Optimum Reception of Binary Sure and Gaussian Signals

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(Manuscript received May 25, 1965)

The problem of optimum reception of binary sure and Gaussian signals is to specify, in terms of the received waveform, a scheme for deciding between two alternative mean and covariance functions with minimum error probability. In the context of a general treatment of the problem, this article presents a solution which is both mathematically rigorous and convenient for physical application. The optimum decision scheme obtained consists in comparing, with a predetermined threshold c, the sum of a linear and a quadratic form in the received waveform $x(t)$*; namely, choose* $m_0(t)$ *and* $r_0(s,t)$ *if*

$$2\int x(t)g(t)\,dt + \iint [x(s) - m_1(s)]h(s,t)[x(t) - m_1(t)]\,ds\,dt < c,$$

choose $m_1(t)$ *and* $r_1(s,t)$ *if otherwise, where* $m_0(t)$, $m_1(t)$, $r_0(s,t)$ *and* $r_1(s,t)$ *are the two mean and covariance functions, and* $g(t)$ *is the square-integrable solution of*

$$\int r_0(s,t)g(s)\,ds = m_1(t) - m_0(t),$$

while $h(s,t)$ *is the symmetric and square-integrable solution of*

$$\iint r_0(s,u)h(u,v)r_1(v,t)\,du\,dv = r_1(s,t) - r_0(s,t).$$

Note that under the assumption of zero mean functions, i.e., $m_0(t) = m_1(t) = 0$, *the above result is reduced to the one in a previous article by this author, while with the assumption of identical covariance functions, i.e.,* $r_0(s,t) = r_1(s,t)$, *it is reduced to the classical result essentially obtained by Grenander.*

Sections I and II introduce the problem and summarize the main results with certain pertinent remarks, while a detailed mathematical treatment is given in Section III. Although Appendices A–D are not directly required *for solution of the problem, they are added to provide a tutorial background for the results on equivalence and singularity of two Gaussian measures obtained by Grenander, Root and Pitcher as well as some generalization of their results.*

I. INTRODUCTION

Suppose the received waveform $x(t)$ observed during the interval $0 \leqq t \leqq 1$ is the sample function of a Gaussian process, whose mean and covariance functions are either $m_0(t)$ and $r_0(s,t)$ or $m_1(t)$ and $r_1(s,t)$. We assume that $m_0(t)$ and $m_1(t)$ are continuous while $r_0(s,t)$ and $r_1(s,t)$ are positive-definite as well as continuous. Denote by H_k, $k = 0,1$, the hypothesis that $m_k(t)$ and $r_k(s,t)$ are the mean and covariance functions of the Gaussian process $\{x_t, 0 \leqq t \leqq 1\}$. Suppose further that α, $0 < \alpha < 1$, and $1 - \alpha$ are the *a priori* probabilities associated with the two hypotheses H_0 and H_1 respectively. Then, reception of binary sure and Gaussian signals may be regarded as a problem of deciding between two hypotheses H_0 and H_1 upon observation of the sample function $x(t)$. Thus, the problem of optimum reception of binary sure and Gaussian signals is to specify a decision scheme in terms of $x(t)$ such that its error probability is minimum.*

In the previous article,[1] a general treatment of the problem was made under the assumption that $m_0(t) = m_1(t) = 0$, and several forms of the optimum decision schemes were given under additional conditions with varying degrees of restriction. The following is most restrictive but most convenient for physical application:

$$\begin{aligned} &\text{choose } H_0 \text{ if } \int_0^1 \int_0^1 x(s)h(s,t)x(t)\, ds\, dt < k, \\ &\text{choose } H_1 \text{ if otherwise,} \end{aligned} \tag{1}$$

where $h(s,t)$ is the solution of the integral equation.†

$$\int_0^1 \int_0^1 r_0(s,u)h(u,v)r_1(v,t)\, du\, dv = r_1(s,t) - r_0(s,t) \tag{2}$$

satisfying

$$\int_0^1 \int_0^1 h^2(s,t)\, ds\, dt < \infty, \tag{3}$$

and k is a positive constant (the predetermined threshold); provided the

* A more complete motivation of the problem is given in Ref. 1.

† Existence of such a solution is a part of the condition for (1) to be the optimum decision scheme.

following additional conditions are satisfied for all $i,j = 1,2, \cdots$:

$$a_{ii} > \sum_{j=1}^{\infty}{}' \mid a_{ij} \mid, \quad \frac{\left| \dfrac{a_{ij}}{\lambda_i} - \delta_{ij} \right|}{a_{jj} - \sum_{k=1}^{\infty}{}' \mid a_{jk} \mid} \leqq K, \tag{4}$$

where λ_i, $i = 1,2, \cdots$, are the eigenvalues of the covariance kernel $r_0(s,t)$ and a_{ij}; $i,j = 1,2, \cdots$, are defined by

$$a_{ij} = \int_0^1 \int_0^1 \psi_i(s) r_1(s,t) \psi_j(t) \, ds \, dt,$$

with $\psi_i(t)$, $i = 1,2, \cdots$, being the orthonormalized eigenfunctions corresponding to λ_i, $i = 1,2, \cdots$, and K is a constant independent of i and j, and the prime above the summation sign signifies omission of the term $j = i$ or $k = j$, whichever the case may be.

As remarked in the previous article, the conditions (4) are not essential to the nature of the problem but are imposed for the sake of mathematical proof. Moreover, they are undesirable from the application viewpoint since they not only are restrictive but also require the explicit knowledge of the eigenvalues and eigenfunctions of the kernel $r_0(s,t)$. Recently, Rao and Varadarajan[2]* and Pitcher[3] have obtained certain general results (on the expression of Radon-Nikodym derivatives), which indicate that such conditions are unnecessary and can be replaced by more meaningful ones. In fact, Rao and Varadarajan extend to the general case where the assumption $m_0(t) = m_1(t) = 0$ is no longer made. The purpose of this article is to generalize the previous results[1] by removing the assumption $m_0(t) = m_1(t) = 0$ and replacing the conditions (4) with more appropriate ones. The first half of the development is a direct generalization of the former Solution — I (the "sampling" approach), while the second half is the application of the results of Grenander[4] and Pitcher to the problem of optimum reception.†

II. SUMMARY AND DISCUSSION OF MAIN RESULTS

As previously stated,[1] the foundation for solution of the problem of optimum reception consists of the following (measure theoretical) facts:

* This article appeared even before the author's previous one,[1] although the current result as well as the previous one were obtained independently.

† The results of Grenander and Pitcher are better suited for this problem than those of Rao and Varadarajan since the former readily yield a concrete specification of the optimum decision scheme comparable to (1)–(3). Although the problem stated at the beginning is solved by a particular combination of Grenander's and Pitcher's result, we have added in appendices an extension of Pitcher's results on equivalence and singularity of two Gaussian measures to the general case where $m_0(t) \neq 0 \neq m_1(t)$ for its own interest.

(1.) Two Gaussian probability measures P_0 and P_1, corresponding respectively to $m_0(t)$ and $r_0(s,t)$ and to $m_1(t)$ and $r_1(s,t)$, can be either "equivalent" or "singular".

(2.) If P_0 and P_1 are equivalent, then there is a certain random variable dP_1/dP_0 called the Radon-Nikodym derivative of P_1 with respect to P_0, which is a function of the sample function $x(t)$, and the following decision scheme yields the minimum non-zero error probability:

$$\begin{aligned} &\text{choose} \quad H_0 \quad \text{if} \quad \frac{dP_1}{dP_0}(x) < \frac{\alpha}{1-\alpha}, \\ &\text{choose} \quad H_1 \quad \text{if} \quad \text{otherwise.} \end{aligned} \tag{5}$$

On the other hand, if P_0 and P_1 are singular, then there is a set N of sample functions such that $P_0(N) = 0$ and $P_1(N) = 1$, thus the error probability of the following decision scheme:

$$\begin{aligned} &\text{choose} \quad H_0 \quad \text{if} \quad x(t) \text{ does not belong to } N, \\ &\text{choose} \quad H_1 \quad \text{if} \quad \text{otherwise,} \end{aligned} \tag{6}$$

is exactly zero, regardless of the *a priori* probabilities, thus resulting in the case of "perfect reception".

Hence, the problem of specifying the optimum decision scheme becomes the problem of finding such a random variable dP_1/dP_0 and a set N as well as a criterion to tell whether P_0 and P_1 are equivalent or singular.

2.1 *Solutions — I*

Suppose $x(t_1), \cdots, x(t_n)$, $0 \leqq t_1 < \cdots < t_n \leqq 1$, are the values of the sample function (the received waveform) sampled at $t_1, \cdots, t_n$, where each sampling interval is to become infinitesimal as $n \to \infty$. Likewise, let $m_0(t_1), \cdots, m_0(t_n)$ be the sampled values of $m_0(t)$. Then, the joint probability density functions for $x(t_1) - m_0(t_1), \cdots, x(t_n) - m_0(t_n)$ under the two hypotheses H_0 and H_1 are obtained by using the mean and ovariance functions $m_0(t)$ and $r_0(s,t)$ (under H_0) and $m_1(t)$ and $r_1(s,t)$ (under H_1).* Then, by forming the ratio of the two density functions, the likelihood ratio l_n of $x(t_1) - m_0(t_1), \cdots, x(t_n) - m_0(t_n)$ is obtained as follows:

* A rather artificial choice of $x(t_i) - m_0(t_i)$, instead of $x(t_i)$, $i = 1, \cdots, n$, is purely for a notational convenience later, and other choices are equally acceptable at this point.

$$\begin{aligned} l_n(x) = {} & | R_0^{(n)}(R_1^{(n)})^{-1} |^{\frac{1}{2}} \exp \Bigg[\frac{1}{2} \sum_{i,j=1}^{n} [x_{t_i} - m_0(t_i)] \\ & \times [(R_0^{(n)})^{-1} - (R_1^{(n)})^{-1}]_{ij} [x_{t_j} - m_0(t_j)] \\ & + \sum_{i,j=1}^{n} \left(x_{t_i} - \frac{m_0(t_i) + m_1(t_i)}{2} \right) [(R_1^{(n)})^{-1}]_{ij} \\ & \times [m_1(t_j) - m_0(t_j)] \Bigg], \end{aligned} \tag{7}$$

where $R_k^{(n)}$, $k = 0, 1$, are $n \times n$ covariance matrices defined by

$$(R_k^{(n)})_{ij} = r_k(t_i, t_j), \qquad k = 0, 1; \qquad i,j = 1, \cdots, n.$$

Next, through the use of martingale theory, the following facts can be established:
P_0 and P_1 are equivalent (the case of non-perfect reception), if and only if

$$\begin{aligned} \lim_{n \to \infty} | \operatorname{tr}[R_0^{(n)}(R_1^{(n)})^{-1} - 2I + R_1^{(n)}(R_0^{(n)})^{-1} + (R_0^{(n)})^{-1} M^{(n)} \\ + (R_1^{(n)})^{-1} M^{(n)}] | < \infty, \end{aligned} \tag{8}$$

where $(M^{(n)})_{ij} = m_i m_j$; $i,j = 1, \cdots, n$,* and m_i, $i = 1, \cdots, n$, are given by

$$m_i = m_1(t_i) - m_0(t_i).$$

In this case

$$\lim_{n \to \infty} l_n(x) = \frac{dP_1}{dP_0}(x) \tag{9}$$

for almost all sample functions under both hypotheses H_0 and H_1.
P_0 and P_1 are singular (the case of perfect reception) if and only if (8) is not satisfied.† In this case, for almost all sample functions,

$$\lim_{n \to \infty} l_n(x) = \begin{cases} 0 \text{ under } H_0, \\ \infty \text{ under } H_1. \end{cases}$$

That is, (8) is a necessary and sufficient condition for the perfect reception to be impossible. The crucial random variable dP_1/dP_0, by which the optimum decision scheme is specified in this case, can be expressed as the limit of the likelihood ratio $l_n(x)$ for almost all sample functions

* "tr" denotes "trace", and I is the $n \times n$ identity matrix.
† In this case, the left-hand side of (8) becomes $+\infty$ necessarily.

$x(t)$. Likewise, negation of (8) is a necessary and sufficient condition for the perfect reception to be possible, and the critical set N can be specified as the set of all sample functions for which the limit of the likelihood ratio is not smaller than any positive constant, say $\alpha/(1 - \alpha)$. Therefore, we conclude, in conjunction with (5) and (6), that irrespective of whether or not the condition (8) is satisfied, the optimum decision scheme can be specified as follows:

$$\begin{aligned} &\text{choose} \quad H_0 \quad \text{if} \quad \lim_{n\to\infty} l_n(x) < \frac{\alpha}{1-\alpha}, \\ &\text{choose} \quad H_1 \quad \text{if} \quad \text{otherwise.} \end{aligned} \tag{10}$$

We note in (8) that, if $m_0(t) = m_1(t) = 0$, the trace of the last two terms in the bracket vanishes, thus the necessary and sufficient condition for equivalence of P_0 and P_1 is reduced to

$$\lim_{n\to\infty} \operatorname{tr}[R_0^{(n)}(R_1^{(n)})^{-1} - 2I + R_1^{(n)}(R_0^{(n)})^{-1}] < \infty, \tag{11}$$

which agrees with the previous result.[1] Similarly, if $r_0(s,t) = r_1(s,t) = r(s,t)$, the trace of the first three terms vanishes and the necessary and sufficient condition is reduced to

$$\lim_{n\to\infty} \operatorname{tr}[(R^{(n)})^{-1} M^{(n)}] \equiv \lim_{n\to\infty} (m^{(n)}, (R^{(n)})^{-1} m^{(n)}) < \infty,$$

where $m^{(n)} = (m_i, \cdots, m_n)$.

