Spectra for a Class of Asynchronous FM Waves

By R. R. ANDERSON, J. E. MAZO and J. SALZ

(Manuscript received July 6, 1965)

The spectral density of a frequency modulated carrier is evaluated for the case when the modulating baseband wave is a "quantized" random facsimile signal. By this nonsynchronous form of modulation we mean holding one of the allowed set of transmitted frequencies for a finite, but randomly distributed, time before switching to another frequency while maintaining phase continuity. Emphasis is given to the Poisson case of exponentially distributed intervals between transitions, and some typical curves for discrete level and continuous level situations are included.

I. INTRODUCTION

In facsimile data transmission systems, printed or pictorial information is converted into electrical signals by optical means. At any instant the signal corresponds to a definite grey level of the facsimile copy. The resulting electrical wave is an analog signal and can be transmitted as such. In some applications only black-and-white images need be transmitted and therefore the electrical signal may be quantized into only two levels. If more detail is desired, multilevel quantization can be applied.

It is possible to model such a quantized signal by considering a random sequence of points on the time (t) axis. At each point a transition may occur in the signal. The value of the signal between transitions is a constant, taking on one of N different values. For the black-and-white case, there are only two permissible values, either $+1$ or -1. The quantized facsimile signal is statistically characterized by specifying the distribution of the points on the t axis and the distribution of the amplitudes between transitions.

In this paper we concern ourselves with the spectral density of a carrier wave whose frequency is modulated by quantized facsimile signals. The spectral density is a useful item in the statistical description of such a signal in that it furnishes an estimate of bandwidth requirements. It often is also used to evaluate mutual interference between channels.

So far as is known, the amplitude modulation case is the only one hitherto covered in literature. However, we were prompted to examine the FM case as FM is currently used in facsimile data sets. From a practical point of view the most interesting case is that in which the phase is continuous at the transitions, as may be obtained from keying a single oscillator. This case differs from previous results[1] in that the transitions occur at random times.

The present paper gives a complete solution for the spectrum for an arbitrary distribution of the interval between transitions as well as arbitrary distribution of amplitudes. We treat an important special case of Poisson transitions, for which we present our results graphically in terms of the important parameters of the process.

An interesting feature is the rapidity with which the spectral density falls off with frequency measured from midband as compared with the AM case. The extent to which spectral peaking occurs at the average signaling frequency for some range of parameters is another curious feature. As would be expected from the asynchronous nature of the modulation there can be no steady sine-wave components in the process and therefore there are no discrete components in the spectrum.

II. ANALYSIS

The baseband facsimile signal is constructed in the following form. Pick a finite set $\{\mathbf{t}\}$ at random and arrange the points such that

$$0 = t_0 < t_1 < t_2 < \cdots t_N = T. \tag{1}$$

Define a set of functions

$$g_{\Delta_n}(t - t_n) = \begin{cases} 1, & t_n \leqq t \leqq t_{n+1} \\ 0, & \text{elsewhere} \end{cases} \tag{2}$$

where

$$\Delta_n = t_{n+1} - t_n .$$

In terms of (1) and (2), construct the baseband signal $x(t)$ as the following time series

$$x(t) = \sum_{n=0}^{n=N-1} a_n g_{\Delta_n}(t - t_n) \tag{3}$$

where $\mathbf{a} = (a_0, a_1, \cdots, a_{N-1})$ is an additional arbitrary set of identically distributed random variables.

The instantaneous phase $\psi(t)$ is represented as follows

$$\psi(t) = \omega_c t + \omega_d \int_0^t x(t')dt' + \varphi \equiv \psi_1(t) + \varphi, \qquad 0 \leqq t \leqq T \tag{4}$$

where φ is uniformly distributed on $[0,2\pi]$, giving the value of the phase at $t = 0$. The instantaneous frequency is $d\psi(t)/dt = \omega_c + \omega_d x(t)$, where ω_c is the angular carrier frequency and ω_d is the minimum angular frequency deviation.

