradation of the average system performance. It would be of considerable interest to determine the receiver which minimizes the effect of this uncertainty on system error performance. A special case of this problem will be considered here.

Helstrom[1] has studied the detection of signals of unknown arrival time using the method of maximum likelihood with particular emphasis on the radar problem. Brown and Palermo[2] consider system performance in the presence of random delays with applications including least squares filtering and sampling with time jitter. Balakrishnan[3] and other

* This work is based on a portion of a thesis entitled "Binary Signal and Receiver Design for Linear Time Invariant Channels". This thesis was accepted by the faculty of the Graduate Division of the School of Engineering and Science of New York University in partial fulfillment of the requirements for the degree of Doctor of Engineering Science, Oct. 1964.

researchers have also considered the problem of time jitter in sampling. However, to the best of this authors knowledge, no optimum statistical test, or approximation thereof, has been determined for detection in the presence of a random delay.

In the subsequent analysis we investigate binary communication for the case where the variance of the random delay is "very small." It is assumed that the transmitted signals and the channel impulse response are such that the received signal satisfies appropriate smoothness conditions, and that the statistics of the random delay δ satisfy the relation that $E[\,|\,\delta - \bar{\delta}\,|^k] \leqq h^k \lambda^k$; where h is some constant, E denotes expectation, $\bar{\delta} \triangleq E\delta$ and λ^2 is the variance of δ. The requirement on the random delay will always be satisfied when δ is restricted to a bounded interval. Our model will also assume that intersymbol interference is negligible, or equivalently, that we are dealing with a single transmission.

Under these assumptions an approximation is obtained for the test statistic which is optimum, in the Neyman-Pearson sense, for the case of gaussian receiver noise with exponential covariance. Generally the test statistic involves a nonlinear operation. However, for the case of phase reversal signals, only linear operations are required.

The form that the test statistic takes for "white noise," which is considered as a limit of the exponential covariance case, is obtained. It is shown that for phase reversal signaling the optimum receiver is a cross-correlator and that a portion of the correlation waveform is the expected value of the received signal itself.

Fig. 1 depicts the communication system under consideration. A signal, $s^{(i)}(t)$ ($i = 1$ or 2), which is non-zero only over an interval $[0,T]$, is transmitted through a channel. The channel consists of a random delay δ and a linear time invariant filter whose output, $x^{(i)}(t - \delta)$, is disturbed by an additive noise source, $n(t)$. It is assumed that the variance of the random delay, denoted by λ^2, is small. Furthermore, the noise, $n(t)$, will

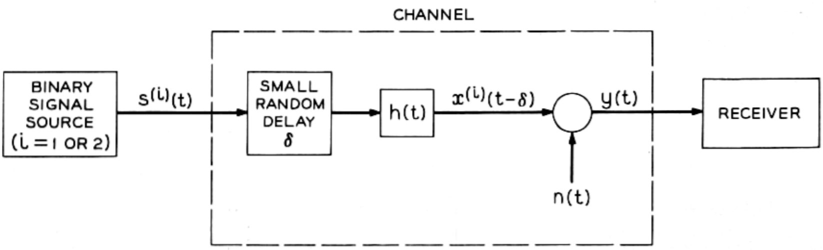

Fig. 1 — Model of Binary Communications System with a Small Random Delay.

be assumed to be a sample function of a stationary gaussian random process with mean zero and covariance $R(\tau) = R(0)e^{-\beta|\tau|}$. For this system we seek the test statistic, based on an observation of the received signal over a fixed interval of length equal to the duration of the transmitted signal, which gives rise to the minimum error probability in the receiver decision process.

II. DERIVATION OF THE TEST STATISTIC

It is known that in testing between two simple hypotheses a Neyman-Pearson test will give rise to minimum error probability. Furthermore, Grenander[4] has shown that in the "regular case" the desired test statistic, which is a random variable called the likelihood function, l, can be obtained as the limit of an N dimensional likelihood ratio. In the subsequent development, in which it is assumed that we deal only with the regular case, the receiver test statistic is obtained as the limit of such an N-dimensional likelihood ratio.

The receiver input, $y(t)$, is given by

$$y(t) = x^{(i)}(t - \delta) + n(t), \qquad i = 1,2 \tag{1}$$

where $x^{(i)}(t - \delta)$ is the portion of the input resulting from sending the signal $s^{(i)}(t)$ when the random delay is δ, and $n(t)$ is the receiver noise. The noise is assumed to be gaussian with covariance $R(\tau) = R(0)e^{-\beta|\tau|}$.