Now, since the trace of the last two terms in the bracket of (8) is always positive as indicated above, (11) is a necessary condition for (8). Also, since the left-hand side of (11) is known to be either finite or $+\infty$, the conditions

$$\lim_{n\to\infty} \operatorname{tr}[(R_k^{(n)})^{-1} M^{(n)}] < \infty, \qquad k = 0,1 \tag{12}$$

are necessary for (8). Thus, we conclude that a necessary and sufficient condition for equivalence of P_0 and P_1 is that P_0 and P_1 be equivalent in the following three special cases:

(i) $m_0(t) = m_1(t) = 0$,
(ii) $r_0(s,t)$ is substitued for $r_1(s,t)$,
(iii) $r_1(s,t)$ is substituted for $r_0(s,t)$.*

* It can easily be shown that the cases (ii) and (iii) can be combined to the case (iv) where $r_0(s,t) + r_1(s,t)$ is substituted for both $r_0(s,t)$ and $r_1(s,t)$. Thus, the necessary and sufficient condition for equivalence of P_0 and P_1 becomes that they be equivalent in the special cases (i) and (iv). This condition has already been reported elsewhere.[2,8] Furthermore, as it turns out, either the case (ii) or the case (iii) is redundant. That is, P_0 and P_1 are equivalent in general if they are so either in the special cases (i) and (ii) or in (i) and (iii), as shown in Appendix D.

It may be illuminating to rephrase this in terms of the perfect reception of binary (sure and Gaussian) signals, though the use of terms is slightly inconsistent with the remainder of this article. Suppose we consider $m_0(t)$ and $m_1(t)$ as binary sure signals and $r_0(s,t)$ and $r_1(s,t)$ as the covariance functions of binary Gaussian signals or noise whichever the case may be. Then, the perfect reception of the binary sure and Gaussian signals is possible if any one of the following three conditions is satisfied by the constituent signals and noise:

(i′) the perfect reception is possible between the two Gaussian signals alone,

(ii′) the perfect reception of the binary sure signals is possible in the presence of the Gaussian noise with the covariance function $r_0(s,t)$.

(iii′) the condition identical to (ii′) except for $r_0(s,t)$ being replaced by $r_1(s,t)$.

Examination of the form of the likelihood ratio l_n in (7) in conjunction with the decision scheme (10) indicates that, if the exponent and the factor before the exponential converge separately, (10) can be rewritten in terms of their limits. Namely, if there exist a positive constant β and a random variable θ such that

$$\beta = \lim_{n\to\infty} | R_0^{(n)}(R_1^{(n)})^{-1} |, \tag{13}$$

and

$$\begin{aligned}\theta(x) = \lim_{n\to\infty}\Bigg[\sum_{i,j=1}^{n} [x_{t_i} - m_0(t_i)]\,[(R_0^{(n)})^{-1} - (R_1^{(n)})^{-1}]_{ij} \\ \times [x_{t_j} - m_0(t_j)] + 2\sum_{i,j=1}^{n}\left(x_{t_i} - \frac{m_0(t_i) + m_1(t_i)}{2}\right) \\ \times [(R_1^{(n)})^{-1}]_{ij}\,[m_1(t_j) - m_0(t_j)]\Bigg],\end{aligned} \tag{14}$$

for almost all sample functions under both hypotheses H_0 and H_1, then (10) is reduced to the following:

$$\begin{aligned}&\text{choose}\quad H_0\quad \text{if}\quad \theta(x) < \log\left[\frac{1}{\beta}\left(\frac{\alpha}{1-\alpha}\right)^2\right],\\ &\text{choose}\quad H_1\quad \text{if}\quad \text{otherwise.}\end{aligned} \tag{15}$$

It can be shown that such β and θ exist if and only if

$$\begin{aligned}&\lim_{n\to\infty} | \operatorname{tr}[R_0^{(n)}(R_1^{(n)})^{-1} - I + (R_1^{(n)})^{-1} M^{(n)}] | < \infty,\\ &\lim_{n\to\infty} | \operatorname{tr}[R_1^{(n)}(R_0^{(n)})^{-1} - I + (R_0^{(n)})^{-1} M^{(n)}] | < \infty.\end{aligned} \tag{16}$$

Note that the above implies the condition (8) as it should. In fact the condition (16) requires not only that the sum of two traces should converge but also that the two traces should converge individually. As we have observed earlier, the condition (8) is equivalent to those of (11) and (12). Hence, the portion of the condition (16) which is additional to (8) is

$$\begin{aligned} &\lim_{n\to\infty} \left| \operatorname{tr}[R_0^{(n)}(R_1^{(n)})^{-1} - I] \right| < \infty, \\ &\lim_{n\to\infty} \left| \operatorname{tr}[R_1^{(n)}(R_0^{(n)})^{-1} - I] \right| < \infty. \end{aligned} \tag{17}$$

But, according to the previous result,[1] (17) is the necessary and sufficient condition for existence of β and θ when $m_0(t) = m_1(t) = 0$. This is no surprise. For, according to (9), $l_n(x)$ converges for almost all sample functions under both hypotheses when the condition (8) is satisfied. Hence, if in addition the factor before the exponential converges, the exponential must also converge (for almost all sample functions). Thus, the additional condition required is the convergence condition of the factor before the exponential alone. But this factor is obviously independent of $m_0(t)$ and $m_1(t)$. In summary therefore, if and only if the conditions (12) and (17) are satisfied, there exist such β and θ as defined by (13) and (14) and the optimum decision scheme can be specified by (15).

Although the decision scheme (15) is certainly simpler than (10), it is still inconvenient for physical application since it requires the limit operation for each received waveform. What is highly desirable is to express θ of (14) not in terms of the infinite sum but in terms of integrals involving $x(t)$ explicitly. It is completely possible to achieve this objective through a straightforward generalization of Solutions — II of the previous article[1] by removing the assumption $m_0(t) = m_1(t) = 0$.* But, as we have remarked in the Introduction, this method cannot avoid the undesirable accompanying conditions analogous to (4). Hence, in the next subsection, we shall obtain the expression of dP_1/dP_0 directly through a particular combination of the results of Grenander and Pitcher.

2.2 *Solutions — II*

Let us introduce a third Gaussian probability measure P_{10} corresponding to $m_1(t)$ and $r_0(s,t)$. Then, just as equivalence of P_0 and P_1 implies existence of a random variable dP_1/dP_0 (the Radon-Nikodym deriva-

* This generalization has been carried out in detail and the result is contained in an unpublished article by this author.

tive of P_1 with respect to P_0) as stated at the beginning of Section II, equivalence of P_0 and P_{10} and equivalence of P_{10} and P_1 imply existence of dP_{10}/dP_0 and dP_1/dP_{10} respectively. Now recall that the key to the solution is to find an expression of dP_1/dP_0 in terms of $x(t)$, in the case where the condition (8) is satisfied. Note that the term, Radon-Nikodym *derivative*, and its symbol immediately suggest the following formalism which is analogous to the chain rule in calculus:

$$\frac{dP_1}{dP_0} = \frac{dP_1}{dP_{10}} \frac{dP_{10}}{dP_0}. \tag{18}$$

According to measure theory, P_0 and P_1 are equivalent and (18) is valid for almost all sample functions under the hypotheses H_0, H_{10} and H_1,* if P_0 and P_{10} as well as P_{10} and P_1 are equivalent. Thus, the task of finding an expression for dP_1/dP_0 in terms of $x(t)$ is equivalent to that of finding such expressions for dP_{10}/dP_0 and dP_1/dP_{10} together with the conditions for equivalence.

Now, through the application of the condition (8) to the case of two Gaussian measures P_0 and P_{10}, it is seen that P_0 and P_{10} are equivalent if and only if (12) with $k = 0$ is satisfied. Note that this is the special case (ii) in the preceding subsection, namely, that perfect reception of the binary sure signals $m_0(t)$ and $m_1(t)$ is not possible in the presence of Gaussian noise with the covariance function $r_0(s,t)$. Then, according to Grenander,[4] if the integral equation

$$\int_0^1 r_0(s,t)g(s)ds = m_1(t) - m_0(t), 0 \leqq t \leqq 1, \tag{19}$$

has a square-integrable solution $g(t)$, then dP_{10}/dP_0 can be expressed as

$$\frac{dP_{10}}{dP_0}(x) = \exp\left\{\int_0^1 \left[x(t) - \frac{m_0(t) + m_1(t)}{2}\right] g(t)\, dt\right\} \tag{20}$$

for almost all sample functions under the hypotheses H_0 and H_{10}. As we may recall, it is through the substitution of (20) into (5) that the well-known optimum receiver (decision scheme) of binary sure signals in noise is obtained; namely,

choose H_0 if

$$\int_0^1 x(t)g(t)\, dt < \frac{1}{2}\int_0^1 [m_0(t) + m_1(t)]g(t)\, dt + \log\frac{\alpha}{1-\alpha}, \tag{21}$$

choose H_{10} if otherwise.

* H_{10} is the hypothesis that $m_1(t)$ and $r_0(s,t)$ are the mean and covariance functions of the Gaussian process $\{x_t, 0 \leqq t \leqq 1\}$.

Similarly, from the condition (8), two Gaussian measures P_{10} and P_1 are equivalent if and only if (11) is satisfied. This is essentially equal to the special case (i), namely, that the perfect reception is not possible between two Gaussian signals with $r_0(s,t)$ and $r_1(s,t)$, where $x(t) - m_1(t)$ instead of $x(t)$ is to be regarded as the sample function in this case. Then, according to the previous result,[1] which is improved by Pitcher,[3] if the integral equation (2) has a solution $h(s,t)$ which is symmetric and satisfies (3), dP_1/dP_{10} can be expressed as

$$\frac{dP_1}{dP_{10}}(x) = \left(\prod_{i=1}^{\infty} \rho_i\right)^{-\frac{1}{2}} \exp\left\{\frac{1}{2}\int_0^1\int_0^1 [x(s) - m_1(s)]h(s,t)[x(t) - m_1(t)]\,ds\,dt\right\} \tag{22}$$

for almost all sample functions under the hypotheses H_{10} and H_1, where $\rho_i > 0$, $i = 1, 2, \cdots$, are the eigenvalues of a certain operator defined in terms of $r_0(s,t)$ and $r_1(s,t)$. As in the preceding case, it is seen that substitution of (22) into (5) yields the optimum decision scheme (1).

In summary, therefore, if the integral equations (19) and (2) have a square-integrable solution $g(t)$ and a symmetric and square-integrable (in the sense of (3)) solution $h(s,t)$ respectively, then the crucial random variable dP_1/dP_0 can be expressed as the product of the right-hand sides of (20) and (22) for almost all sample functions under H_0 and H_1. Thus, the desired optimum decision scheme becomes the following:

choose H_0 if

$$2\int_0^1 x(t)g(t) + \int_0^1\int_0^1 [x(s) - m_1(s)]h(s,t)[x(t) - m_1(t)]\,ds\,dt < \int_0^1 [m_0(t) + m_1(t)]g(t)dt + \log\left[\frac{1}{\hat{\mathfrak{B}}}\left(\frac{\alpha}{1-\alpha}\right)^2\right], \tag{23}$$

choose H_1 if otherwise,

where

$$\hat{\mathfrak{B}}^{-1} = \prod_{i=1}^{\infty} \rho_i .$$

It should be remarked that the indices 0 and 1 can be consistently interchanged throughout. This follows from the symmetry of the problem with respect to the indices. Moreover, by virtue of the symmetry of $h(s,t)$ in s and t, the indices on the left-hand side of (2) can be interchanged while the right-hand side remains unchanged. We also remark that the solutions $g(t)$ and $h(s,t)$ of the integral equations (19) and (2) respectively are unique under the constraints of square-integrability for $g(t)$ and symmetry and square-integrability in the sense of (3) for $h(s,t)$.

Physical interpretation of the optimum decision scheme (23) is obvious, at least, in principle. Given two alternative mean and covariance functions $m_0(t)$ and $r_0(s,t)$, and $m_1(t)$ and $r_1(s,t)$, the optimum receiver consists of a linear and a quadratic filter whose impulse responses are $g(t)$ and $h(s,t)$, respectively, and whose inputs are $2x(t)$ and $x(t) - m_1(t)$ respectively. The outputs of the two filters are sampled at the end of the observation interval, and the decision is made by comparing the sum of the two sampled outputs with the predetermined threshold c, namely, the right-hand side of the inequality in (23).

Finally, although somewhat redundant, it seems instructive to examine the optimum decision scheme in the two special cases which have already been considered.

Case 1:

$$r_0(s,t) = r_1(s,t) = r(s,t),$$

namely, the case of reception of binary sure signals $m_0(t)$ and $m_1(t)$ in the presence of Gaussian noise with the covariance function $r(s,t)$. In this case, the second integral in the inequality of the optimum decision scheme (23) vanishes, since the right-hand side of the integral equation (2) becomes identically zero, thus yielding the identically vanishing function as the only solution satisfying the conditions of symmetry and square-integrability (3), i.e., $h(s,t) = 0$. Moreover, $\hat{\beta}$ becomes unity since all ρ_i, $i = 1,2, \cdots$, are unity. Hence, the optimum decision scheme (23) is reduced to that of (21) where $g(t)$ is the square-integrable solution of (19) with $r_0(s,t)$ replaced by $r(s,t)$.

Case 2:

$$m_0(t) = m_1(t) = 0,$$

namely, the case of reception of binary Gaussian signals with the covariance functions $r_0(s,t)$ and $r_1(s,t)$. In this case, the first and the third integrals in the inequality of (23) vanish, since the right-hand side of the integral equation (19) becomes identically zero, thus admitting the trivial solution as the only square-integrable solution, i.e., $g(t) = 0$. Hence, the optimum decision scheme (23) is reduced essentially to (1).