The FM wave whose spectrum we wish to examine has the following representation

$$\begin{aligned} S(t) &= A \cos [\psi(t)] \\ &= (A/2) \exp \{i\psi(t)\} + (A/2) \exp \{-i\psi(t)\}, \end{aligned} \tag{5}$$

where A is a real amplitude. The spectral density of $S(t)$, $G(\omega)$, defines the average power in a unit bandwidth. Formally, it may be obtained as

$$G(\omega) = \lim_{T\to\infty} (2/T) \langle | S(\omega,T) |^2 \rangle, \qquad \omega > 0,$$

where $S(\omega,T)$ is the Fourier transform of $S(t)$ given by

$$S(\omega,T) = \int_0^T S(t) \exp(-i\omega t)\, dt \tag{7}$$

and the symbol $\langle \cdot \rangle$ denotes the ensemble average over all the random variables in $S(\omega,T)$.

One may write $S(\omega,T)$ as

$$S(\omega,T) = (A/2)e^{i\varphi} W_1(\omega,T) + (A/2)e^{-i\varphi} W_2(\omega,T) \tag{8}$$

where W_1 and W_2 are the Fourier transforms of $\exp [i\psi_1(t)]$ and $\exp [-i\psi_1(t)]$ respectively.

The ensemble average of $| S(\omega,T) |^2$ over φ is readily carried out to obtain

$$\langle | S(\omega,T) |^2 \rangle_\varphi = (A^2/4) \, | W_1(\omega,T) |^2 + (A^2/4) \, | W_2(\omega,T) |^2. \tag{9}$$

We proceed to evaluate $\langle | W_1(\omega,T) |^2 \rangle$:

$$\begin{aligned} W_1(\omega,T) &= \int_0^T \exp i\{\psi_1(t) - \omega t\} dt \\ &= \sum_{k=0}^{k=N-1} \int_{t_k}^{t_{k+1}} \exp i\{\psi_1(t) - \omega t\} dt. \end{aligned} \tag{10}$$

We observe from (4) that in the interval $[t_k, t_{k+1}]$,

$$\psi_1(t) = \omega_d \sum_{n=0}^{n=k-1} a_n \Delta_n + \omega_d a_k (t - t_k) + \omega_c t. \tag{11}$$

Inserting this expression into (10), and using

$$t_k = \sum_{n=0}^{n=k-1} \Delta_n ,$$

we find that

$$W_1(\omega,T) = \sum_{k=0}^{k=N-1} \frac{1}{i\lambda_k} \left[\exp \left\{ i \sum_{n=0}^{n=k} \lambda_n \Delta_n \right\} - \exp \left\{ i \sum_{n=0}^{n=k-1} \lambda_n \Delta_n \right\} \right] \tag{12}$$

where

$$\lambda_n = \omega_d a_n - \omega + \omega_c .$$

Multiplying (12) by its complex conjugate we obtain

$$\begin{aligned} | W_1(\omega,T) |^2 = 2\mathrm{Re} \Bigg[& \sum_{k=0}^{k=N-1} \frac{1 - \exp i\lambda_k \Delta_k}{\lambda_k^2} \\ & + \sum_{\substack{k,s=0 \\ k>s}}^{N-1} \frac{1}{\lambda_k \lambda_s} \left(\exp\left\{ i \sum_{n=s+1}^{k} \lambda_n \Delta_n \right\} + \exp\left\{ i \sum_{n=s}^{k-1} \lambda_n \Delta_n \right\} \right. \\ & \left. - \exp \left\{ i \sum_{n=s+1}^{n=k-1} \lambda_n \Delta_n \right\} - \exp \left\{ i \sum_{n=s}^{n=k} \lambda_n \Delta_n \right\} \right) \Bigg], \end{aligned} \tag{13}$$

where Re(·) denotes the real part.

At this point we must specify in more statistical detail the sets of random vectors $\boldsymbol{\Delta} = (\Delta_1, \Delta_2, \cdots, \Delta_N)$ and $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_N)$. In our original representation (3), the Δ_n's are the intervals between transitions. We adopt the reasonable assumption that these intervals are independent. On the other hand the random variables λ_n which are related to the amplitude a_n by (12) are not independent if we consider only observable transitions. Clearly for observable transitions one insists that $a_n \neq a_{n-1}$, and thus adjacent amplitudes are dependent. To remedy this awkwardness in the analysis, we construct an alternate random process with independent a_n's by admitting virtual transitions. That is, we do not require that the signal change at every t_n given in the sequence (1). We will show later how such a process, entirely equivalent to the original, may be constructed. For the present, we shall merely assume that the λ_n's are independent.