Using a theorem due to Belayev[5] the noise sample functions can be shown to be almost surely continuous. Furthermore, almost all sample functions can be expanded almost surely in a pointwise convergent series in terms of the eigenfunctions of the noise covariance kernel. That is,

$$n(t) = \sum_k n_k \varphi_k(t), \tag{2}$$

with

$$E(n_j n_k) = \sigma_j^2 \delta_{jk},$$

where the φ_k satisfy the integral equation

$$\sigma_k^2 \varphi_k(t) = \int_{t_0}^{t_1} du\, \varphi_k(u) R(t - u), \qquad t_0 < t < t_1 \tag{3}$$

and

$$n_k = \int_{t_0}^{t_1} dt\, \varphi_k(t) n(t).$$

Here t_0 and $t_1 \triangleq t_0 + T$ mark the beginning and end of the receiver processing interval, $\sum_k$ is used to denote $\sum_{k=1}^{\infty}$, the symbol E signifies mathematical expectation, and

$$\delta_{jk} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

Since the $\{n_k\}$ are themselves gaussian and uncorrelated they are statistically independent. Assuming that the noise has zero mean, the joint density function of the first N coefficients, p_N, can be written as

$$p_N(n_k, k = 1, \cdots, N) \triangleq \prod_{k=1}^{N} \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \frac{1}{\sigma_k} \exp\left\{-\frac{{n_k}^2}{2{\sigma_k}^2}\right\}, \tag{4}$$

where the n_k are ordered corresponding to the relationship that

$$\sigma_1 \geqq \sigma_2 \geqq \cdots .$$

Now consider a formal expansion of the receiver input in terms of the eigenfunctions of (3). One can write

$$\begin{aligned} y(t) &\sim \sum_k y_k \varphi_k(t) = \sum_k {\psi_k}^{(i)} \varphi_k(t) + \sum_k n_k \varphi_k(t) \\ y_k &\triangleq \int_I dt\, \varphi_k(t) y(t) = \int_I dt\, \varphi_k(t) [{x_\delta}^{(i)}(t) + n(t)] \\ {\psi_k}^{(i)}(\delta) &\triangleq \int_I dt\, \varphi_k(t) x^{(i)}(t - \delta), \end{aligned} \tag{5}$$

where by definition $I \triangleq [t_0, t_1]$. Since the series $\sum_k n_k \varphi_k(t)$ is almost surely pointwise convergent to $n(t)$ we need only investigate the sense in which $\sum_k {\psi_k}^{(i)} \varphi_k(t)$ converges to $x^{(i)}(t - \delta)$. With this in mind we digress to consider some of the properties associated with the eigenfunctions of the integral (3) with $R(t - u) = R(0)e^{-\beta|t-u|}$. It is easy to show that in this case the solutions of the integral equation are identical to those which satisfy the following differential equations and boundary conditions:

$$\begin{aligned} \frac{d^2}{dt^2} \varphi_k(t) - \frac{\beta(\beta{\sigma_k}^2 - 2R(0))}{\sigma_k^2} \varphi_k(t) &= 0 \\ \beta \varphi_k(t_0) &= \frac{d}{dt} \varphi_k(t) \Big|_{t=t_0} \\ \beta \varphi_k(t_1) &= -\frac{d}{dt} \varphi_k(t) \Big|_{t=t_1} . \end{aligned} \tag{6}$$

The solutions of this system are proportional to

$$\cos \gamma_k \left[t - \left(\frac{t_0 + t_1}{2}\right)\right]$$

for k even, and

$$\sin \gamma_k \left[t - \left(\frac{t_0 + t_1}{2}\right)\right]$$

for k odd. Here γ_k satisfies the relation $(\beta^2 + \gamma_k^2) = 2\beta R(0)/\sigma_k^2$.

The differential equation (6) and the associated boundary conditions together form a Sturm-Liouville eigenvalue problem. The convergence properties of expansions in terms of the resulting eigenfunctions, the $\{\varphi_k(t)\}$, are stronger than those generally associated with expansions in terms of the eigenfunctions of the integral equation (3). An expansion of an integrable function $f(t)$ on the interval (t_0, t_1) in terms of the eigenfunctions of a Sturm-Liouville system, possesses the following property:[6]

In every interval where $f(t)$ is continuous and of bounded variation, the expansion converges uniformly and absolutely to $f(t)$. If at the ends of the interval there are neighborhoods in which $f(t)$ is of bounded variation then the series converges at these points to $f(t_{0+})$ and $f(t_{1-})$.

It will be assumed that the transmitted signal $s^{(i)}(t)$ and $h(t)$ are such that $x^{(i)}(t - \delta)$ is continuous and of bounded variation. In fact to make the succeeding development valid we shall have to impose more stringent requirements on $s^{(i)}(t)$ and $h(t)$. Under this assumption the expansion of $x^{(i)}(t - \delta)$ in terms of eigenfunctions of the integral equation (3) will converge uniformly and absolutely to $x^{(i)}(t - \delta)$.

Returning to (5) we have established that $\sum_k y_k\varphi_k(t)$ converges pointwise to $y(t)$ for almost all sample functions. That is $y(t) = \sum_k y_k\varphi_k(t)$, almost surely.

Choosing the values of the $\{y_k\}$ set as the observable coordinates, the likelihood function, l, can now be determined as the limit of the ratio of two N-dimensional density functions evaluated at the sample values, the y_j, $j = 1, \cdots N$. Here we have used the same symbol, y_j, to represent the sample and the random variable itself.