III. MATHEMATICAL THEORY

3.1 *Statement of Problem*

Definitions

Let Ω be the space of all real-valued functions $\omega(\cdot)$ on [0,1], and let $\tilde{x}_t(\cdot)$ be a real-valued function defined on Ω such that the value of $\tilde{x}_t(\cdot)$ at ω is equal to $\omega(t)$. Let $\tilde{\mathfrak{B}}$ be the σ-field generated by the class of all sets of the form

$$\{\omega: (\tilde{x}_{t_1}(\omega), \cdots, \tilde{x}_{t_n}(\omega)) \in A\}, \tag{24}$$

where n and $t_i \in [0,1]$, $i = 1, \cdots, n$ are arbitrary and A is any n-dimensional Borel set. Finally, let $\tilde{P}_0$, $\tilde{P}_{10}$ and $\tilde{P}_1$ be Gaussian measures induced on $\tilde{\mathfrak{B}}$ respectively by m_0 and r_0, by m_1 and r_0, and by m_1 and r_1, where m_k, $k = 0,1$, are real-valued, continuous functions on [0,1], while r_k, $k = 0,1$, are real-valued, symmetric, positive-definite, continuous functions on $[0,1] \times [0,1]$.* Then, $\tilde{x}_t$ is obviously $\tilde{\mathfrak{B}}$-measurable for every $t \in [0,1]$, thus $\{\tilde{x}_t, 0 \leqq t \leqq 1\}$ is a real Gaussian process whose finite dimensional distributions are given by the values of $\tilde{P}_0$, $\tilde{P}_{10}$ and $\tilde{P}_1$ on the set defined by (24). Since m_k and r_k, $k = 0,1$, are continuous, there always exists a separable (with respect to all $\tilde{P}_0$, $\tilde{P}_{10}$ and $\tilde{P}_1$) and measurable version of $\{\tilde{x}_t, 0 \leqq t \leqq 1\}$, which we denote by $\{x_t, 0 \leqq t \leqq 1\}$.† Let $\mathfrak{B}$ be the minimal σ-field with respect to which x_t is measurable for every $t \in [0,1]$, and let P_0, P_{10} and P_1 be the restrictions of $\tilde{P}_0$, $\tilde{P}_{10}$ and $\tilde{P}_1$ respectively on $\mathfrak{B}$.

Next, define a set function $P_\alpha(\Lambda)$, $0 < \alpha < 1$, $\Lambda \in \mathfrak{B}$, by

$$P_\alpha(\Lambda) = \alpha P_0(\Lambda) + (1 - \alpha) P_1(\Omega - \Lambda).$$

Let Λ_α be such a set that

$$P_\alpha(\Lambda_\alpha) \leqq P_\alpha(\Lambda) \quad \text{for all} \quad \Lambda \in \mathfrak{B}.$$

Problem

Given α, $0 < \alpha < 1$, specify such a set Λ_α in terms of x_t.

---

* See Ref. 5, pp. 609–610 and p. 72.

† Let $\tilde{P}$ be a probability measure on $\tilde{\mathfrak{B}}$ with respect to which all $\tilde{P}_0$, $\tilde{P}_{10}$ and $\tilde{P}_1$ are absolutely continuous, e.g., $\tilde{P} = \frac{1}{3}(\tilde{P}_0 + \tilde{P}_{10} + \tilde{P}_1)$. Now continuity of m_k and r_k, $k = 0,1$, implies continuity in probability of $\tilde{x}_t$ on [0,1] with respect to $\tilde{P}_0$, $\tilde{P}_{10}$ and $\tilde{P}_1$, hence with respect to $\tilde{P}$. Then, there exists a separable (with respect to $\tilde{P}$) and measurable version of $\{\tilde{x}_t, 0 \leqq t \leqq 1\}$, (see Ref. 5, pp. 54–59). But, because $\tilde{P}_0$, $\tilde{P}_{10}$, $\tilde{P}_1 \ll \tilde{P}$, the same version is separable with respect to $\tilde{P}_0$, $\tilde{P}_{10}$ and $\tilde{P}_1$ also.

3.2 *Solution*

Preliminaries

The foundation for solving the above problem consists of the following two measure theoretical facts:

(a.) The Gaussian measures P_0 and P_1 can be either equivalent, $P_0 \equiv P_1$, or singular, $P_0 \perp P_1$.[2,6,7,8,9 *]

$$\text{(b.)} \qquad \text{If } P_0 \equiv P_1, \quad \text{then} \quad \Lambda_\alpha = \left\{\omega \colon \frac{dP_1}{dP_0}(\omega) \geqq \frac{\alpha}{1-\alpha}\right\}, \tag{25}$$

$$\text{if } P_0 \perp P_1, \quad \text{then} \quad \Lambda_\alpha = N,$$

where dP_1/dP_0 is the Radon-Nikodym derivative of P_1 with respect to P_0 and N is a $\mathcal{B}$-measurable set such that $P_0(N) = 0 = P_1(\Omega - N)$.[1]

Thus, the problem stated in the preceding subsection is reduced to that of finding dP_1/dP_0 if $P_0 \equiv P_1$ and N is $P_0 \perp P_1$, which are expressible in terms of x_t.

Solutions — I

Let $\{\tau_k\}$ be a sequence of points in [0,1], which is dense in [0,1]. Let $\mathcal{B}_n$ be the minimal σ-field with respect to which all x_{τ_i}, $i = 1, \cdots, n$, are measurable, and let $\mathcal{B}_\infty$ be the minimal σ-field containing $\bigcup_{n=1}^{\infty} \mathcal{B}_n$. Obviously,

$$\mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots \subset \mathcal{B}_\infty \subset \mathcal{B}. \tag{26}$$

Then, since $\{x_t, 0 \leqq t \leqq 1\}$ is continuous in probability (with respect to P_0), it follows that, for every set $\Lambda \in \mathcal{B}$, there exists a set $\Lambda' \in \mathcal{B}_\infty$ such that

$$P_0(\Lambda \Delta \Lambda') = 0. \tag{27}$$

Now, from the fact that m_k and r_k, $k = 0,1$, are two alternative mean and covariance functions of $\{x_t, 0 \leqq t \leqq 1\}$, the density functions p_0 and p_1 of the random variables $x_{\tau_i}(\omega) - m_0(\tau_i)$, $i = 1, \cdots, n$, corresponding to P_0 and P_1 respectively, are obtained as follows:

---

* Also see Theorem 3 in Appendix D.

$$p_0(\nu_1, \cdots, \nu_n) = (2\pi)^{-n/2} \mid R_0^{(n)} \mid^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\sum_{i,j=1}^{n} \nu_i[(R_0^{(n)})^{-1}]_{ij}\, \nu_j\right],$$

$$p_1(\nu_1, \cdots, \nu_n) = (2\pi)^{-n/2} \mid R_1^{(n)} \mid^{-\frac{1}{2}} \exp \left[-\frac{1}{2}\sum_{i,j=1}^{n} (\nu_i - m_i)\, [(R_1^{(n)})^{-1}]_{ij}(\nu_j - m_j)\right],$$

where $R_k^{(n)}$, $k = 0,1$, are $n \times n$ symmetric, positive-definite matrices defined by

$$(R_k^{(n)})_{ij} = r_k(\tau_i, \tau_j), \qquad k = 0, 1; \qquad i,j = 1, \cdots, n,$$

and

$$m_i = m_1(\tau_i) - m_0(\tau_i), \qquad i = 1, \cdots, n.$$

Then, define a random variable l_n by

$$\begin{aligned} l_n(\omega) &= \frac{p_1[x_{\tau_1}(\omega) - m_0(\tau_1), \cdots, x_{\tau_n}(\omega) - m_0(\tau_n)]}{p_0[x_{\tau_1}(\omega) - m_0(\tau_1), \cdots, x_{\tau_n}(\omega) - m_0(\tau_n)]} \\ &= \mid R_0^{(n)}(R_1^{(n)})^{-1} \mid^{\frac{1}{2}} \exp\Bigg[\frac{1}{2}\sum_{i,j=1}^{n} [x_{\tau_i}(\omega) - m_0(\tau_i)] \\ &\quad \times [(R_0^{(n)})^{-1} - (R_1^{(n)})^{-1}]_{ij}\, [x_{\tau_j}(\omega) - m_0(\tau_j)] \\ &\quad + \sum_{i,j=1}^{n} \left[x_{\tau_i}(\omega) - \frac{m_0(\tau_i) + m_1(\tau_i)}{2}\right] [(R_1^{(n)})^{-1}]_{ij} \\ &\quad \times [m_1(\tau_j) - m_0(\tau_j)]\Bigg]. \end{aligned} \tag{28}$$

Note that $l_n(\omega) \geqq 0$ for all n, and $p_1 = 0$ whenever $p_0 = 0$ and vice versa. Hence, the processes $\{l_n, n \geqq 1\}$ and $\{1/l_n, n \geqq 1\}$ are martingales with respect to P_0 and P_1 respectively.* Then, $\lim_{n\to\infty} l_n$ exists a.e. (P_0) and is denoted by l_∞, and also $\lim_{n\to\infty} 1/l_n$ exists a.e. (P_1).† Furthermore, it can be shown that

(i) if $P_0 \equiv P_1$, then (26) and (27) imply

$$l_\infty = \frac{dP_1}{dP_0}, \quad \text{a.e. } (P_0), \tag{29}$$

(ii) if $P_0 \perp P_1$, then

$$P_0(\{\omega: \lim_{n\to\infty} l_n(\omega) \geqq c\}) = 0 = P_1(\{\omega: \lim l_n(\omega) < c\}) \tag{30}$$

for an arbitrary constant $c > 0$.‡

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* See Ref. 5, pp. 91–93.
† See Ref. 5, p. 319.
‡ See Ref. 1, pp. 2783–2784. Although the definition of l_n is slightly different from the one in Ref. 1 the derivation procedure is identical.

Thus, upon combination of (29) and (30) in conjunction with (25), the desired set Λ_α can be given by

$$\Lambda_\alpha = \left\{\omega: \lim_{n\to\infty} l_n(\omega) \geqq \frac{\alpha}{1-\alpha}\right\},$$

irrespective of whether $P_0 \equiv P_1$ or $P_0 \perp P_1$.

Under certain restrictive conditions, the set Λ_α can be specified in terms of well defined functions of x_t. It is of interest to obtain such specifications as well as the accompanying conditions in terms of the given mean and covariance functions m_k and r_k, $k = 0,1$.

If $P_0 \equiv P_1$, it has already been shown that

$$\Lambda_\alpha = \left\{\omega: l_\infty(\omega) \geqq \frac{\alpha}{1-\alpha}\right\}.$$

Furthermore, it can be shown through the use of martingale theory that $P_0 \equiv P_1$ if and only if (8) is satisfied.*

Next, examination of (28) indicates that, in addition to the condition (8), if there exists a positive constant β such that (13) holds, then there exists a random variable θ such that

$$\begin{aligned}\theta(\omega) = \lim_{n\to\infty}\Bigg[\sum_{i,j=1}^{n} [x_{\tau_i}(\omega) - m_0(\tau_i)]\,[(R_0^{(n)})^{-1} - (R_1^{(n)})^{-1}]_{ij} \\ \times [x_{\tau_j}(\omega) - m_0(\tau_j)] + 2\sum_{i,j=1}^{n}\left(x_{\tau_i}(\omega) - \frac{m_0(\tau_i) + m_1(\tau_i)}{2}\right) \\ \times [(R_1^{(n)})^{-1}]_{ij}\,[m_1(\tau_j) - m_0(\tau_j)]\Bigg], \quad \text{a.e.}(P_0).\end{aligned}$$

Thus, the set Λ_α can be specified as follows:

$$\Lambda_\alpha = \left\{\omega: \theta(\omega) \geqq \log\left[\frac{1}{\beta}\left(\frac{\alpha}{1-\alpha}\right)^2\right]\right\}.$$

It can be shown through the use of martingale theory that the conditions (8) and (13) are equivalent to those of (16).†

Solutions — II

Let R_0 and R_1 be the integral operators whose kernels are r_0 and r_1 respectively, that is, for any real-valued function f,

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* See Ref. 1, pp. 2784–2785, with the definition of l_n replaced by (28) of this article.

† See Ref. 1, pp. 2786–2787, with the definition of l_n replaced by (28) of this article.

$$(R_k f)(t) = \int_0^1 r_k(s,t) f(s)\, ds, \qquad 0 \leqq t \leqq 1, \qquad k = 0,1,$$

whenever the right-hand side is well defined. Then, Grenander shows that*

if there exists $g \in \mathfrak{L}_2(0,1)$† satisfying the integral equation (19), then $P_0 \equiv P_{10}$ and

$$\frac{dP_{10}}{dP_0} = \exp\left\{\int_0^1 \left[x_t - \frac{m_0(t) + m_1(t)}{2}\right] g(t)\, dt\right\}, \qquad \text{a.e.}(P_0).$$

On the other hand, according to the previous result, improved by Pitcher,‡

if there exists a symmetric function h on $[0,1] \times [0,1]$ satisfying (3) and the integral equation (2), then $P_{10} \equiv P_1$ and

$$\frac{dP_1}{dP_{10}} = |R_0^{-\frac{1}{2}} R_1 R_0^{-\frac{1}{2}}|^{-\frac{1}{2}} \exp\left\{\frac{1}{2}\int_0^1 \int_0^1 [x_s - m_1(s)] h(s,t) \times [x_t - m_1(t)]\, ds\, dt\right\}, \qquad \text{a.e.}(P_{10})§.$$

Since $P_0 \equiv P_{10}$ and $P_{10} \equiv P_1$ imply $P_0 \equiv P_1$ and

$$\frac{dP_1}{dP_0} = \frac{dP_1}{dP_{10}} \frac{dP_{10}}{dP_0}, \qquad \text{a.e.}(P_0),$$

we conclude that

if there exist $g \in \mathfrak{L}_2(0,1)$ satisfying (19) and symmetric h satisfying (3) and (2), then $P_0 \equiv P_1$ and

$$\frac{dP_1}{dP_0} = |R_0^{-\frac{1}{2}} R_1 R_0|^{-\frac{1}{2}} \exp\left\{\int_0^1 \left[x_t - \frac{m_0(t) + m_1(t)}{2}\right] g(t)\, dt + \frac{1}{2}\int_0^1 \int_0^1 [x_s - m_1(s)] h(s,t) [x_t - m_1(t)]\, ds\, dt\right\}, \qquad \text{a.e. } (P_0).$$

Therefore, through the substitution of the above into (25), the desired set Λ_α can be specified as follows:

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* See Appendix B.