For fixed N the average of (13) with respect to $\boldsymbol{\Delta}$ and $\boldsymbol{\lambda}$ becomes

$$\langle | W_1(\omega,T) |^2 \rangle_{\Delta,\lambda} = 2\mathrm{Re}\Bigg[N \left\langle \frac{1}{\lambda^2} \right\rangle - N \left\langle \frac{\exp i\lambda\Delta}{\lambda^2} \right\rangle + \sum_{\substack{k,s=0 \\ k>s}}^{k,s=N-1} \left\{ \left\langle \frac{\exp i \sum_{s+1}^{k} \lambda_n \Delta_n}{\lambda_s \lambda_k} \right\rangle + \left\langle \frac{\exp i \sum_{s}^{k-1} \lambda_n \Delta_n}{\lambda_s \lambda_k} \right\rangle - \left\langle \frac{\exp i \sum_{s+1}^{k-1} \lambda_n \Delta_n}{\lambda_s \lambda_k} \right\rangle - \left\langle \frac{\exp i \sum_{s}^{k} \lambda_n \Delta_n}{\lambda_s \lambda_k} \right\rangle \right\} \Bigg]. \tag{14}$$

The respective averages in (14) may be expressed in terms of the characteristic function of Δ. A typical calculation yields

$$\begin{aligned} \left\langle \frac{\exp i \sum_{s+1}^{k} \lambda_n \Delta_n}{\lambda_s \lambda_k} \right\rangle_{\lambda,\Delta} &= \left\langle \frac{1}{\lambda_s \lambda_k} \prod_{n=s+1}^{n=k} \exp i\lambda_n \Delta_n \right\rangle_{\lambda,\Delta} \\ &= \left\langle \frac{1}{\lambda_s \lambda_k} \left\langle \prod_{n=s+1}^{n=k} \exp i\lambda_n \Delta_n \right\rangle_{\Delta} \right\rangle_{\lambda} \\ &= \left\langle \frac{1}{\lambda_s \lambda_k} \prod_{n=s+1}^{n=k} C_\Delta(\lambda_n) \right\rangle_{\lambda} \\ &= \left\langle \frac{1}{\lambda} \right\rangle_{\lambda} \left\langle \frac{C_\Delta(\lambda)}{\lambda} \right\rangle_{\lambda} [\langle C_\Delta(\lambda) \rangle_\lambda]^{k-s-1} \end{aligned} \tag{15}$$

where $C_\Delta(\lambda) = \langle \exp(i\lambda\Delta) \rangle_\Delta$ is the characteristic function of the random variable Δ.

Using the same procedure as above on every term in (14) we obtain

$$\langle | W_1(\omega,T) |^2 \rangle_{\Delta,\lambda} = 2\mathrm{Re}\left[N \left\langle \frac{1 - C_\Delta(\lambda)}{\lambda^2} \right\rangle_\lambda - \left(\left\langle \frac{1 - C_\Delta(\lambda)}{\lambda} \right\rangle \right)^2 \sum_{\substack{k,s \\ k>s}} \rho^{k-s-1} \right] \tag{16}$$

where

$$\rho \equiv \langle C_\Delta(\lambda) \rangle_\lambda \,.$$

Since $|\rho| < 1$, the series in (16) can be summed. If we designate the average of N by $\bar{N}$, divide (16) by T and take the limit as $T \to \infty$ such that

$$\nu = \lim_{T\to\infty} (\bar{N}/T)$$

we obtain

$$\lim_{T\to\infty} \frac{1}{T} \langle | W_1(\omega,T) |^2 \rangle_{\Delta,\lambda,N}$$
$$= 2\nu \mathrm{Re} \left[\left\langle \frac{1 - C_\Delta(\lambda)}{\lambda^2} \right\rangle_\lambda - \left(\left\langle \frac{1 - C_\Delta(\lambda)}{\lambda} \right\rangle_\lambda \right)^2 \frac{1}{1-\rho} \right], \tag{17}$$

where we made use of the identity

$$\sum_{\substack{k,s=0 \\ k>s}}^{N-1} \rho^{k-s-1} = \sum_{n=1}^{N-1} (N - n) \rho^{n-1}.$$

We can repeat the identical operations on $W_2(\omega,T)$ in (9) that we have just concluded on $W_1(\omega,T)$ and obtain an identical expression except that $\omega - \omega_c$ in W_1 will be replaced by $\omega + \omega_c$.