Thus l can be written as

$$l = \lim_{N\to\infty} \frac{\tilde{p}^{(1)}(y_1, \cdots, y_N)}{\tilde{p}^{(2)}(y_1, \cdots, y_N)}, \tag{7}$$

where $\tilde{p}^{(i)}(y_1, \cdots, y_N)$ is the joint probability density function of the first N members of the $\{y_k\}$ set.

Noting that $y_k = \psi_k^{(i)} + n_k$ and using the fact that the signal and noise components are statistically independent the joint probability density of the first N of the y_k can be written as a convolution. Thus

$$\begin{aligned} \tilde{p}_N^{(i)}(y_1, \cdots, y_N) &= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} dz_1, \cdots, dz_N \zeta^{(i)}(z_1, \cdots, z_N) \\ &\quad \cdot p_N(y_k - z_k, k = 1, \cdots, N) \\ \tilde{p}_N^{(i)}(y_1, \cdots, y_N) &= E_\psi(i) p_N(y_k - \psi_k^{(i)}, k = 1, \cdots, N) \end{aligned} \tag{8}$$

where

$\zeta^{(i)}(z_1, \cdots, z_N)$ is the joint probability density function of the first N of the $\psi_k^{(i)}(\delta)$,

$E_{\psi^{(i)}}$ denotes an expectation with respect to the random vector, $\psi^{(i)} = (\psi_1^{(i)}(\delta) \cdots \psi_N^{(i)}(\delta))$.

Since the $\psi_k^{(i)}(\delta)$ are all functions of the random variable δ, the averaging process can be performed with respect to δ instead of the $\psi_k^{(i)}(\delta)$. Thus

$$\tilde{p}_N^{(i)}(y_1, \cdots, y_N) = E_\delta p_N(y_k - \psi_k^{(i)}(\delta), k = 1, \cdots, N). \tag{9}$$

Using (4), (7) and (9), and cancelling common factors the likelihood function, l, is given by

$$l = \lim_{N \to \infty} \frac{E_\delta \exp \left\{ \sum_{k=1}^{N} \left[y_k - \frac{\psi_k^{(1)}(\delta)}{2} \right] \frac{\psi_k^{(1)}(\delta)}{\sigma_k^2} \right\}}{E_\delta \exp \left\{ \sum_{k=1}^{N} \left[y_k - \frac{\psi_k^{(2)}(\delta)}{2} \right] \frac{\psi_k^{(2)}(\delta)}{\sigma_k^2} \right\}}. \tag{10}$$

$$\text{Let } Q_N^{(i)}[\delta] \triangleq \sum_{k=1}^{N} \left[y_k - \frac{\psi_k^{(i)}(\delta)}{2} \right] \frac{\psi_k^{(i)}(\delta)}{\sigma_k^2}.$$

Then (10) can be rewritten as

$$l = \lim_{N \to \infty} \frac{E_\delta \exp \{Q_N^{(1)}[\delta]\}}{E_\delta \exp \{Q_N^{(2)}[\delta]\}}. \tag{11}$$

At this point one may develop an integral form for $Q_N^{(i)}[\delta]$. The integral will suggest the form of $\lim_{N \to \infty} Q_N^{(i)}[\delta]$ which will be required in the subsequent development. Define

$$\begin{aligned} q_N^{(i)}(t;\delta) &= \sum_{k=1}^{N} \frac{\psi_k^{(i)}(\delta)}{\sigma_k^2} \varphi_k(t) \\ x_N^{(i)}(t;\delta) &= \sum_{k=1}^{N} \psi_k^{(i)}(\delta) \varphi_k(t) \\ y_N(t) &= \sum_{k=1}^{N} y_k \varphi_k(t). \end{aligned} \tag{12}$$

Then using the orthonormality of the $\{\varphi_k(t)\}$ one has the relation

$$Q_N^{(i)}[\delta] = \int_I dt \left[y_N(t) - \frac{x_N(t;\delta)}{2} \right] q_N^{(i)}(t;\delta). \tag{13}$$

It also follows from (3), (12), and the orthonormality relations, that $q_N^{(i)}(t;\delta)$ satisfies the integral equation

$$x_N^{(i)}(t;\delta) = \int_I du\, q_N^{(i)}(u;\delta) R(0) \exp(-\beta \mid t - u \mid). \tag{14}$$

The solution of the integral equation is

$$q_N^{(i)}(t;\delta) = \frac{1}{2\beta R(0)} \left(\beta^2 - \frac{\partial^2}{\partial t^2} \right) x_N(t;\delta). \tag{15}$$

Thus (13) can be rewritten as

$$Q_N^{(i)}[\delta] = \int_I dt \left[y_N(t) - \frac{x_N(t;\delta)}{2} \right] \frac{1}{2\beta R(0)} \left(\beta^2 - \frac{\partial^2}{\partial t^2} \right) x_N(t;\delta). \tag{16}$$

Let us return to the study of the likelihood function of (11). In general it seems that the expectations appearing in (11) cannot be evaluated. However it is possible to evaluate the required expectations when the variance of the random delay δ is very small. With this in mind we drop the superscript and expand $Q_N(\delta)$ in a power series with remainder around $\delta = \bar{\delta}$.