† $\mathfrak{L}_2(0,1)$ is the space of all square-integrable functions on $[0,1]$.

‡ See Appendix C.

§ $| R_0^{-\frac{1}{2}} R_1 R_0^{-\frac{1}{2}} | = \prod_{i=1}^{\infty} \rho_i$ where ρ_i, $i = 1, 2, \cdots$, are the eigenvalues of $R_0^{-\frac{1}{2}} R_1 R_0^{-\frac{1}{2}}$.

$$\Lambda_\alpha = \Big\{\omega: 2\int_0^1 x_t(\omega)g(t)\,dt + \int_0^1\int_0^1 [x_s(\omega) - m_1(s)]h(s,t)[x_t(\omega) - m_1(t)]\,ds\,dt \geqq \int_0^1 [m_0(t) + m_1(t)]g(t)\,dt + \log\left(\frac{\alpha}{1-\alpha}\right)^2 \mid R_0^{-\frac{1}{2}} R_1 R_0^{-\frac{1}{2}} \mid \Big\},$$

if there exist $g \in \mathfrak{L}_2(0,1)$ satisfying (19) and symmetric h satisfying (3) and (2).

IV. ACKNOWLEDGMENT

The author has greatly benefitted from discussions with T. S. Pitcher (MIT Lincoln Laboratory) and W. L. Root (University of Michigan) as well as with S. P. Lloyd and J. McKenna (Bell Telephone Laboratories, Inc.).

APPENDICES

These appendices are given primarily for a tutorial reason. The majority of the theorems and lemmas here are taken from two articles by Root[9] and Pitcher,[3] in the original, modified or extended forms.* Lemmas 1, 2, and 3 are in the original modified and extended form respectively. Theorem 1 is supplemented by (iii) and a corollary. A more significant supplement, however, is in its proof. While the extended portion of Lemma 4 is a routine matter (hence its proof is omitted), Lemma 5 is significantly extended and strengthened. Lemmas 6 and 7† are added as a supplementary part of the proof of Theorem 2. Although Theorem 2 is stated somewhat differently and in much more detail, its main content remains the same. While the first corollary to Theorem 2 is almost obvious, the proof of the second is considerably involved and is given as "Theorem 3" in Ref. 3. Lemmas 8 and 9 and Theorem 3, which is a generalization of Theorems 1 and 2, are the author's addition. However, their major contents have already been reported elsewhere in different forms, e.g., Ref. 2, including the two corollaries to Theorem 3.

* The term "extended" refers to the extension of the results in Refs. 9 and 3 to the case where the assumption $m_0 = m_1 = 0$ is no longer made.

† The proof of Lemma 7 is supplied by both Root and Pitcher.

APPENDIX A

Preliminaries

Let ρ and μ be probability measures defined on a σ-field $\mathfrak{F}$ of subsets of an infinite set (uncountable in general). Let $\bar{\rho}$ and $\bar{\mu}$ be the completions of ρ and μ on σ-fields $\bar{\mathfrak{F}}_\rho$ and $\bar{\mathfrak{F}}_\mu$ respectively.

Lemma 1: Let $\mathfrak{F}_0$ be a σ-field such that

$$\mathfrak{F}_0 \subset \bar{\mathfrak{F}}_\rho \quad \textit{and} \quad \mathfrak{F}_0 \subset \bar{\mathfrak{F}}_\mu ,$$

and let ρ_0 and μ_0 be the restrictions of $\bar{\rho}$ and $\bar{\mu}$ on $\mathfrak{F}_0$. Then,

$$\rho_0 \perp \mu_0 \Rightarrow \rho \perp \mu.$$

Lemma 2: Assume

$$\rho_0 \equiv \mu_0 .$$

Let $\bar{\mathfrak{F}}_0$ be a σ-field of sets of the form $\Lambda \triangle N$, $\Lambda \in \mathfrak{F}_0$, $\bar{\rho}(N) = 0$. Assume

$$\mathfrak{F} \subset \bar{\mathfrak{F}}_0 .$$

Let ρ'_0 and μ'_0 be the restrictions of $\bar{\rho}$ and $\bar{\mu}$ on $\mathfrak{F}_0$, and let ρ' and μ' be the restrictions of ρ'_0 and μ'_0 on $\mathfrak{F}$. Then,

(*i*) $\rho = \rho'$ *and* $\mu = \mu'$,
(*ii*) $\rho \equiv \mu$,
(*iii*) $\bar{\mathfrak{F}}_0 = \bar{\mathfrak{F}}_\rho = \bar{\mathfrak{F}}_\mu$,
(*iv*) $\dfrac{d\mu}{d\rho} = \dfrac{d\mu_0}{d\rho_0}$, *a.e.* ($\rho$).

Lemma 3: Let $\theta_1, \theta_2, \cdots$, be a sequence of Gaussian variables ($\mathfrak{F}$-measurable) with respect to both ρ and μ such that

$$E_{\hat{\rho}}\{\theta_i\} = 0, \qquad E_{\hat{\mu}}\{\theta_i\} = \nu_i ,$$

$$E_\rho\{\theta_i\theta_j\} = \alpha_i\delta_{ij} , \qquad E_\mu\{(\theta_i - \nu_i)(\theta_j - \nu_j)\} = \beta_i\delta_{ij} ,$$

where E_ρ and E_μ denote the expectations with respect to ρ and μ respectively and α_i, β_i, $i = 1, 2, \cdots$, are arbitrary positive numbers. Let $\hat{\mathfrak{F}}$ be the minimal σ-field with respect to which all θ_i, $i = 1, 2, \cdots$, are measurable, and let $\hat{\rho}$ and $\hat{\mu}$ be the restrictions of ρ and μ on $\hat{\mathfrak{F}}$. Then,

(*i*) *either* $\hat{\rho} \equiv \hat{\mu}$ *or* $\hat{\rho} \perp \hat{\mu}$,
(*ii*) $\hat{\rho} \equiv \hat{\mu}$ *if and only if*

$$\sum_{i=1}^{\infty} \left(1 - \frac{\alpha_i}{\beta_i}\right)^2 < \infty \quad \textit{and} \quad \sum_{i=1}^{\infty} \frac{\nu_i^2}{\alpha_i + \beta_i} < \infty .$$

(iii) if $\hat{\rho} \equiv \hat{\mu}$,

$$\frac{d\hat{\mu}}{d\hat{\rho}} = \exp\left\{\sum_{i=1}^{\infty}\left[\frac{1}{2}\left(\frac{1}{\alpha_i} - \frac{1}{\beta_i}\right)\theta_i^2 + \frac{\nu_i}{\beta_i}\left(\theta_i - \frac{\nu_i}{2}\right) + \tfrac{1}{2}\log\frac{\alpha_i}{\beta_i}\right]\right\}, \quad \text{a.e. } (\hat{\rho}).$$

Proof:

(i) Let $\hat{\mathfrak{F}}_i$, $i = 1, 2, \cdots$, be the minimal σ-field with respect to which θ_i is measurable, and let $\hat{\rho}^{(i)}$ and $\hat{\mu}^{(i)}$ be the restrictions of ρ and μ on $\hat{\mathfrak{F}}_i$. Then, from the hypothesis of the lemma,

$$\hat{\rho}^{(i)} \equiv \hat{\mu}^{(i)}, \qquad i = 1, 2, \cdots,$$

and

$$\hat{\mathfrak{F}} = \prod_{i=1}^{\infty} \hat{\mathfrak{F}}_i, \qquad \hat{\rho} = \prod_{i=1}^{\infty} \hat{\rho}^{(i)}, \qquad \hat{\mu} = \prod_{i=1}^{\infty} \hat{\mu}^{(i)}.$$

Hence, the assertion (i) follows from Kakutani's theorem.*

(ii) From the hypothesis of the lemma,

$$\frac{d\hat{\mu}^{(i)}}{d\hat{\rho}^{(i)}} = \left(\frac{\alpha_i}{\beta_i}\right)^{\frac{1}{2}} \exp\left[\frac{1}{2}\left(\frac{1}{\alpha_i} - \frac{1}{\beta_i}\right)\theta_i^2 + \frac{\nu_i}{\beta_i}\theta_i - \frac{\nu_i^2}{2\beta_i}\right], \qquad \text{a.e. } (\rho). \tag{31}$$

Thus,†

$$E_{\hat{\rho}^{(i)}}\left\{\left(\frac{d\hat{\mu}^{(i)}}{d\hat{\rho}^{(i)}}\right)^{\frac{1}{2}}\right\} = \int_{-\infty}^{\infty}\left(\frac{\alpha_i}{\beta_i}\right)^{\frac{1}{4}} \exp\left[\frac{1}{4}\left(\frac{1}{\alpha_i} - \frac{1}{\beta_i}\right)\zeta^2 + \frac{\nu_i}{2\beta_i}\zeta - \frac{\nu_i^2}{4\beta_i}\right] \cdot (2\pi\alpha_i)^{-\frac{1}{2}} \exp\left(-\frac{\zeta^2}{2\alpha_i}\right) d\zeta$$

$$= \frac{(4\alpha_i\beta_i)^{\frac{1}{4}}}{(\alpha_i + \beta_i)^{\frac{1}{2}}} \exp\left(-\frac{1}{4}\frac{\nu_i^2}{\alpha_i + \beta_i}\right).$$

Note, for all $i = 1, 2, \cdots$,

$$\frac{4\alpha_i\beta_i}{(\alpha_i + \beta_i)^2} \leqq 1 \quad \text{and} \quad 0 < \exp\left(-\frac{1}{4}\frac{\nu_i^2}{\alpha_i + \beta_i}\right) < 1.$$

Hence

$$\prod_{i=1}^{\infty} E_{\hat{\rho}^{(i)}}\left\{\left(\frac{d\hat{\mu}^{(i)}}{d\hat{\rho}^{(i)}}\right)^{\frac{1}{2}}\right\}$$

converges to a positive number if both

* See Ref. 9, pp. 295–296.

† $E_{\hat{\rho}^{(i)}}$ denotes expectation with respect to $\hat{\rho}^{(i)}$.

$$\prod_{i=1}^{\infty} \frac{4\alpha_i\beta_i}{(\alpha_i + \beta_i)^2} \quad \text{and} \quad \sum_{i=1}^{\infty} \frac{\nu_i^2}{\alpha_i + \beta_i}$$

converge. Therefore, according to Kakutani's theorem, $\hat{\rho} \equiv \hat{\mu}$ if and only if these infinite product and sum converge.
Now,

$$\prod_{i=1}^{\infty} \frac{4\alpha_i\beta_i}{(\alpha_i + \beta_i)^2}$$

converges if and only if*

$$\sum_{i=1}^{\infty} \left[1 - \frac{4\alpha_i\beta_i}{(\alpha_i + \beta_i)^2}\right] < \infty.$$

But

$$1 - \frac{4\alpha_i\beta_i}{(\alpha_i + \beta_i)^2} = \left(1 - \frac{\alpha_i}{\beta_i}\right)^2 \Big/ \left(1 + \frac{\alpha_i}{\beta_i}\right)^2$$

and the infinite sum of this converges if and only if

$$\sum_{i=1}^{\infty} \left(1 - \frac{\alpha_i}{\beta_i}\right)^2 < \infty,$$

hence, the assertion (ii) follows.

(iii) Note

$$\prod_{i=1}^{n} \frac{d\hat{\mu}^{(i)}}{d\hat{\rho}^{(i)}} = E_{\hat{\rho}} \left\{ \frac{d\hat{\mu}}{d\hat{\rho}} \,\middle|\, \prod_{i=1}^{n} \mathfrak{F}_i \right\}, \qquad \text{a.e. } (\hat{\rho}).$$

Hence,*

$$\prod_{i=1}^{\infty} \frac{d\hat{\mu}^{(i)}}{d\hat{\rho}^{(i)}} = \frac{d\hat{\mu}}{d\hat{\rho}}, \qquad \text{a.e. } (\hat{\rho}).$$

Then, the assertion (iii) is obtained through substitution of (31) into the above.