Combining (17) with (9) and (6), we write down the positive image spectrum as our general result, namely

$$G_+(\omega) = A^2 \nu$$
$$\mathrm{Re} \left\langle \left[\frac{1 - C_\Delta(\lambda)}{\lambda^2} \right\rangle_\lambda - \frac{1}{1 - \langle C_\Delta(\lambda) \rangle_\lambda} \cdot \left(\left\langle \frac{1 - C_\Delta(\lambda)}{\lambda} \right\rangle_\lambda \right)^2 \right]. \tag{18}$$

This result is general and applies when the choice of amplitudes is made independently at every t_n point. As remarked earlier, in a real facsimile process the choice of amplitudes is constrained. If a transition is to occur at every t_n point, the adjacent amplitudes must be correlated. We now show that the real process can indeed be represented in terms of independent amplitudes by the expediency of introducing virtual transitions. We write down the following equality

$$\sum_{n=0}^{n=N-1} a_n g_{\Delta_n}(t - t_n) = \sum_{n=0}^{n=N'-1} b_n g_{\Delta'_n}(t - t_n'). \tag{19}$$

The process on the left is the artificial process with independent a_n's and Δ_n's, whereas the process on the right has the b_n's correlated, as observation would require, and a different set of t_n''s representing the real process. Clearly the two processes are equivalent if one can find a transformation from the primed set of variables on the right to the unprimed set on the left.

We observe the following characteristics of the two representations. The representation on the right demands that a transition occur at every $t = t_n'$ for all n. To accomplish this b_n must be different from b_{n-1}, thus restricting the choice of the b_n's. The representation on the left admits independent choices of the a_n's, thus giving rise to masking of some transitions, since in fact, if $a_{n-1} = a_n$ there cannot be a transition at t_n. Furthermore, the set of t_n''s is a proper subset of the set of t_n's. To make the representation on the left useful, we must find how the two parameter sets transform. Toward this end define the following set of random variables

$$X_n = f(a_n, a_{n-1}) = \begin{cases} 1, & a_n = a_{n-1} \\ 0, & a_n \neq a_{n-1} \end{cases} \tag{20}$$

for $n = 1, 2, 3 \cdots$.

Let P_j be the probability of obtaining a sequence of exactly jX's out of $N + j$ taking on the value unity. Then the probability $P(N)$ of obtaining exactly N real transitions in T seconds in the process on the left of (19) is

$$P(N) = \sum_{j=0}^{j=\infty} f(N + j)P_j, \tag{21}$$

where $f(N + j)$ is the probability of exactly $N + j$ transitions in the process on the left of (19). Equation (21) is a linear summand equation from which we would like to find a suitable $f(\cdot)$ from the knowledge of $P(\cdot)$ and P_j.

Not intending to make a general study of solutions of (21), one simple solution is presented in the next section for the case of exponentially distributed intervals Δ. In preparation for this discussion, we point out that if a_n is a discrete multilevel random variable with equally likely probabilities the set of random variables $\{X_n\}$ in (20) is independent and therefore P_j is the binomial probability distribution. To demonstrate that indeed the set $\{X_n\}$ is independent we have to show that the conditional probability distribution of X_n given X_{n-1} does not depend on X_{n-1}.