$$\begin{aligned} Q_N[\delta] = Q_N[\bar{\delta}] + (\delta - \bar{\delta}) Q_N'[\bar{\delta}] + \frac{(\delta - \bar{\delta})^2}{2!} Q_N''[\bar{\delta}] \\ + \frac{(\delta - \bar{\delta})^3}{3!} Q_N'''[\bar{\delta} + \theta(\delta - \bar{\delta})], \qquad 0 \leqq \theta \leqq 1 \end{aligned} \tag{17}$$

where we have used the notation

$$Q_N'[\bar{\delta}] \triangleq \frac{d}{d\delta} Q_N(\delta) |_{\delta = \bar{\delta}}.$$

It follows that $\exp\{Q_N[\delta]\}$ can be expressed as,

$$\exp\{Q_N[\delta]\} = \exp\{Q_N[\bar{\delta}]\} \exp\left\{ (\delta - \bar{\delta})\, a_N + \frac{(\delta - \bar{\delta})^2}{2!} b_N + \frac{(\delta - \bar{\delta})^3}{3!} c_N(\delta) \right\}.$$

where

$$a_N \triangleq Q_N'[\bar{\delta}] \qquad b_N \triangleq Q_N''[\bar{\delta}] \qquad c_N(\delta) \triangleq Q_N'''[\bar{\delta} + \theta(\delta - \bar{\delta})].$$

On expanding the exponential in a powers series and averaging the uniformly convergent series term by term with respect to δ one obtains

$$E_\delta \exp \{Q_N[\delta]\} = \exp \{Q_N[\bar{\delta}]\} \left[1 + \sum_{k=1}^{\infty} \frac{d_{kN}}{k!}\right], \tag{18}$$

where

$$d_{kN} = E_\delta \left\{\left[(\delta - \bar{\delta}) a_N + \frac{(\delta - \bar{\delta})^2}{2!} b_N + \frac{(\delta - \bar{\delta})^3}{3!} c_N(\delta) \right]^k\right\}.$$

The limit of the likelihood ratio of (10), l, can be expressed as

$$l = \lim_{N\to\infty} \exp \{Q_N^{(1)}[\bar{\delta}] - Q_N^{(2)}[\bar{\delta}]\} \frac{\left(1 + \sum_{k=1}^{\infty} \frac{d_{kN}^{(1)}}{k!}\right)}{\left(1 + \sum_{k=1}^{\infty} \frac{d_{kN}^{(2)}}{k!}\right)}. \tag{19}$$

Note that if for some number $D(N)$

$$| d_{kN} | < [D(N)]^k \quad \text{for all } k,$$

then the infinite sums each converge since the series is majorized by $\exp [D(N)]$. It is observed that

$$| z_1 + z_2 + z_3 |^k \leqq [3 \max_i | z_i |]^k$$

which implies

$$| z_1 + z_2 + z_3 |^k \leqq 3^k (| z_1 |^k + | z_2 |^k + | z_3 |^k).$$

Using the definition of d_{kN} and the above inequality in conjunction with the Schwarz inequality yields

$$\begin{aligned} | d_{kN} | \leqq 3^k \Bigg\{ | a_N |^k E | (\delta - \bar{\delta})^k | + \left|\frac{b_N}{2!}\right|^k E | (\delta - \bar{\delta})^{2k} | \\ + \left(E\left|\frac{c_N(\delta)}{3!}\right|^{2k} \cdot E | \delta - \bar{\delta} |^{6k}\right)^{\frac{1}{2}}\Bigg\}. \end{aligned} \tag{20}$$

Now we restrict our investigation to the class of random delays whose probability distributions satisfy the relationship that for some number h and all k,

$$E_\delta | (\delta - \bar{\delta})^k | \leqq h^k \lambda^k, \tag{21}$$

where λ^2 is the variance of the distribution. This condition will be satisfied by all probability distribution functions which take on the values zero and one in a bounded region of the real line. That is when the values of δ are essentially restricted to a bounded region.

For distributions which satisfy (21) one finds

$$| d_{kN} | \leqq 3^k \left\{ | a_N |^k h^k \lambda^k + \left| \frac{b_N}{2!} \right|^k h^{2k} \lambda^{2k} + \frac{h^{3k} \lambda^{3k}}{3!} [E | c_N(\delta) |^{2k}]^{\frac{1}{2}} \right\}.$$

However if $Q_N'''[\delta]$ is bounded, $E | c_N(\delta) |^{2k}. \leqq (\bar{c}_N)^{2k}$ and

$$| d_{kN} | \leqq 3^k \{ | a_N | h\lambda + | b_N | h^2\lambda^2 + (\bar{c}_N)^2 h^3 \lambda^3 \}^k, \qquad | d_{kN} | \leqq A_N^k$$

where

$$\bar{c}_N \triangleq \sup Q_N'''[\delta],$$

$$A_N \triangleq 3 \{ | a_N | h\lambda + | b_N | h^2\lambda^2 + (\bar{c}_N)^2 h^3 \lambda^3 \} .$$

Now it can be shown that under certain restrictions on the channel and the class of transmitted signals the following limits exist almost surely:

$$\begin{aligned} \lim_{N\to\infty} a_N &\triangleq a, \\ \lim_{N\to\infty} b_N &\triangleq b, \\ \lim_{N\to\infty} c_N(\delta) &\triangleq c(\delta) \\ \lim_{N\to\infty} E | c_N(\delta) |^{2k} &\leqq \bar{c}^{2k} \end{aligned} \tag{22}$$

for all k.