APPENDIX B

First Theorem on Equivalence and Singularity

Theorem 1: (*Grenander*)

(*i*) *Either* $P_0 \equiv P_{10}$ *or* $P_0 \perp P_{10}$,
(*ii*) $P_0 \equiv P_{10}$ *if and only if* $R_0^{-\frac{1}{2}}m \in \mathfrak{L}_2(0,1)$,
(*iii*) *if* $P_0 \equiv P_{10}$,

* See Ref. 11, p. 381.
† See Ref. 5, p. 331.

$$\frac{dP_{10}}{dP_0} = \exp\left\{\sum_{i=1}^{\infty}\left[\frac{\nu_i}{\lambda_i}\,\xi_i - \frac{\nu_i^2}{2\lambda_i}\right]\right\}, \qquad a.e.\ (\bar{P}_0),$$

where $m(t) = m_1(t) - m_0(t)$, $0 \leqq t \leqq 1$, *and* ξ_i *and* ν_i, $i = 1, 2, \cdots$ *are defined by*

$$\xi_i(\omega) = (x(\omega) - m_0, \psi_1) \equiv \int_0^1 [x_t(\omega) - m_0(t)]\psi_i(t)\, dt,$$

$$a.e.\ (\bar{P}_0, \bar{P}_{10})$$

$$\nu_i = (m,\psi_i) \equiv \int_0^1 m(t)\psi_i(t)\, dt.$$

Proof: Let $\bar{P}_0$ and $\bar{P}_{10}$ be the completions of P_0 and P_{10} on $\bar{\mathcal{B}}_{P_0}$ and $\bar{\mathcal{B}}_{P_{10}}$ respectively. Then, from the definition, ξ_i, $i = 1, 2, \cdots$, are measurable with respect to both $\bar{\mathcal{B}}_{P_0}$ and $\bar{\mathcal{B}}_{P_{10}}$, and Gaussian distributed with respect to both $\bar{P}_0$ and $\bar{P}_{10}$ such that*

$$E_0\{\xi_i\} = E_{10}\{\xi_i - \nu_i\} = 0,$$

$$E_0\{\xi_i\xi_j\} = E_{10}\{(\xi_i - \nu_i)(\xi_j - \nu_j)\} = \lambda_i\delta_{ij}.$$

Furthermore, a modified version of Kauhunen-Loéve theorem† holds; namely, for every $t \in [0,1]$,

$$x_t - m_0(t) = \lim_{n\to\infty}\sum_{i=1}^{n}\xi_i\psi_i(t), \qquad \text{a.e. } (P_0). \tag{32}$$

Now, let $\mathcal{B}_i'$, $i = 1, 2, \cdots$, be the minimal σ-field with respect to which ξ_i is measurable, and let

$$\mathcal{B}' = \prod_{i=1}^{\infty}\mathcal{B}_i'.$$

Then,

$$\mathcal{B}_i' \subset \bar{\mathcal{B}}_{P_0}, \mathcal{B}_i' \subset \bar{\mathcal{B}}_{P_{10}}, \mathcal{B}' \subset \bar{\mathcal{B}}_{P_0}, \mathcal{B}' \subset \bar{\mathcal{B}}_{P_{10}}. \tag{33}$$

Let $P_0'^{(i)}$ and $P_{10}'^{(i)}$ be the restrictions of $\bar{P}_0$ and $\bar{P}_{10}$ on $\mathcal{B}_i'$, and P_0' and P_{10}' on $\mathcal{B}'$. Then, it is readily seen that $P_0'^{(i)} \equiv P_{10}'^{(i)}$, $i = 1, 2, \cdots$, and

$$\frac{dP_{10}'^{(i)}}{dP_0'^{(i)}} = \exp\left[\frac{\xi_i^2}{2\lambda_i} - \frac{(\xi_i - \nu_i)^2}{2\lambda_i}\right] = \exp\left[\frac{\nu_i}{\lambda_i}\,\xi_i - \frac{\nu_i^2}{2\lambda_i}\right], \qquad \text{a.e. } (\bar{P}_0).$$

* E_0, E_{10} and E_1 denote expectations with respect to $\bar{P}_0$, $\bar{P}_{10}$ and $\bar{P}_1$ in general. However, if the function whose expectation is in question is $\mathcal{B}$-measurable, the same symbols are used for expectations with respect to P_0, P_{10} and P_1 also.

† See Ref. 1, pp. 2801–2802.

Hence,

$$\int_\Omega \left(\frac{dP_{10}'^{(i)}}{dP_0'^{(i)}}\right)^{\frac{1}{2}} dP_0'^{(i)} = (2\pi\lambda_i)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp\left(\frac{\nu_i}{2\lambda_i}\zeta - \frac{\nu_i^2}{4\lambda_i}\right) \exp\left(-\frac{\zeta^2}{2\lambda_i}\right) d\zeta$$

$$= \exp\left(-\frac{\nu_i^2}{8\lambda_i}\right).$$

Thus,

$$\prod_{i=1}^{\infty} \int_\Omega \left(\frac{dP_{10}'^{(i)}}{dP_0'^{(i)}}\right)^{\frac{1}{2}} dP_0'^{(i)} = \exp\left(-\frac{1}{8}\sum_{i=1}^{\infty}\frac{\nu_i^2}{\lambda_i}\right).$$

Hence, from Kakutani's theorem, either

$$P_0' \equiv P_{10}' \quad \text{or} \quad P_0' \perp P_{10}', \tag{34}$$

and $P_0' \equiv P_{10}'$ if and only if

$$\sum_{i=1}^{\infty} \frac{\nu_i^2}{\lambda_i} < \infty, \quad \text{i.e.,} \quad R_0^{-\frac{1}{2}}m \in \mathfrak{L}_2(0,1). \tag{35}$$

Next, for an arbitrary $t \in [0,1]$, define

$$\Gamma_t = \left\{\omega: x_t(\omega) - m_0(t) = \sum_{i=1}^{\infty} \xi_i(\omega)\psi_i(t)\right\},$$

$$\Lambda_t = \{\omega: x_t(\omega) - m_0(t) \in A\}, \qquad \Lambda_t' = \left\{\omega: \sum_{i=1}^{\infty} \xi_i(\omega)\psi_i(t) \in A\right\},$$

where A is an arbitrary Borel set. Put

$$\Lambda_t = (\Lambda_t \cap \Gamma_t) \cup (\Lambda_t \cap \Gamma_t^c), \qquad \Lambda_t' = (\Lambda_t' \cap \Gamma_t) \cup (\Lambda_t' \cap \Gamma_t^c).$$

Then, from (32)

$$\Lambda_t \cap \Gamma_t = \Lambda_t' \cap \Gamma_t, \qquad \bar{P}_0(\Lambda_t \cap \Gamma_t^c) = 0 = \bar{P}_0(\Lambda_t' \cap \Gamma_t^c).$$

Hence,

$$\bar{P}_0(\Lambda_t \triangle \Lambda_t') = 0.$$

Let $\mathfrak{B}'$ be a σ-field of sets of the form $\Lambda' \triangle N, K \in \mathfrak{B}', \bar{P}_0(N) = 0$. Then

$$\Lambda_t \in \overline{\mathfrak{B}}', 0 \leqq t \leqq 1.$$

That is, $x_t - m_0(t)$ is $\overline{\mathfrak{B}}'$-measurable for every $t \in [0,1]$. Hence, x_t is $\overline{\mathfrak{B}}'$-measurable for every t. But, since $\mathfrak{B}$ is the minimal σ-field with respect to which x_t is measurable for every t, we have

$$\mathfrak{B} \subset \overline{\mathfrak{B}}'. \tag{36}^*$$

(ii) *Necessity:* Assume $P_0 \equiv P_{10}$. Then, $\bar{P}_0 \equiv \bar{P}_{10}$, thus $P_0' \equiv P_{10}'$.

* This part of the proof, i.e., establishment of (36), is not given in Ref. 4. In fact, Grenander's assertion is only on the primed measures P_0' and P_{10}'.

Hence, from (35),

$$R_0^{-\frac{1}{2}}m \in \mathfrak{L}_2(0,1).$$

Sufficiency: Assume $R_0^{-\frac{1}{2}}m \in \mathfrak{L}_2(0,1)$.

Then, from (35), $P_0' \equiv P_{10}'$. Then, from Lemma 2 (ii) together with (36),

$$P_0 \equiv P_{10}\,.$$

(i) *Dichotomy:* Assume P_0 and P_{10} are not equivalent. Then, from (ii), $R_0^{-\frac{1}{2}}m \notin \mathfrak{L}_2(0,1)$. Hence, from (35) and (34), $P_0' \perp P_{10}'$. Then, from Lemma 1 together with (33),

$$P_0 \perp P_{10}\,.$$

(iii) *Radon-Nikodym Derivative:* From (ii) and (35), $P_0 \equiv P_{10} \Rightarrow P_0' \equiv P_{10}'$. Then, from Lemma 2 (iv) together with (36), $dP_{10}/dP_0 = dP_{10}'/dP_0'$, a.e. $(\bar{P}_0)$. Then, the assertion follows from Lemma 3 (iii) with $\alpha_i = \beta_i = \lambda_i$, $i = 1, 2, \cdots$.

Corollary (Grenander):

If $R_0^{-1}m \in \mathfrak{L}_2(0,1)$, then $P_0 \equiv P_{10}$ and

$$\frac{dP_{10}}{dP_0} = \exp\left(x - \frac{m_0 + m_1}{2}, R_0^{-1}m\right), \qquad a.e.\ (P_0).$$

Proof: The first assertion is obvious from Theorm 1 (ii) since

$$R_0^{-1}m \in \mathfrak{L}_2(0,1) \Rightarrow R_0^{-\frac{1}{2}}m \in \mathfrak{L}_2(0,1).$$

To prove the second assertion, note that

$$\sum_{i=1}^{n}\left(\xi_i - \frac{\nu_i}{2}\right)\frac{\nu_i}{\lambda_i} = \left(\sum_{i=1}^{n}\left(\xi_i - \frac{\nu_i}{2}\right)\psi_i\,, R_0^{-1}m\right).$$

Then, from (32) and the definition of ν_i, $i = 1, 2, \cdots$,

$$\left(x - \frac{m_0 + m_1}{2}, R_0^{-1}m\right) = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\left(\xi_i - \frac{\nu_i}{2}\right)\psi_i\,, R_0^{-1}m\right),$$

a.e. $(\bar{P}_0)$.

APPENDIX C

Second Theorem on Equivalence and Singularity

Lemma 4: If either $R_1^{\frac{1}{2}}R_0^{-\frac{1}{2}}$ or $R_0^{\frac{1}{2}}R_1^{-\frac{1}{2}}$ is unbounded, then $P_0 \perp P_1$ and $P_{10} \perp P_1$.

Lemma 5: If $R_1^{\frac{1}{2}}R_0^{-\frac{1}{2}}$ is bounded and $R_0^{-\frac{1}{2}}m \in \mathfrak{L}_2(0,1)$, then, for any sequence of functions $f_i \in \mathfrak{L}_2(0,1)$, $i = 1, 2, \cdots$, there exists a corresponding sequence of Gaussian variables θ_i, $i = 1, 2, \cdots$, (measurable with respect to $\bar{\mathfrak{B}}_{P_0}$, $\bar{\mathfrak{B}}_{P_{10}}$ and $\bar{\mathfrak{B}}_{P_1}$) such that for $i, j = 1, 2, \cdots$,

$$E_0\{\theta_i + \nu_i\} = E_{10}\{\theta_i\} = E_1\{\theta_i\} = 0,$$

$$E_0\{(\theta_i + \nu_i)(\theta_j + \nu_j)\} = E_{10}\{\theta_i\theta_j\} = (f_i, f_j), \tag{37}$$

$$E_1\{\theta_i\theta_j\} = (f_i, X^*Xf_j),$$

where X is the bounded extension of $R_1^{\frac{1}{2}}R_0^{-\frac{1}{2}}$ to the whole of $\mathfrak{L}_2(0,1)$ and ν_i, $i = 1, 2, \cdots$, are defined as

$$\nu_i = (f_i, R_0^{-\frac{1}{2}}m).$$

Proof: Since $R_0^{\frac{1}{2}}(\mathfrak{L}_2(0,1))$ is dense in $\mathfrak{L}_2(0,1)$, there exists a sequence $\{f_{ij}\}_j$ for each f_i, $i = 1, 2, \cdots$, such that

$$R_0^{-\frac{1}{2}}f_{ij} \in \mathfrak{L}_2(0,1),\ j = 1, 2, \cdots, \text{ and } \lim_{j\to\infty} \| f_i - f_{ij} \| = 0,$$

where $\| f \|$ is the norm of f in the space $\mathfrak{L}_2(0,1)$. Then, through elementary steps, it can be shown that

$$\lim_{m,n\to\infty} (f_{im}, f_{jn}) = \lim_{n\to\infty}\lim_{m\to\infty} (f_{im}, f_{jn}) = (f_i, f_j). \tag{38}$$

1° Let θ_{ij} ; $i,j = 1,2, \cdots$, be $\mathfrak{B}$-measurable functions such that

$$\theta_{ij} = (x - m_1, R_0^{-\frac{1}{2}}f_{ij}), \qquad \text{a.e. } (P_0, P_{10}, P_1).$$

Then, there exist random variables θ_i, $i = 1,2, \cdots$, which are measurable with respect to $\bar{\mathfrak{B}}_{P_0}$, $\bar{\mathfrak{B}}_{P_{10}}$ and $\bar{\mathfrak{B}}_{P_1}$, and Gaussian distributed with respect to $\bar{P}_0$, $\bar{P}_{10}$ and $\bar{P}_1$ such that

$$\theta_i = \operatorname*{l.i.m.}_{j\to\infty} \theta_{ij}, \qquad (\bar{P}_0, \bar{P}_{10}, \bar{P}_1).$$

To prove 1°, consider expectation with respect to P_0, P_{10} and P_1 of $| \theta_{ij} - \theta_{ik} |^2 = \theta_{ij}^2 - 2\theta_{ij}\theta_{ik} + \theta_{ik}^2$, $i = 1,2, \cdots$. First, note

$$E_0\{\theta_{ij}\theta_{ik}\} = E_0\{(R_0^{-\frac{1}{2}}f_{ij}, x - m_0 - m)(x - m_0 - m, R_0^{-\frac{1}{2}}f_{ik})\}$$
$$= (R_0^{-\frac{1}{2}}f_{ij}, R_0R_0^{-\frac{1}{2}}f_{ik}) - (R_0^{-\frac{1}{2}}f_{ij}, m)(R_0^{-\frac{1}{2}}f_{ik}, m).$$

Thus, from (38),

$$\lim_{j,k\to\infty} E_0\{\theta_{ij}\theta_{ik}\} = \lim_{j,k\to\infty} [(f_{ij}, f_{ik}) - (f_{ij}, R_0^{-\frac{1}{2}}m)(f_{ik}, R_0^{-\frac{1}{2}}m)]$$
$$= \| f_i \|^2 - (f_i, R_0^{-\frac{1}{2}}m)^2.$$