With this in mind consider the joint characteristic function $C(\omega_1, \omega_2)$ of X_n and X_{n+1}:

$$\begin{aligned} C(\omega_1, \omega_2) &= \langle \exp(i\omega_1 X_n + i\omega_2 X_{n+1}) \rangle_{X_n, X_{n+1}} \\ &= \langle \exp\{i\omega_1 f(a_n, a_{n-1})\} \cdot \exp\{i\omega_2 f(a_n, a_{n+1})\} \rangle_{a_n, a_{n-1}, a_{n+1}}. \end{aligned} \tag{22}$$

From (20), fixing a_n and averaging first over a_{n+1} and then over a_{n-1}, we obtain

$$\begin{aligned} C(\omega_1, \omega_2) = \langle \langle \exp\{i\omega_1 f(a_n, a_{n-1})\} \rangle_{a_{n-1}} \\ \cdot \langle \exp\{i\omega_2 f(a_n, a_{n+1})\} \rangle_{a_{n+1}} \rangle_{a_n}. \end{aligned} \tag{23}$$

Since $P\{a_n = y_k\} = (1/M)$, $k = 1, 2, \cdots, M$, independent of k we can write the last equation as

$$\begin{aligned} C(\omega_1, \omega_2) = (1/M) \sum_{k=1}^{k=M} \langle \exp\{i\omega_1 f(y_k, a_{n-1})\} \rangle_{a_{n-1}} \\ \cdot \langle \exp\{i\omega_2 f(y_k, a_{n+1})\} \rangle_{a_{n+1}}. \end{aligned} \tag{24}$$

Now

$$\langle \exp\{i\omega_1 f(y_k, a_{n-1})\} \rangle_{a_{n-1}} = \frac{1}{M} \exp(i\omega_1) + \left(1 - \frac{1}{M}\right), \tag{25}$$

and likewise

$$\langle \exp\{i\omega_2 f(y_k, a_{n+1})\} \rangle_{a_{n+1}} = \frac{1}{M} \exp(i\omega_2) + \left(1 - \frac{1}{M}\right).$$

Since neither of the above averages under the summation sign in (24) depend on k, the joint characteristic function $C(\omega_1, \omega_2) = C(\omega_1)C(\omega_2)$ which says that the random variables X_n and X_{n-1} are independent for all n. Clearly the above arguments still hold if a_n is allowed to take on a continuum of values.

III. POISSON TRANSITIONS

As a special case we assume that the number N of t points in a fixed interval T obeys the Poisson probability law; consequently the probability density of the intervals Δ between transitions is exponentially distributed, namely

$$P(\Delta) = \begin{cases} \nu e^{-\nu\Delta}, & \Delta \geqq 0 \\ 0, & \Delta < 0. \end{cases} \tag{26}$$

The characteristic function of Δ is then

$$C_\Delta(\lambda) = \langle \exp i\lambda\Delta \rangle = [\nu/(\nu - i\lambda)]. \tag{27}$$

In particular it follows from (27) that

$$\lambda = \frac{\nu}{i} \frac{C_\Delta(\lambda) - 1}{C_\Delta(\lambda)}. \tag{28}$$

When (28) is substituted into (18) we obtain a very simple result for the spectrum, namely

$$G_+(\omega) = \frac{A^2}{\nu} \operatorname{Re}\left[\frac{\langle C_\Delta(\lambda)\rangle_\lambda}{1 - \langle C_\Delta(\lambda)\rangle_\lambda}\right], \tag{29}$$

where we made use of the fact that

$$\left\langle \frac{C_\Delta(\lambda)}{1 - C_\Delta(\lambda)} \right\rangle_\lambda = i\nu \left\langle \frac{1}{\lambda} \right\rangle_\lambda$$

which is purely imaginary.

For black and white transmission $a_n = \pm 1$ with equal probability, and the spectrum reduces to

$$G_+(\omega) = \frac{A^2}{\nu} \operatorname{Re}\left[\frac{\frac{1}{2}\frac{\nu}{\nu - i\lambda_1} + \frac{1}{2}\frac{\nu}{\nu - i\lambda_2}}{1 - \frac{1}{2}\frac{\nu}{\nu - i\lambda_1} - \frac{1}{2}\frac{\nu}{\nu - i\lambda_2}}\right], \tag{30}$$

where $\lambda_1 = \omega_d - \omega + \omega_c$ and $\lambda_2 = -\omega_d - \omega + \omega_c$ from (12). By algebraic manipulation (30) is reduced to

$$G_+(\omega) = \frac{A^2}{\nu} \frac{\nu^2 \omega_d^2}{[\omega_d^2 - (\omega - \omega_c)^2]^2 + \nu^2(\omega - \omega_c)^2}. \tag{31}$$

We see from this expression that the spectrum falls off as the fourth power of frequency. In a forthcoming section we shall present graphs of the various spectra.