The convergence of $\lim_{N\to\infty} \sum_{k=1}^{\infty} (d_{kN}/k!)$ can now be demonstrated. In this and the subsequent development we will consider convergence in the almost sure sense. Breaking the sum into a finite sum from k equal 1 through m and a sum from $m + 1$ to ∞ and taking magnitudes gives

$$\left| \lim_{N\to\infty} \sum_{k=1}^{\infty} \frac{d_{kN}}{k!} - \sum_{k=1}^{m} \frac{d_k}{k!} \right| = \left| \lim_{N\to\infty} \sum_{k=m+1}^{\infty} \frac{d_{kN}}{k!} \right|$$

where $d_k \triangleq \lim_{N\to\infty} d_{kN}$. Since $| d_{kN} |$ is less than or equal to some number A_N^k, then

$$\left| \lim_{N\to\infty} \sum_{k=1}^{\infty} \frac{d_{kN}}{k!} - \sum_{k=1}^{m} \frac{d_k}{k!} \right| \leqq \lim_{N\to\infty} \sum_{k=m+1}^{\infty} \frac{A_N^k}{k!} .$$

But $\lim_{N\to\infty} A_N = A$ exists, thus for all $N > N_0$

$$| A - A_N | < \epsilon, \quad | A_N | < | A | + \epsilon,$$

$$\Rightarrow \left| \lim_{N\to\infty} \sum_{k=1}^{\infty} \frac{d_{kN}}{k!} - \sum_{k=1}^{m} \frac{d_k}{k!} \right| \leqq \sum_{k=m+1}^{\infty} \frac{[| A | + \epsilon]^k}{k!}.$$

By choosing m sufficiently large the right hand side of the inequality can be made less than any positive constant. Therefore

$$\lim_{N\to\infty} \sum_{k=1}^{\infty} \frac{d_{kN}}{k!} = \sum_{k=1}^{\infty} \frac{d_k}{k!}. \tag{23}$$

For small λ one makes the approximation

$$\lim_{N\to\infty} \left[1 + \sum_{k=1}^{\infty} \frac{d_{kN}}{k!} \right] \sim 1 + d_1 + \frac{d_2}{2!}, \tag{24}$$

where terms involving λ to powers greater than λ^2 have been neglected, and

$$d_k = \lim_{N\to\infty} d_{kN}.$$

Using the definition of d_{kN}, which follows (18), we find

$$d_1 \sim (\lambda^2/2!)b,$$

$$d_2 \sim \lambda^2 a^2.$$

Restoring the superscript notation and using (22) and (24), (19) becomes

$$l \sim \exp \left[\lim_{N\to\infty} \{ Q_N^{(1)}[\bar{\delta}] - Q_N^{(2)}[\bar{\delta}] \} \right]$$

$$\cdot \{ 1 + (\lambda^2/2)[(a^{(1)})^2 - (a^{(2)})^2 + b^{(1)} - b^{(2)}] \} \tag{25}$$

Equation (25) is the desired approximation for the likelihood function for the case of small λ.

III. STRUCTURE OF THE OPTIMUM RECEIVER

The structure of the approximation to the optimum receiver statistic, the likelihood function, can be determined from (25). The quantities appearing there can all be expressed in terms of the received signal, $y(t)$ and the noise-free filter outputs $x^{(1)}(t)$ and $x^{(2)}(t)$, which are assumed to be known. In the Appendix it is shown that almost surely

$$\lim_{N\to\infty} Q_N[\bar\delta] = \frac{1}{2\beta R(o)}\left\{\int_I \left[y(t) - \frac{x(t-\bar\delta)}{2}\right]\right.$$
$$\cdot[\beta^2 x(t-\bar\delta) - x''(t-\bar\delta)]\,dt$$
$$- [x'(t_0-\bar\delta) - \beta x(t_0-\bar\delta)]\left[y(t_0) - \frac{x(t_0-\bar\delta)}{2}\right] \quad (26)$$
$$+ [x'(t_1-\bar\delta) + \beta x(t_1-\bar\delta)]$$
$$\left.\cdot\left[y(t_1) - \frac{x(t_1-\bar\delta)}{2}\right]\right\},$$

$$a = \lim_{N\to\infty} a_N = \frac{1}{2\beta R(0)}\left\{\int_I \left[y(t) - \frac{x(t-\bar\delta)}{2}\right]\right.$$
$$\cdot[-\beta^2 x'(t-\bar\delta) + x'''(t-\bar\delta)]\,dt$$
$$- [x''(t_1-\bar\delta) + \beta x'(t_1-\bar\delta)]\,[y(t_1) - \tfrac{1}{2}x(t_1-\bar\delta)] \quad (27)$$
$$+ [x''(t_0-\bar\delta) - \beta x'(t_0-\bar\delta)]\,[y(t_0) - \tfrac{1}{2}x(t_0-\bar\delta)]$$
$$+ \tfrac{1}{2}[x'(t_1-\bar\delta) + \beta x(t_1-\bar\delta)]\,[x'(t_1-\bar\delta)]$$
$$\left. - \tfrac{1}{2}[x'(t_0-\bar\delta) - \beta x(t_0-\bar\delta)]\,[x'(t_0-\bar\delta)]\right\},$$