Hence,

$$\lim_{j,k\to\infty} E_0\{ | \theta_{ij} - \theta_{ik} |^2 \} = 0. \tag{39}$$

Secondly, note

$$\lim_{j,k\to\infty} E_{10}\{\theta_{ij}\theta_{ik}\} = \lim_{j,k\to\infty} (R_0^{-\frac{1}{2}} f_{ij}, R_0 R_0^{-\frac{1}{2}} f_{ik}) = \lim_{j,k\to\infty} (f_{ij}, f_{ik}) = \| f_i \|^2.$$

Hence,

$$\lim_{j,k\to\infty} E_{10}\{ | \theta_{ij} - \theta_{ik} |^2 \} = 0. \tag{40}$$

Thirdly, note

$$E_1\{\theta_{ij}\theta_{ik}\} = E_1\{ (R_0^{-\frac{1}{2}} f_{ij}, R_1 R_0^{-\frac{1}{2}} f_{ik})\} = (X f_{ij}, X f_{ik}).$$

Since X is bounded, it is continuous. Thus, from (38),

$$\lim_{j,k\to\infty} (X f_{ij}, X f_{ik}) = \| X f_i \|^2.$$

Hence,

$$\lim_{j,k\to\infty} E_1\{ | \theta_{ij} - \theta_{ik} |^2 \} = 0. \tag{41}$$

Next, upon combination of (39), (40) and (41), $\{\theta_{ij}\}_j$, $i = 1,2, \cdots$, are seen to be mean fundamental sequences with respect to $P_0 + P_{10} + P_1$. Hence, there exist θ_i, $i = 1,2, \cdots$, measurable with respect to $\bar{\mathfrak{B}}_{P_0+P_{10}+P_1}$ such that

$$\theta_i = \operatorname*{l.i.m.}_{j\to\infty} \theta_{ij}, \qquad (\bar{P}_0 + \bar{P}_{10} + \bar{P}_1).$$

But, since this implies

$$\theta_i = \operatorname*{l.i.m.}_{j\to\infty} \theta_{ij}, \qquad (\bar{P}_0, \bar{P}_{10}, \bar{P}_1), \qquad i = 1,2, \cdots,$$

θ_i, $i = 1,2, \cdots$, are measurable with respect to $\bar{\mathfrak{B}}_{P_0}$, $\bar{\mathfrak{B}}_{P_{10}}$ and $\bar{\mathfrak{B}}_{P_1}$, and are Gaussian distributed with respect to $\bar{P}_0$, $\bar{P}_{10}$ and $\bar{P}_1$.

2° To prove (37), simply note

$$\begin{aligned} E_0 \{\theta_i + \nu_i\} &= \lim_{j\to\infty} E_0\{\theta_{ij}\} + \nu_i \\ &= \lim_{j\to\infty} E_0\{(x - m_0 - m, R_0^{-\frac{1}{2}} f_{ij})\} + \nu_i = 0, \end{aligned}$$

$$E_{10}\{\theta_i\} = \lim_{j\to\infty} E_{10}\{\theta_{ij}\} = 0,$$

$$\begin{aligned}
E_1\{\theta_i\} &= \lim_{j\to\infty} E_1\{\theta_{ij}\} = 0, \\
E_0\{(\theta_i + \nu_i)(\theta_j + \nu_j)\} &= \lim_{n\to\infty}\lim_{m\to\infty} E_0\{(\theta_{im} + \nu_i)(\theta_{jn} + \nu_j)\} \\
&= \lim_{n\to\infty}\lim_{m\to\infty} (R_0^{-\frac{1}{2}} f_{im}, R_0 R_0^{-\frac{1}{2}} f_{jn}) \\
&= (f_i, f_j), \\
E_{10}\{\theta_i \theta_j\} &= \lim_{n\to\infty}\lim_{m\to\infty} E_{10}\{\theta_{im}\theta_{jn}\} \\
&= \lim_{n\to\infty}\lim_{m\to\infty} (R_0^{-\frac{1}{2}} f_{im}, R_0 R_0^{-\frac{1}{2}} f_{jn}) \\
&= (f_i, f_j) \\
E_1\{\theta_i \theta_j\} &= \lim_{n\to\infty}\lim_{m\to\infty} E_1\{\theta_{im}\theta_{jn}\} \\
&= \lim_{n\to\infty}\lim_{m\to\infty} (R_0^{-\frac{1}{2}} f_{im}, R_1 R_0^{-\frac{1}{2}} f_{jn}) \\
&= (Xf_i, Xf_j),
\end{aligned}$$

where (38) is used for the last three calculations.

Remark 1: The assertion of Lemma 5 with respect to P_{10} and P_1 only, is valid without the condition $R_0^{-\frac{1}{2}} m \in \mathfrak{L}_2(0,1)$.

Remark 2: Suppose $R_0^{-\frac{1}{2}} m \notin \mathfrak{L}_2(0,1)$ but there exist a sequence $\{f_{ij}\}_j$ for each f_i, $i = 1,2, \cdots$, such that $R_0^{-\frac{1}{2}} f_{ij} \in \mathfrak{L}_2(0,1)$, $j = 1,2, \cdots$,

$$\lim_{j\to\infty} \|f_i - f_{ij}\| = 0, \quad \lim_{j\to\infty} (m, R_0^{-\frac{1}{2}} f_{ij}) = \nu_i'$$

for some real number ν_i'. Then, Lemma 5 is still valid if ν_i is replaced by ν_i', $i = 1, 2, \cdots$.

Remark 3: Suppose $R_0^{-\frac{1}{2}} m \notin \mathfrak{L}_2(0,1)$ and there is no such sequence. Then,

$$P_0 \perp P_1.$$

Proof: Let

$$\nu_{ij} = (m, R_0^{-\frac{1}{2}} f_{ij}); \qquad i,j = 1,2, \cdots,$$

where

$$\lim_{j\to\infty} \|f_i - f_{ij}\| = 0$$

for each $i = 1,2, \cdots$. Without loss of generality, we assume that

$$\lim_{j\to\infty} |\nu_{ij}| = \infty$$

for some i. Define for such i,

$$\tilde{\theta}_{ij} = \theta_{ij}/\nu_{ij}, \qquad j = 1,2, \cdots .$$

Then

$$E_0\{\tilde{\theta}_{ij} + 1\} = E_1\{\tilde{\theta}_{ij}\} = 0, \qquad j = 1,2, \cdots .$$

Put

$$\sigma_{ij}^{\ 0} = E_0\{(\tilde{\theta}_{ij} + 1)^2\}, \qquad \sigma_{ij}^{\ 1} = E_1\{(\tilde{\theta}_{ij})^2\}.$$

Then,

$$\lim_{j\to\infty} \sigma_{ij}^{\ k} = 0, \qquad k = 0,1.$$

Thus, there exists a subsequence $\{\sigma_{ij_n}^{\ k}\}$ such that

$$\sum_{n=1}^{\infty} \sigma_{ij_n}^{\ k} < \infty, \qquad k = 0,1.$$

But, from Tchebycheff inequality, we have for some ε, $0 < \varepsilon < \frac{1}{2}$,

$$P_0(\{\omega: |\tilde{\theta}_{ij_n}(\omega)| < \varepsilon\}) < P_0(\{\omega: |\tilde{\theta}_{ij_n}(\omega) + 1| \geqq \varepsilon\}) \leqq \frac{\sigma_{ij_n}^{\ 0}}{\varepsilon^2},$$

$$P_1(\{\omega: |\tilde{\theta}_{ij_n}(\omega)| \geqq \varepsilon\}) \leqq \frac{\sigma_{ij_n}^{\ 1}}{\varepsilon^2}.$$

Hence, by Borel-Cantalli lemma,

$$P_0(\liminf_n\{\omega: |\tilde{\theta}_{ij_n}(\omega)| < \varepsilon\}) \leqq P_0(\limsup_n\{\omega: |\tilde{\theta}_{ij_n}(\omega)| < \varepsilon\}) = 0,$$

$$P_1(\limsup_n\{\omega: |\tilde{\theta}_{ij_n}(\omega)| \geqq \varepsilon\}) = 0.$$

Hence, by noting that

$$\limsup_n \{\omega: |\tilde{\theta}_{ij_n}(\omega)| \geqq \varepsilon\} = \Omega - \liminf_n\{\omega: |\tilde{\theta}_{ij_n}(\omega)| < \varepsilon\},$$

we have

$$P_0 \perp P_1 .$$

Lemma 6: If $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ *is a densely defined, bounded, completely continuous operator on* $\mathfrak{L}_2(0,1)$, *then*

$$x_t - m_1(t) = \lim_{m,n\to\infty} \left[\sum_{i=1}^{m} (R_0^{\frac{1}{2}}\varphi_i)(t)\eta_i + \sum_{i=1}^{n} (R_0^{\frac{1}{2}}\tilde{\varphi}_i)(t)\tilde{\eta}_i\right], \qquad (\mu \times \bar{P}_{10})$$

where φ_i, $i = 1,2, \cdots$, *are the orthonormal eigenfunctions corresponding to nonzero eigenvalues of* $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ *and* $\tilde{\varphi}_i$, $i = 1,2, \cdots$, *are an orthonormal basis of the null space of* $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$,* *and* η_i *and* $\tilde{\eta}_i$ *are defined by*

$$\begin{aligned} \eta_i &= \underset{j\to\infty}{\text{l.i.m.}}\ (x - m_1, R_0^{-\frac{1}{2}}\varphi_{ij}), \\ \tilde{\eta}_i &= \underset{j\to\infty}{\text{l.i.m.}}\ (x - m_1, R_0^{-\frac{1}{2}}\tilde{\varphi}_i), \end{aligned} \qquad (P_0, P_{10}, P_1),$$

where φ_{ij}, $\tilde{\varphi}_{ij} \in \mathfrak{L}_2(0,1)$; $i,j = 1,2, \cdots$, *are chosen in such a way that* $R_0^{-\frac{1}{2}}\varphi_{ij}$, $R_0^{-\frac{1}{2}}\tilde{\varphi}_{ij} \in \mathfrak{L}_2(0,1)$ *and, for each* i,

$$\lim_{j\to\infty} \| \varphi_i - \varphi_{ij} \| = 0, \qquad \lim_{j\to\infty} \| \tilde{\varphi}_i - \tilde{\varphi}_{ij} \| = 0,$$

and finally μ *is Lebesgue measure on Borel field* $\mathcal{A}$ *of the subsets of* [0,1].

Proof: Note that φ_i and $\tilde{\varphi}_i$, $i = 1,2, \cdots$, exist since $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ is densely defined, bounded, self-adjoint and completely continuous.

Consider

$$I_{m,n} = E_{10}\left\{\int_0^1 \left| x_t - m_1(t) - \sum_{i=1}^m (R_0^{\frac{1}{2}}\varphi_i)(t)\eta_i - \sum_{i=1}^n (R_0^{\frac{1}{2}}\tilde{\varphi}_i)(t)\tilde{\eta}_i \right|^2 dt\right\}.$$

By expanding the bracket,

$$\begin{aligned} I_{m,n} = &\int_0^1 r_0(t,t)\, dt - 2\sum_{i=1}^m \int_0^1 (R_0^{\frac{1}{2}}\varphi_i)(t)E_{10}\{[x_t - m_1(t)]\eta_i\}\, dt \\ &- 2\sum_{i=1}^n \int_0^1 (R_0^{\frac{1}{2}}\tilde{\varphi}_i)(t)E_{10}\{[x_t - m_1(t)]\tilde{\eta}_i\}\, dt \\ &+ 2\sum_{i=1}^m \sum_{j=1}^n E_{10}\{\eta_i\tilde{\eta}_j\}(R_0^{\frac{1}{2}}\varphi_i, R_0^{\frac{1}{2}}\tilde{\varphi}_j) \\ &+ \sum_{i,j=1}^m E_{10}\{\eta_i\eta_j\}(R_0^{\frac{1}{2}}\varphi_i, R_0^{\frac{1}{2}}\varphi_j) + \sum_{i,j=1}^n E_{10}\{\tilde{\eta}_i\tilde{\eta}_j\}(R_0^{\frac{1}{2}}\tilde{\varphi}_i, R_0^{\frac{1}{2}}\tilde{\varphi}_j). \end{aligned}$$

Note

$$\begin{aligned} E_{10}\{[x_t - m_1(t)]\eta_i\} = E_{10}\Big\{&\underset{j\to\infty}{\text{l.i.m.}} \int_0^1 (R_0^{-\frac{1}{2}}\varphi_{ij})(s)[x_s - m_1(s)] \\ &\cdot [x_t - m_1(t)]\, ds\Big\} \end{aligned}$$

* If the null space is finite dimensional, then $\{\tilde{\varphi}_i\}$ can be incorporated into $\{\varphi_i\}$ and there is no need to treat $\{\tilde{\varphi}_i\}$ separately.

$$= \lim_{j\to\infty} (R_0 R_0^{-\frac{1}{2}} \varphi_{ij})(t)$$

$$= (R_0^{\frac{1}{2}} \varphi_i)(t);$$

similarly,

$$E_{10}\{[x_t - m_1(t)]\tilde{\eta}_i\} = (R_0^{\frac{1}{2}} \tilde{\varphi}_i)(t).$$

Also, from Remark 1 of Lemma 5,*

$$E_{10}\{\eta_i \tilde{\eta}_j\} = (\varphi_i, \tilde{\varphi}_j) = 0,$$

$$E_{10}\{\eta_i \eta_j\} = (\varphi_i, \varphi_j) = \delta_{ij},$$

$$E_{10}\{\tilde{\eta}_i \tilde{\eta}_j\} = (\tilde{\varphi}_i, \tilde{\varphi}_j) = \delta_{ij}.$$

Therefore,

$$I_{m,n} = \int_0^1 r_0(t,t)\, dt - \sum_{i=1}^{m} (\varphi_i, R_0 \varphi_i) - \sum_{i=1}^{n} (\tilde{\varphi}_i, R_0 \tilde{\varphi}_i).$$

Now,

$$\int_0^1 r_0(t,t)\, dt = \sum_{k=1}^{\infty} \lambda_k.$$

On the other hand,

$$\begin{aligned}
\sum_{i=1}^{\infty} (\varphi_i, R_0\varphi_i) + \sum_{i=1}^{\infty} (\tilde{\varphi}_i, R_0\tilde{\varphi}_i) &= \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} (\varphi_i, \psi_k)(R_0\varphi_i, \psi_k) \\
&\quad + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} (\tilde{\varphi}_i, \psi_k)(R_0\tilde{\varphi}_i, \psi_k) \\
&= \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \lambda_k (\varphi_i, \psi_k)^2 + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} \lambda_k (\tilde{\varphi}_i, \psi_k)^2 \\
&= \sum_{k=1}^{\infty} \lambda_k \left[\sum_{i=1}^{\infty} (\varphi_i, \psi_k)^2 + \sum_{i=1}^{\infty} (\tilde{\varphi}_i, \psi_k)^2 \right] \\
&= \sum_{k=1}^{\infty} \lambda_k.
\end{aligned}$$

Hence,

$$\lim_{m,n\to\infty} I_{m,n} = 0.$$

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* Note that, it $R_0^{-\frac{1}{2}} R_1 R_0^{-\frac{1}{2}}$ is densely defined and bounded, then $R_1^{\frac{1}{2}} R_0^{-\frac{1}{2}}$ is bounded.