It is instructive to examine the physical meaning of the parameter ν. This parameter is the average number of transitions per unit time of the virtual process. In fact the average number of transitions in the real process is $\nu[1 - (1/M)]$, with the Poisson form of the density being preserved. To show that this is so, we observe that a solution of (21) is a Poisson probability distribution. In general, for M levels with Pr $[a_n = k] = 1/M$, $k = 1, 2, \cdots, M$ we have, from the previous section,

$$P_j = \frac{(N + j)!}{j!N!} \left(\frac{1}{M}\right)^j \left(1 - \frac{1}{M}\right)^N.$$

If $f(N + j)$ is assumed to be

$$f(N + j) = \frac{e^{-\nu} \nu^{N+j}}{(N + j)!}, \tag{32}$$

we find from (21) that

$$P(N) = \sum_{j=0}^{j=\infty} f(N + j) P_j = \frac{\nu_1^N e^{-\nu_1}}{N!} \tag{33}$$

where $\nu_1 = \nu[1 - (1/M)]$. Thus Poisson transitions with parameter ν in the simple representation correspond to Poisson transitions with parameter ν_1 in the real process.

IV. GRAPHICAL REPRESENTATION

In this section we present graphical results for the case of Poisson distributed transitions. It is important to bear in mind the distinction between the parameters for the virtual transitions, with which the calculations are done, and the parameters of the real process, for which the results are reported. Here, we shall reverse the convention of (19) and use primes to distinguish "virtual" parameters. Frequency and fequency deviation are normalized to the average transition rate, i.e.,

$$\begin{aligned} \beta' &= \frac{M - 1}{M} \beta = \frac{M - 1}{M} \cdot \frac{\omega - \omega_c}{2\pi\nu} \\ K' &= \frac{M - 1}{M} K = \frac{M - 1}{M} \frac{\omega_d}{\pi\nu}. \end{aligned} \tag{34}$$

The normalized characteristic function, with λ_n defined as in (12), may then be written as

$$C_\Delta(a) = \frac{1}{1 - 2\pi i \left(\frac{aK'}{2} - \beta'\right)}. \tag{35}$$

As our first example we consider a to be a discrete random variable taking on the possible values $2n - (M + 1)$, $n = 1, 2, \cdots, M$ with equal probability $(1/M)$. Using these facts

$$\langle C_\Delta(a) \rangle_a = \bar{x}(1 - iZ\pi\beta') + i\pi k'\bar{y}, \tag{36}$$

where

$$\bar{x} = \frac{1}{M} \sum_{n=1}^{n=M} \frac{1}{1 + 4\pi^2 \left(\frac{a_n K'}{2} - \beta'\right)^2},$$

$$\bar{y} = \frac{1}{M} \sum_{n=1}^{n=M} \frac{a_n}{1 + 4\pi^2 \left(\frac{a_n K'}{2} - \beta'\right)^2},$$

where we let $a_n = 2n - (M + 1)$. Using (36) in (29) we obtain, for the virtual process

$$\frac{\nu G_+(\beta')}{A^2} = \frac{1 - \bar{x}}{(1 - \bar{x})^2 + \pi_2(2\bar{x}\beta' - K'\bar{y})^2} - 1, \quad (37)$$

and for the real process

$$\frac{\nu G_+(\beta)}{A^2} = \frac{M - 1}{M} \cdot \frac{\nu G_+(\beta')}{A^2}. \quad (38)$$