$$b = \lim_{N\to\infty} b_N = \frac{1}{2\beta R(0)}\left\{\int_I [y(t) - x(t-\bar\delta)]\right.$$
$$\cdot[\beta^2 x''(t-\bar\delta) - x''''(t-\bar\delta)]\,dt$$
$$+ [\beta x''(t_0-\bar\delta) - x'''(t_0-\bar\delta)]\,[y(t_0) - x(t_0-\bar\delta)]$$
$$+ [\beta x''(t_1-\bar\delta) + x'''(t_1-\bar\delta)]\,[y(t_1) - x(t_1-\bar\delta)] \quad (28)$$
$$- \int_I dt[(\beta x'(t-\bar\delta))^2 - x'(t-\bar\delta)x'''(t-\bar\delta)]$$
$$+ [x''(t_0-\bar\delta) - \beta x'(t_0-\bar\delta)]x'(t_0-\bar\delta)$$
$$\left. - [x''(t_1-\bar\delta) + \beta x'(t_1-\bar\delta)]x'(t_1-\bar\delta)\right\}.$$

It is also found that $c(\delta)$ is a piecewise continuous function of δ.

In obtaining these results it is assumed that the second derivative of $x(t)$ is continuous and that the quantity $[-\beta^2 x'(t-\bar\delta) + x'''(t-\bar\delta)]$ appearing in the integral in (27) is of bounded variation and continuous except at a finite number of points. Similar assumptions are made on

$[\beta^2 x''(t - \bar{\delta}) - x''''(t - \bar{\delta})]$ in the expression for b and a similar term in the expression for $c(\delta)$. One requires that the fifth derivative of $x(t)$ be of bounded variation and be continuous except at a finite number of points. These assumptions allow Sturm-Liouville expansions of the integrands involved in (27) and (28) which, using the orthonormality of the $\{\varphi_k(t)\}$, yield the proof of (27) and (28). Thus a sufficient condition for the validity of these results is that the second derivative of $x(t)$ be continuous and that the fifth derivative of $x(t)$ be of bounded variation and continuous except at a finite number of points.

The desired test statistic, l, is obtained by substituting (26), (27) and (28) in (25) with $x^{(i)}(t)$, $i = 1$ or 2, replacing $x(t)$ in the appropriate places. The resulting expression is nonlinear in $y(t)$ and rather lengthy and will not be explicitly stated here. Observe however from (27) that if phase reversal signaling is used, that is $x^{(2)}(t) = -x^{(1)}(t)$, the quantity $(a^{(1)})^2 - (a^{(2)})^2$ will be linear in $y(t)$. Furthermore under this condition, to first order in λ^2, $\ln l$ will be linear in $y(t)$. Therefore for phase reversal signaling the receiver correlates $y(t)$ with a signal related to $x(t - \delta)$ and its derivatives evaluated at $\delta = \bar{\delta}$.

The "white noise case" will now be obtained as a limit of the exponential covariance case. Setting $R(0) = N_0\beta/2$ and letting $\beta \to \infty$ in the expressions for Q, a and b one obtains

$$\ln l \sim \left\{\frac{1}{N_0}\int_I dt\left[y(t) - \frac{x(t-\bar{\delta})}{2}\right]x(t-\bar{\delta}) + \frac{\lambda^2}{2N_0^2}\left(-\int_I dt\left[y(t) - \frac{x(t-\bar{\delta})}{2}\right]x'(t-\bar{\delta})\right)^2 + \frac{\lambda^2}{2N_0}\left(\int_I dt[y(t) - x(t-\bar{\delta})]\,x''(t-\bar{\delta}) - \int_I dt[x'(t-\bar{\delta})]^2\right)\right\}_{x=x^{(2)}}^{x=x^{(1)}} \tag{29}$$

In the above expression the notation $\{\ \ \}_{x=x^{(2)}}^{x=x^{(1)}}$ is used to indicate that the expression in braces is evaluated with $x(t) = x^{(2)}(t)$ and the result is subtracted from the result which obtains when $x(t) = x^{(1)}(t)$. For phase reversal signalling, $x^{(1)}(t) = -x^{(2)}(t) = x(t)$, (29) becomes

$$\ln l \sim \frac{2}{N_0}\int_I dt\, y(t)\left\{x(t-\bar{\delta}) - \frac{\lambda^2}{2}\left[\frac{1}{N_0}\,(x^2(t_1-\bar{\delta}) - x^2(t_0-\bar{\delta}))\,x'(t-\bar{\delta}) - x''(-\bar{\delta})\right]\right\}. \tag{30}$$

Let us examine the correlation waveform appearing inside the braces in (30). The first term represents the receiver input when the random delay is equal to its mean value, $\delta = \bar{\delta}$. If λ is set equal to zero the remaining terms vanish and one obtains the standard result which is that the receiver input should be correlated with the signal portion of the input, $x(t - \bar{\delta})$. The remaining terms inside the braces in (30) are the perturbations introduced by the random delay.