Lemma 7: Under the hypothesis of Lemma 6,

$$\mathcal{B} \subset \bar{\hat{\mathcal{B}}}$$

where $\bar{\hat{\mathcal{B}}}$ *is a* σ*-field of sets of the form* $\hat{\Lambda} \triangle N$, $\hat{\Lambda} \in \hat{\mathcal{B}}$, $\bar{P}_{10}(N) = 0$, *and* $\hat{\mathcal{B}}$ *is the minimal* σ*-field with respect which all* η_i *and* $\tilde{\eta}_i$, $i = 1,2, \cdots$, *are measurable.*

Proof: It suffices to prove that x_t is $\bar{\hat{\mathcal{B}}}$-measurable for every $t \in [0,1]$, since $\mathcal{B}$ is the minimal σ-field with respect to which x_t is measurable for every t.

1° x_t is $\bar{\hat{\mathcal{B}}}$-measurable for almost every t (with respect to μ).

To prove 1°, define

$$s_n(t,\omega) = \sum_{i=1}^{n} [(R_0^{\frac{1}{2}}\varphi_i)(t)\eta_i(\omega) + (R_0^{\frac{1}{2}}\tilde{\varphi}_i)(t)\tilde{\eta}_i(\omega)].$$

Then, from Lemma 6, there exists a subsequence $\{s_{n_k}(t,\omega)\}$ which converges to $x_t - m_1(t)$, a.e. $(\mu \times \bar{P}_{10})$. Namely, if

$$D = \{(t,\omega): x_t(\omega) - m_1(t) \neq \lim_{k\to\infty} s_{n_k}(t,\omega)\},$$

then

$$D \in \mathcal{A} \times \bar{\mathcal{B}}_{P_{10}} \quad \text{and} \quad (\mu \times \bar{P}_{10})(D) = 0.$$

Hence, from Fubini's theorem,* for almost every t

$$P_{10}(D_t) = 0,$$

where D_t is the section of D determined by t. In other words, $s_{n_k}(t,\omega)$ converges to $x_t(\omega) - m_1(t)$, a.e. $(\bar{P}_{10})$, for almost every t. Then, since each $s_{n_k}(t,\omega)$, $k = 1,2, \cdots$, is $\hat{\mathcal{B}}$-measurable for every t, an argument analogous to the one in the proof of Theorem 1 (p. 1642) shows that $\Lambda_t = \{\omega: x_t(\omega) - m_1(t) \in A\}$ is $\bar{\hat{\mathcal{B}}}$-measurable for almost every t, where A is any Borel set. Namely, $x_t - m_1(t)$ and, hence, x_t are $\bar{\hat{\mathcal{B}}}$-measurable for almost every t.

2° x_t is $\bar{\hat{\mathcal{B}}}$-measurable for every t.

To prove 2°, let $T \in \mathcal{A}$ be a set of t for which x_t is $\bar{\hat{\mathcal{B}}}$-measurable. Then, $\mu(T) = 1$. Since r_0 is continuous on $[0,1] \times [0,1]$ and T is dense in [0,1], there exists for every $t \in [0,1]$ a sequence $\{t_n\}$, $t_n \in T$, converging to t such that

$$\lim_{n\to\infty} E_{10}\{ | x_t - m_1(t) - x_{t_n} + m_1(t_n) |^2\} = 0.$$

* See Ref. 10, p. 147.

Hence, there exists a subsequence $\{t_{n_k}\}$ such that

$$\lim_{k\to\infty} [x_{t_{n_k}} - m_1(t_{n_k})] = x_t - m_1(t), \qquad \text{a.e. } (P_{10}).$$

Then, since each $x_{t_{n_k}} - m_1(t_{n_k})$, $k = 1,2, \cdots$, is $\bar{\bar{\mathcal{B}}}$-measurable for every t, the same argument used above shows that Λ_t is $\bar{\bar{\mathcal{B}}}$-measurable for every t. Namely, $x_t - m_1(t)$ and, hence, x_t are $\bar{\bar{\mathcal{B}}}$-measurable for every $t \in [0,1]$.

Theorem 2 (Pitcher):

(*i*) *Either* $P_{10} \equiv P_1$ *or* $P_{10} \perp P_1$,
(*ii*) $P_{10} \equiv P_1$ *if and only if* $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ *is a densely defined, bounded, completely continuous, Hilbert-Schmidt operator on* $\mathfrak{L}_2(0,1)$,
(*iii*) *if* $P_{10} \equiv P_1$,

$$\frac{dP_1}{dP_{10}} = \lim_{n\to\infty} \exp\left\{\frac{1}{2}\sum_{i=1}^{n}\left[\left(1 - \frac{1}{\rho_i}\right)\eta_i^2 - \log \rho_i\right]\right\}, \qquad \text{a.e. } (\bar{P}_{10}),$$

where ρ_i, $i = 1,2, \cdots$, *are the eigenvalues of* $R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$.

Proof:

(ii) *Necessity:* Assume $P_{10} \equiv P_1$.
Then, from Lemma 4, $R_1^{\frac{1}{2}}R_0^{-\frac{1}{2}}$ is bounded. Hence, $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ is densely defined and bounded.

The above statement implies that $R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ is self-adjoint and positive-definite, and its bounded extension to the whole of $\mathfrak{L}_2(0,1)$ is equal to X^*X. Let $\int \nu dP_\nu$ be the spectral representation of X^*X. We now show by contradiction that X^*X has a purely discrete spectrum. Suppose for some $\varepsilon > 0$, $I - P_{1+\varepsilon}$ is infinite dimensional. Then, there exists a sequence $\{\nu_i\}$, $1 + \varepsilon \leq \nu_1 < \nu_2 < \cdots$, and a sequence of orthonormal functions $f_i \in \mathfrak{L}_2(0,1)$, $i = 1, 2, \cdots$, such that

$$(P_{\nu_{i+1}} - P_{\nu_i})f_i = f_i. \tag{42}$$

Hence, from Remark 1 of Lemma 5, there exists a sequence of random variables θ_i, $i = 1,2, \cdots$, which are measurable with respect to $\bar{\mathcal{B}}_{P_{10}}$ and $\mathcal{B}_{P_1}$ and Gaussian distributed with respect to both $\bar{P}_{10}$ and $\bar{P}_1$, such that

$$E_{10}\{\theta_i\} = E_1\{\theta_i\} = 0,$$

$$E_{10}\{\theta_i\theta_j\} = (f_i, f_j) = \delta_{ij},$$

$$E_1\{\theta_i\theta_j\} = (f_i, X^*Xf_j) = \delta_{ij}\int_{\nu_i}^{\nu_i+1} \nu d(f_i, P_\nu f_i) \geqq (1 + \varepsilon)\delta_{ij}.$$

Let $\mathcal{B}^*$ be the minimal σ-field with respect to which all θ_i, $i = 1,2, \cdots$, are measurable, and let P_{10}^* and P_1^* be the restrictions of $\bar{P}_{10}$ and $\bar{P}_1$ on $\mathcal{B}^*$. Then, from Lemma 3, $P_{10}^* \perp P_1^*$. It follows then from Lemma 1 that $P_{10} \perp P_1$, which is a contradiction. Therefore, $I - P_{1+\varepsilon}$ is finite dimensional for every $\varepsilon > 0$. Similarly, it can be shown that $P_{1-\varepsilon}$ is finite dimensional also. Hence, X^*X has a purely discrete spectrum, and 1 is the only limit point of the spectrum. Hence, $I - X^*X$ is completely continuous,* and so is $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$.

It follows from the preceding paragraph that the eigenvalues and the corresponding eigenfunctions, ρ_i and φ_i, $i = 1,2, \cdots$, of $R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ exist. Then, according to Lemma 5, η_i and $\tilde{\eta}_i$, $i = 1,2, \cdots$, defined in Lemma 6 have the following properties:

$$
\begin{aligned}
E_{10}\{\eta_i\} &= E_{10}\{\tilde{\eta}_i\} = E_1\{\eta_i\} = E_1\{\tilde{\eta}_i\} = 0 \\
E_{10}\{\eta_i\eta_j\} &= (\varphi_i, \varphi_j) = \delta_{ij}, \\
E_{10}\{\tilde{\eta}_i\tilde{\eta}_j\} &= (\tilde{\varphi}_i, \tilde{\varphi}_j) = \delta_{ij}, \\
E_{10}\{\eta_i\tilde{\eta}_j\} &= (\varphi_i, \tilde{\varphi}_j) = 0, \\
E_1\{\eta_i\eta_j\} &= (\varphi_i, R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}\varphi_j) = \rho_i\delta_{ij}, \\
E_1\{\tilde{\eta}_i\tilde{\eta}_j\} &= (\tilde{\varphi}_i, R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}\tilde{\varphi}_j) = \delta_{ij} \\
E_1\{\eta_i\tilde{\eta}_j\} &= (\varphi_i, R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}\tilde{\varphi}_j) = (\varphi_i, \tilde{\varphi}_j) = 0.
\end{aligned}
\tag{43}
$$

Let $\hat{P}_{10}$ and $\hat{P}_1$ be the restrictions of $\bar{P}_{10}$ and $\bar{P}_1$ on $\hat{\mathcal{B}}$. Then, since $\rho_i > 0$, $i = 1,2, \cdots$, it follows from Lemma 3 that either $\hat{P}_{10} \equiv \hat{P}_1$ or $\hat{P}_{10} \perp \hat{P}_1$, and $\hat{P}_{10} \equiv \hat{P}_1$ if and only if

$$\sum_{i=1}^{\infty} \left(1 - \frac{1}{\rho_i}\right)^2 < \infty. \tag{44}$$

Furthermore, from Lemma 1, $\hat{P}_{10} \perp \hat{P}_1 \Rightarrow P_{10} \perp P_1$. But, since $P_{10} \equiv P_1$ from the hypothesis, we must have $\hat{P}_{10} \equiv \hat{P}_1$. Hence, (44) is satisfied, or equivalently,

$$\sum_{i=1}^{\infty} (1 - \rho_i)^2 < \infty. \tag{45}$$

Namely, $I - R_0^{-\frac{1}{2}}R_1R_1^{-\frac{1}{2}}$ is of Hilbert-Schmidt type.

Sufficiency: Assume that $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ is a densely defined, bounded, completely continuous, Hilbert-Schmidt operator on $\mathcal{L}_2(0,1)$. Then, $R_1^{\frac{1}{2}}R_0^{-\frac{1}{2}}$ is bounded, and $R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ is self-adjoint and positive-

*See Ref. 12, pp. 234–235.

definite. Thus, we establish η_i and $\tilde{\eta}_i$, $i = 1,2, \cdots$, and (43) as previously done. Now, since $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ is of Hilbert-Schmidt type, (45) is satisfied, and so is (44). Then, since $\rho_i > 0$, $i = 1,2, \cdots$, it follows from Lemma 3 that $\hat{P}_{10} \equiv \hat{P}_1$. Then, from Lemma 7 and Lemma 2 (ii),

$$P_{10} \equiv P_1 .$$

(i) *Dichotomy:* Assume that P_{10} and P_1 are not equivalent. Then, one of the following three cases must hold:

(a) $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ is either not densely defined or unbounded, or both,

(b) it is densely defined and bounded, but not completely continuous,

(c) it is densely defined, bounded and completely continuous, but not of Hilbert-Schmidt type.

In case (a), $R_1^{\frac{1}{2}}R_0^{-\frac{1}{2}}$ is unbounded. Hence, from Lemma 4, $P_{10} \perp P_1$. In case (b), X^*X has a spectral representation, and either $I - P_{1+\varepsilon}$ or $P_{1-\varepsilon}$ must be infinite dimensional for some $\varepsilon > 0$. Then, $P_{10}^* \perp P_1^*$ and, hence, $P_{10} \perp P_1$, as shown in the necessity part of the proof of (ii). In case (c), $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ has the eigenvalues and eigenfunctions $1 - \rho_i$ and φ_i, $i = 1,2, \cdots$, and there are the associated Gaussian variables η_i and $\tilde{\eta}_i$, $i = 1,2, \cdots$, as described previously. But since $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ is not of Hilbert-Schmidt type, (45) and, hence, (44) do not hold. Then, according to Lemma 3, $\hat{P}_{10} \perp \hat{P}_1$. Then, from Lemma 1, $P_{10} \perp P_1$. Therefore, we conclude that if P_{10} and P_1 are not equivalent then they must be singular.*

(ii) *Radon-Nikodym Derivative:* The assertion (iii) is an immediate consequence of Lemma 3 (iii) with $\nu_i = 0$, $i = 1,2, \cdots$, and Lemma 2 (*iv*).