We have plotted this normalized spectral density (38) as a function

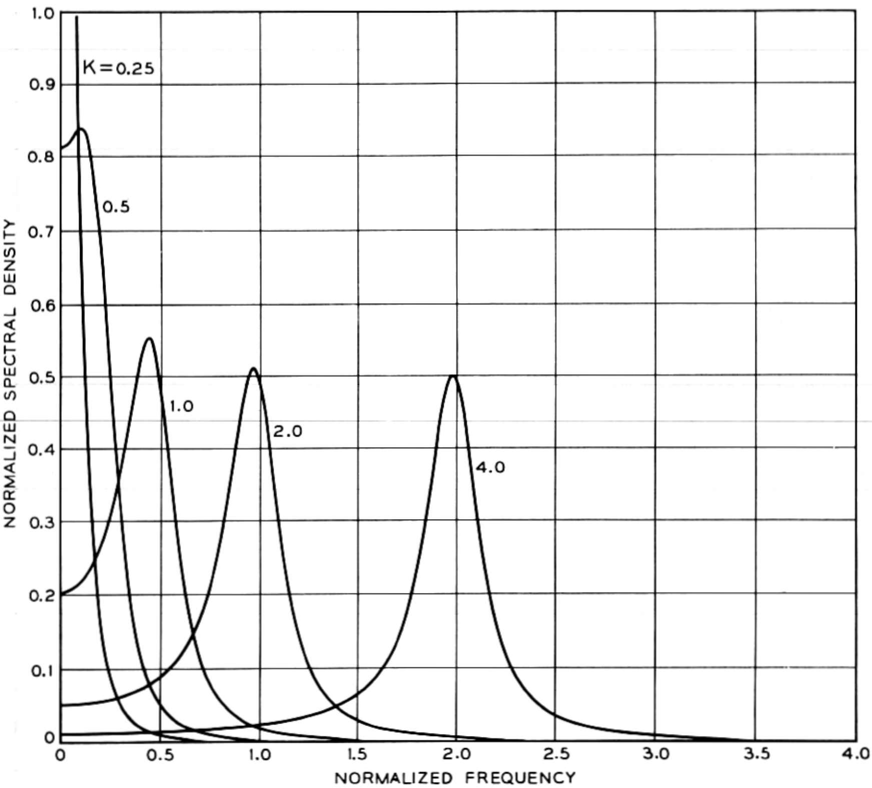

Fig. 1 — Spectra for 2-level FM FAX signals.

of the normalized frequency β for several values of normalized frequency deviation. Only the positive spectrum is shown since it is symmetrical about the normalized carrier, $\beta = 0$.

The binary case is shown on Fig. 1. We note that these spectra contain none of the spectral lines which appear with synchronous modulation. There is, nevertheless, a tendency for the spectrum to be concentrated about the frequency $\beta = k/2$. Unnormalized, this is $\omega = \omega_c \pm K\pi\nu$.

Higher level cases, $M = 4$ and 8, are shown in Figs. 2 and 3, respectively. These are similar to the binary case except that they have M levels and therefore M frequencies where concentration tends to occur. These frequencies are approximately $\beta = (2n - 1)K/2$, $n = 1, 2, \cdots, M/2$.