Let us expand $x(t - \delta)$ in the Taylor series with remainder

$$x(t - \delta) = x(t - \bar{\delta}) + (\delta - \bar{\delta})[-x'(t - \bar{\delta})] + [(\delta - \bar{\delta})^2/2!][x''(t - \bar{\delta})] + \mathcal{R}(\delta - \bar{\delta}, t).$$

If $x'''(t - \bar{\delta})$ is continuous and the moments of δ satisfy

$$E_\delta \mid (\delta - \bar{\delta})^k \mid \leqq h^k \lambda^k,$$

then for small λ, $x(t - \bar{\delta}) + (\lambda^2/2)x''(t - \bar{\delta})$ is the principal part of $E_\delta x(t - \delta)$. Thus, part of the correlation waveform is essentially the expected value of the received signal. The other term in the correlation waveform involves $x'(t - \bar{\delta})$, which is the slope of the received signal when the delay is $\bar{\delta}$. The weight attached to it is proportional to the difference in the squared values of the received signal at time t_0 and at time t_1 when $\delta = \bar{\delta}$. As yet no physical significance has been found for this term.

IV. CONCLUSIONS

An approximation has been obtained for a test statistic that minimizes the error probability of a binary communication system which operates over a linear channel, with a small random delay, in the presence of gaussian noise of covariance $R(0)e^{-\beta|\tau|}$. For the case of phase reversal signaling, the statistic, which in general is nonlinear, is a linear functional of the receiver input. Treating the "white noise" case as a limit of the exponential covariance case, the test statistic is expressed as a cross-correlation operation. The waveform with which the input is correlated is related to the expected value of the received signal plus a term proportional to the slope of the received signal when the delay is equal to its mean value.

V. ACKNOWLEDGMENT

The author wishes to express appreciation to Dr. A. R. Cohen of New York University, Dr. I. Jacobs of Bell Telephone Laboratories and the referees for their valuable suggestions.

APPENDIX

The Evaluation of Terms Arising in the Approximation to the Likelihood Function

The convergence asserted in (22) will now be established and the structure of the terms appearing in the likelihood function, (25), will be exhibited.

First let us note the following property associated with an expansion in terms of the eigenfunctions of a Sturm-Liouville system.

Let $g(t)$ be a piecewise continuous function which is of bounded variation and let $z(t)$ be a piecewise continuous function. If the sets $\{g_k\}$ and $\{z_k\}$ are the expansion coefficients of $g(t)$ and $z(t)$ in terms of the above eigenfunctions, then

$$\int_I dt\, g(t)z(t) = \sum_{k=1}^{\infty} g_k z_k . \tag{31}$$

This expression is obtained by noting that the series expansion of $g(t)$ converges uniformly except at a finite number of points.

One considers the following integral which is suggested by (16),

$$K(\delta) \triangleq \frac{1}{2\beta R(0)} \int_I dt \left[y(t) - \frac{x(t-\delta)}{2} \right] \left(\beta^2 - \frac{\partial^2}{\partial t^2} \right) x(t-\delta), \tag{32}$$

where one requires that $[\beta^2 - (\partial^2/\partial t^2)]x(t-\delta)$ be of bounded variation and piecewise continuous. Let

$$g(t-\delta) \triangleq \left(\beta^2 - \frac{\partial^2}{\partial t^2} \right) x(t-\delta) = \beta^2 x(t-\delta) - x''(t-\delta). \tag{33}$$

$g(t)$ can be expanded in a series in terms of the eigenfunctions of (6). This series converges uniformly to $g(t)$ where $g(t)$ is continuous and to $\frac{1}{2}[g(u_+) + g(u_-)]$ at points where $g(t)$ is not continuous. Thus

$$g(t-\delta) = \sum_{k=1}^{\infty} g_k \varphi_k(t), \tag{34}$$

where

$$g_k = \int_I du\, \varphi_k(u) g(t-\delta).$$

By integration by parts and utilizing (6) one can show that

$$\begin{aligned} g_k = (\beta^2 + \gamma_k^2)\psi_k &+ \varphi_k(t_0)[x'(t_0-\delta) - \beta x(t_0-\delta)] \\ &- \varphi_k(t_1)[x'(t_1-\delta) + \beta x(t_1-\delta)], \end{aligned} \tag{35}$$

where

ψ_k is the kth expansion coefficient of $x(t-\delta)$ in terms of the $\{\varphi_k(t)\}$ set,

$(\beta^2+\gamma_k^2) = [2\beta R(0)/\sigma_k^2]$,

σ_k^2 is the kth eigenfunction of (6).