Corollary 1: If $P_{10} \equiv P_1$ and

$$\left| \sum_{i=1}^{\infty} (1 - \rho_i) \right| < \infty ,$$

then

$$\frac{dP_1}{dP_{10}} = \left(\prod_{i=1}^{\infty} \rho_i \right)^{-\frac{1}{2}} \exp\left[\frac{1}{2} \sum_{i=1}^{\infty} \left(1 - \frac{1}{\rho_i} \right) \eta_i^2 \right], \qquad \text{a.e. } (\bar{P}_{10}).$$

Proof: Note that η_i, $i = 1,2, \cdots$, are mutually independent Gaussian variables with

$$E_{10}\{\eta_i\} = 0, \qquad E_{10}\{\eta_i^2\} = 1, \qquad E_{10}\{\eta_i^4\} = 3.$$

* Note this trivially implies that if P_{10} and P_1 are not singular, then they must be equivalent.

Hence

$$\left| \sum_{i=1}^{\infty} E_{10} \left\{ \left(1 - \frac{1}{\rho_i} \right) \eta_i^2 \right\} \right| = \left| \sum_{i=1}^{\infty} \left(1 - \frac{1}{\rho_i} \right) \right| < \infty,$$

$$\sum_{i=1}^{\infty} E_{10} \left\{ \left(1 - \frac{1}{\rho_i} \right)^2 \eta_i^4 \right\} = 3 \sum_{i=1}^{\infty} \left(1 - \frac{1}{\rho_i} \right)^2 < \infty.$$

Therefore,*

$$\sum_{i=1}^{\infty} \left(1 - \frac{1}{\rho_i} \right) \eta_i^2 < \infty, \qquad \text{a.e. } (\bar{P}_{10}).$$

Then, the assertion follows upon combination of the above and Theorem 2 (iii).

Corollary 2 (Pitcher): If there exists a bounded, self-adjoint operator H on $\mathfrak{L}_2(0,1)$ *satisfying*

$$R_0 H R_1 = R_1 H R_0 = R_1 - R_0 , \tag{46}$$

then

$$P_{10} \equiv P_1$$

and

$$\frac{dP_1}{dP_{10}} = \left(\prod_{i=1}^{\infty} \rho_i \right)^{-\frac{1}{2}} \exp\left[\tfrac{1}{2}(x - m_1 , H(x - m_1)\right], \qquad \text{a.e. } (P_{10}).$$

APPENDIX D

Third Theorem on Equivalence and Singularity

Lemma 8: $P_{10} \perp P_1 \Rightarrow P_0 \perp P_1$.

Proof: If $P_{10} \perp P_1$, it follows from Theorem 2, (i) and (ii), that one of the three cases (a), (b) and (c) listed in the proof of Theorem 2 (i) holds.

In case (a), $R_1^{\frac{1}{2}} R_0^{-\frac{1}{2}}$ is unbounded, then $P_0 \perp P_1$ according to Lemma 4.

In case (b), at least, either $I - P_{1+\varepsilon}$ or $P_{1-\varepsilon}$ must be infinite dimensional for some $\varepsilon > 0$, as shown in the proof of Theorem 2 (ii). Suppose $I - P_{1+\varepsilon}$ is infinite dimensional. Then, there exists a sequence of orthonormal functions f_i , $i = 1,2, \cdots$, satisfying (42). Hence, according to Lemma 5, there exist a corresponding sequence of Gaussian variables

* See Ref. 5, p. 108.

θ_i, $i = 1, 2, \cdots$, such that

$$E_0\{\theta_i + \nu_i'\} = E_1\{\theta_i\} = 0, \qquad E_0\{(\theta_i + \nu_i')(\theta_j + \nu_j')\} = \delta_{ij},$$
$$E_1\{\theta_i\theta_j\} = (f_i, X^*Xf_j) \geqq (1 + \varepsilon)\delta_{ij},$$

provided that either $R_0^{-\frac{1}{2}}m \in \mathfrak{L}_2(0,1)$ or there exists a sequence $\{f_{ij}\}_j$, $i = 1,2, \cdots$, satisfying the conditions of Remark 2 of Lemma 5. Now, let $\mathcal{B}^*$ be the minimal σ-field with respect to which all θ_i, $i = 1,2, \cdots$, are measurable, and let P_0^* and P_1^* be the restrictions of $\bar{P}_0$ and $\bar{P}_1$ on $\mathcal{B}^*$. Then, from Lemma 3 and the above result, it follows that $P_0^* \perp P_1^*$. Hence, from Lemma 1, $P_0 \perp P_1$. On the other hand, suppose neither $R_0^{-\frac{1}{2}}m \in \mathfrak{L}_2(0,1)$ nor there exist such a sequence $\{f_{ij}\}_j$ for some i. Then, from Remark 3 of Lemma 5, $P_0 \perp P_1$ also.

Similarly, if $P_{1-\varepsilon}$ is infinite dimensional, it can be shown that $P_0 \perp P_1$.

In case (c), we can assume existence of the Gaussian variables η_i and $\tilde{\eta}_i$, $i = 1,2, \cdots$, with the properties (43) and the following:

$$E_0\{\eta_i + \gamma_i\} = E_0\{\tilde{\eta}_i + \tilde{\gamma}_i\} = 0,$$
$$E_0\{(\eta_i + \gamma_i)(\eta_j + \gamma_j)\} = E_0\{(\tilde{\eta}_i + \tilde{\gamma}_i)(\tilde{\eta}_j + \tilde{\gamma}_j)\} = \delta_{ij}, \quad (47)$$
$$E_0\{(\eta_i + \gamma_i)(\tilde{\eta}_j + \tilde{\gamma}_j)\} = 0,$$

where $\gamma_i = (\varphi_i, R_0^{-\frac{1}{2}}m)$, $\tilde{\gamma}_i = (\tilde{\varphi}_i, R_0^{-\frac{1}{2}}m)$, $i = 1, 2, \cdots$.

Since $I - R_0^{-\frac{1}{2}}R_1R_0^{-\frac{1}{2}}$ is not of Hilbert-Schmidt type, (45) does not hold. Thus, (44) is not satisfied. Hence, from Lemma 3, $\hat{P}_0 \perp \hat{P}_1$. Then, from Lemma 1, $P_0 \perp P_1$.

Lemma 9: $P_0 \perp P_{10} \Rightarrow P_0 \perp P_1$.

Proof: Since $P_0 \perp P_{10}$, there exist a non-empty set $\Lambda \in \mathcal{B}$ such that

$$P_0(\Omega - \Lambda) = 0 \quad \text{and} \quad P_{10}(\Lambda) = 0.$$

Now, if $P_{10} \equiv P_1$, then $P_1(\Lambda) = 0$. Hence, we have

$$P_0(\Omega - \Lambda) = 0 \quad \text{and} \quad P_1(\Lambda) = 0,$$

namely, $P_0 \perp P_1$. If P_{10} and P_1 are not equivalent, then they must be singular according to Theorem 2 (i), i.e., $P_{10} \perp P_1$. Then, from Lemma 8, $P_0 \perp P_1$.

Theorem 3:

(*i*) *Either* $P_0 \equiv P_1$ *or* $P_0 \perp P_1$,
(*ii*) $P_0 \equiv P_1$ *if and only if*

(*a*) $I - R_0^{-\frac{1}{2}} R_1 R_0^{-\frac{1}{2}}$ *is a densely defined, bounded, completely continuous, Hilbert-Schmidt operator on* $\mathfrak{L}_2(0,1)$,

(*b*) $R_0^{-\frac{1}{2}} m \in \mathfrak{L}_2(0,1)$,

(*iii*) *if* $P_0 \equiv P_1$,

$$\frac{dP_1}{dP_0} = \exp\left\{\frac{1}{2}\sum_{i=1}^{\infty}\left[\left(1 - \frac{1}{\rho_i}\right)\eta_i^2 - \log \rho_i\right]\right\} \cdot \exp\left\{\sum_{i=1}^{\infty}\left[\gamma_i\left(\eta_i + \frac{\gamma_i}{2}\right) + \tilde{\gamma}_i\left(\tilde{\eta}_i + \frac{\tilde{\gamma}_i}{2}\right)\right]\right\}, \qquad \text{a.e. } (\bar{P}_0).$$

(Remark) Note it follows from Theorems 1 and 2 that the necessary and sufficient condition for $P_0 \equiv P_1$ is (a) $P_{10} \equiv P_1$ and (b) $P_0 \equiv P_{10}$.

Proof:

(ii) *Necessity:* Assume $P_0 \equiv P_1$.

Then, from Theorem 2 (i) and Lemma 8,

$$P_{10} \equiv P_1,$$

while, from Theorem 1 (i) and Lemma 9,

$$P_0 \equiv P_{10}.$$

Hence, (a) and (b) follow immediately from Theorem 2 (ii) and Theorem 1 (ii) respectively.

Sufficiency: Obvious since $P_0 \equiv P_{10}$ and $P_{10} \equiv P_1$ imply $P_0 \equiv P_1$.

(i) *Dichotomy:* Assume that P_0 and P_1 are not equivalent.

Then, it follows from the sufficiency part of (ii) as well as from Theorem 2 (i) and Theorem 1 (i) that, at least, either

$$P_{10} \perp P_1 \quad \text{or} \quad P_0 \perp P_{10}.$$

Then, from Lemma 8 and Lemma 9, we have

$$P_0 \perp P_1.$$

Thus, if P_0 and P_1 are not equivalent, then they must be singular.

(iii) *Radon-Nikodym Derivative:* From Lemma 3 (iii) and Lemma 2 (iv), in conjunction with (43) and (47), we have

$$\frac{dP_1}{dP_0} = \exp\left\{\sum_{i=1}^{\infty}\left[\frac{1}{2}\left(1 - \frac{1}{\rho_i}\right)\eta_i^2 - \tfrac{1}{2}\log \rho_i + \gamma_i\left(\eta_i + \frac{\gamma_i}{2}\right) + \tilde{\gamma}_i\left(\tilde{\eta}_i + \frac{\tilde{\gamma}_i}{2}\right)\right]\right\}, \qquad \text{a.e. } (\bar{P}_0). \tag{48}$$

Since $P_0 \equiv P_1 \Rightarrow P_0 \equiv P_{10}$ and $P_{10} \equiv P_1$ according to (ii), it follows from Theorem 2 (iii) that

$$\sum_{i=1}^{\infty} \left[\left(1 - \frac{1}{\rho_i}\right) \eta_i^2 - \log \rho_i \right] < \infty, \qquad \text{a.e. } (\bar{P}_0).$$

Hence, the remainder of the exponent of (48) converges a.e. $(\bar{P}_0)$. This proves (iii).

Corollary 1: If $P_0 \equiv P_1$ *and*

$$\left| \sum_{i=1}^{\infty} (1 - \rho_i) \right| < \infty,$$

then

$$\frac{dP_1}{dP_0} = \left(\prod_{i=1}^{\infty} \rho_i \right)^{-\frac{1}{2}} \exp \left[\frac{1}{2} \sum_{i=1}^{\infty} \left(1 - \frac{1}{\rho_i}\right) \eta_i^2 \right] \cdot \exp \left\{ \sum_{i=1}^{\infty} \left[\gamma_i \left(\eta_i + \frac{\gamma_i}{2} \right) + \tilde{\gamma}_i \left(\tilde{\eta}_i + \frac{\tilde{\gamma}_i}{2} \right) \right] \right\}, \qquad \text{a.e. } (\bar{P}_0).$$

Proof: This follows from Corollary 1 of Theorem 2 and Theorem 3 (iii).

Corollary 2: If there exists a bounded, self-adjoint operator H on $\mathfrak{L}_2(0,1)$ satisfying (46), *and* $R_0^{-\frac{1}{2}} m \in \mathfrak{L}_2(0,1)$, *then*

(i) $P_0 \equiv P_1$,

(ii) $$\frac{dP_1}{dP_0} = \left(\prod_{i=1}^{\infty} \rho_i \right)^{-\frac{1}{2}} \exp \left[\tfrac{1}{2}(x - m_1, H(x - m_1)) + \left(x - \frac{m_0 + m_1}{2}, R_0^{-1} m \right) \right], \qquad \text{a.e. } (P_0),$$

(iii) $R_0(\mathfrak{L}_2(0,1)) = R_1(\mathfrak{L}_2(0,1))$.

Proof:

(i) The assertion is an immediate consequence of combination of Theorem 3 and Corollary 2 of Theorem 2.

(ii) Note

$$\frac{dP_1}{dP_0} = \frac{dP_1}{dP_{10}} \frac{dP_{10}}{dP_0}, \qquad \text{a.e. } (P_0).$$

Then, the assertion follows upon combination of Corollary 2 of Theorem 2 and the corollary to Theorem 1.

(iii) From (46),

$$R_1(\mathfrak{L}_2(0,1)) = [R_0(HR_1 + I)](\mathfrak{L}_2(0,1)) \subset R_0(\mathfrak{L}_2(0,1)),$$

$$R_0(\mathfrak{L}_2(0,1)) = [R_1(I - HR_0)](\mathfrak{L}_2(0,1)) \subset R_1(\mathfrak{L}_2(0,1)).$$

Hence, the assertion follows.

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