Substituting the expansion for $g(t)$ in (32) and using (35) one finds

$$\begin{aligned} K(\delta) = \frac{1}{2\beta R(0)} \Big\{ & 2\beta R(0) \sum_{k=1}^{\infty} \left(y_k - \frac{\psi_k}{2}\right) \frac{\psi_k}{\sigma_k^2} \\ & + [x'(t_0-\delta) - \beta x(t_0-\delta)] \sum_{k=1}^{\infty} \left(y_k - \frac{\psi_k}{2}\right) \varphi_k(t_0) \\ & - [x'(t_1-\delta) + \beta x(t_1-\delta)] \sum_{k=1}^{\infty} \left(y_k - \frac{\psi_k}{2}\right) \varphi_k(t_1) \Big\}. \end{aligned} \tag{36}$$

Noting that $\sum_{k=1}^{\infty} \left(y_k - \frac{\psi_k}{2}\right) \frac{\psi_k}{\sigma_k^2}$ equals $\lim_{N\to\infty} Q_N[\delta]$ one finds

$$\begin{aligned} \lim_{E\to\infty} Q_N[\delta] = K(\delta) & - \frac{[x'(t_0-\delta) - \beta x(t_0-\delta)]}{2\beta R(0)} \sum_{k=1}^{\infty} \left(y_k - \frac{\psi_k}{2}\right) \varphi_k(t_0) \\ & + \frac{[x'(t_1-\delta) + \beta x(t_1-\delta)]}{2\beta R(0)} \sum_{k=1}^{\infty} \left(y_k - \frac{\psi_k}{2}\right) \varphi_k(t_1) \end{aligned} \tag{37}$$

One now can proceed to evaluate $a = \lim_{N=\infty} a_N$.

From the definitions following (17) and (10) one has

$$a_N = Q_N'[\bar{\delta}] = \sum_{i=1}^{N} \frac{(y_k - \psi_k(\bar{\delta}))}{\sigma_k^2} x_k'(\bar{\delta}). \tag{38}$$

Consider

$$K_1(\delta) \triangleq \int_I dt \left[y(t) - \frac{x(t-\delta)}{2} \right] [-\beta^2 x'(t-\delta) + x'''(t-\delta)]. \tag{39}$$

Let us assume that $[-\beta^2 x'(t-\delta) + x'''(t-\delta)]$ is piecewise continuous, $[\beta^2 x(t-\delta) - x''(t-\delta)]$ is continuous and that both are of bounded variation. Define

$$\alpha_k = \int_I dt\, \varphi_k(t)\, [-\beta^2 x'(t-\delta) + x'''(t-\delta)]. \tag{40}$$

Then under these assumptions and using (33) and (34) one finds

$$\alpha_k = g_k'(\delta). \tag{41}$$

Applying (31) to $K_1(\delta)$ gives

$$K_1(\delta) = \sum_{k=1}^{\infty} \left(y_k - \frac{\psi_k(\delta)}{2} \right) \alpha_k . \tag{42}$$

Using (41), (42), (35) and (31) yields

$$\begin{aligned}\sum_{k=1}^{\infty} \left(\frac{y_k - \psi_k(\delta)}{\sigma_k^2} \right) \psi_k'(\delta) &= \frac{K_1(\delta)}{2\beta R(0)} \\ &- \frac{1}{2\beta R(0)} [x''(t_1 - \delta) + \beta x'(t_1 - \delta)] \left[y(t_1) - \frac{x(t_1 - \delta)}{2} \right] \\ &+ \frac{1}{2\beta R(0)} [x''(t_0 - \delta) - \beta x'(t_0 - \delta)] \left[y(t_0) - \frac{x(t_0 - \delta)}{2} \right] \\ &- \tfrac{1}{2} [x'(t_0 - \delta) - \beta x(t_0 - \delta)] x'(t_0 - \delta) \\ &\qquad + \tfrac{1}{2} [x'(t_1 - \delta) + \beta x(t_1 - \delta)] x'(t_1 - \delta). \end{aligned} \tag{43}$$

The left hand side of (43) evaluated at $\delta = \bar{\delta}$ is $\lim_{N \to \infty} a_N$.

Proceeding in a similar manner the $\lim_{N \to \infty} b_N$ and $\lim_{N \to \infty} c_N(\delta)$ are obtained.

REFERENCES

1. Helstrom, Carl W., *Statistical Theory of Signal Detection,* Chapter IX, Pergamon Press, 1960.
2. Brown, W. M., and Palermo, C. J., System Performance In the Presence of Stochastic Delays, IRE Trans. Info. Theory, *IT-8*, Sept. 1962, No. 5.
3. Balakrishnan, A. V. On the Problem of Time Jitter in Sampling, IRE Trans. Info. Theory, *IT-8*, April 1962, No. 3.
4. Grenander, U., Stochastic Processes and Statistical Inference, Arkiv for Matematik, Band 1 nr 17.
5. Belayev Yu. K., *Continuity and Hölder's Conditions for Sample Functions of Stationary Gaussian Processes*, Fourth Berkeley Symposium on Mathematical Statistics and Probability (1960), University of California Press, 1961.
6. Hobson, E. W., *The Theory of Functions of a Real Variable, 2,* p. 772, Dover Publications, 1956.