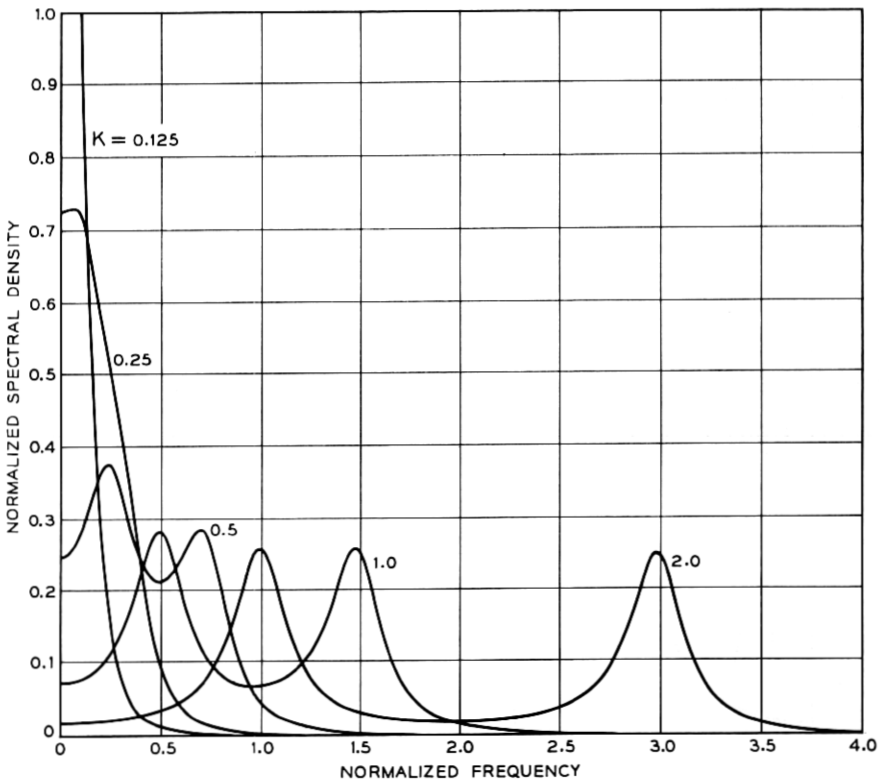

Fig. 2 — Spectra for 4-level FM FAX signals.

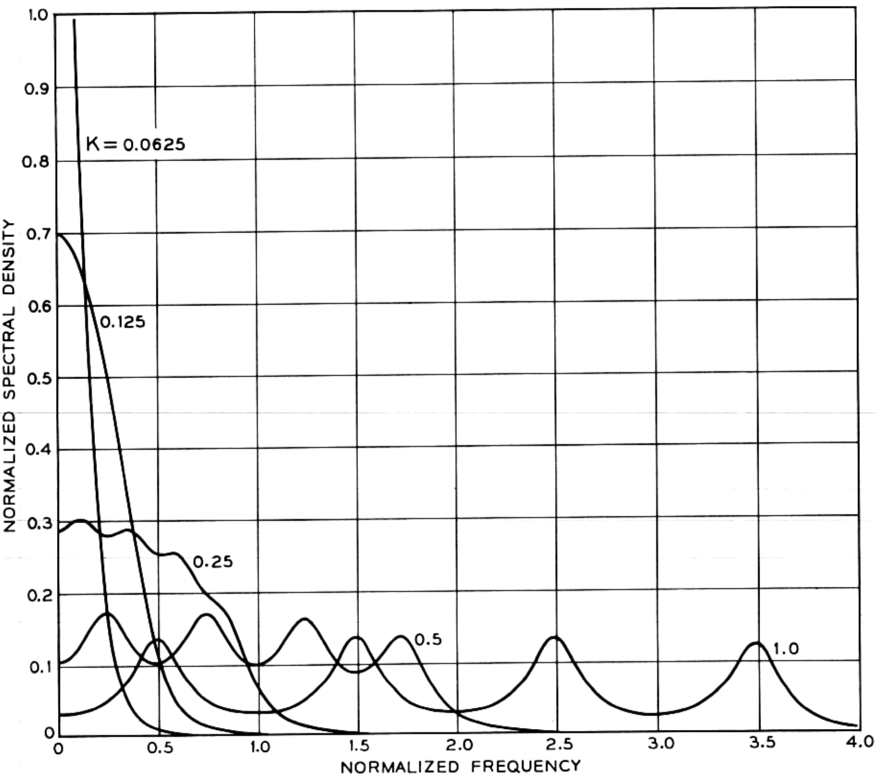

Fig. 3 — Spectra for 8-level FM FAX signals.

As a second example, we shall retain the Poisson distribution of transitions times, but allow the amplitudes to be continuously distributed over the interval $[-r,r]$. This is not true analog representation, but corresponds to "sample-and-hold" operation with exponential holding times. For this case the probability density of a is

$$P(a) = 1/2r, \qquad -r \leqq a \leqq r \tag{39}$$

and the expected value of the characteristic function becomes

$$\langle C \rangle_a = \frac{\nu}{2r} \int_{-r}^{r} \frac{da}{\nu + i\omega - i\omega_d a}$$

$$= i \frac{\nu}{2r\omega_d} \ln \frac{\nu + i\omega - i\omega_d r}{\nu + i\omega + i\omega_d r} .$$

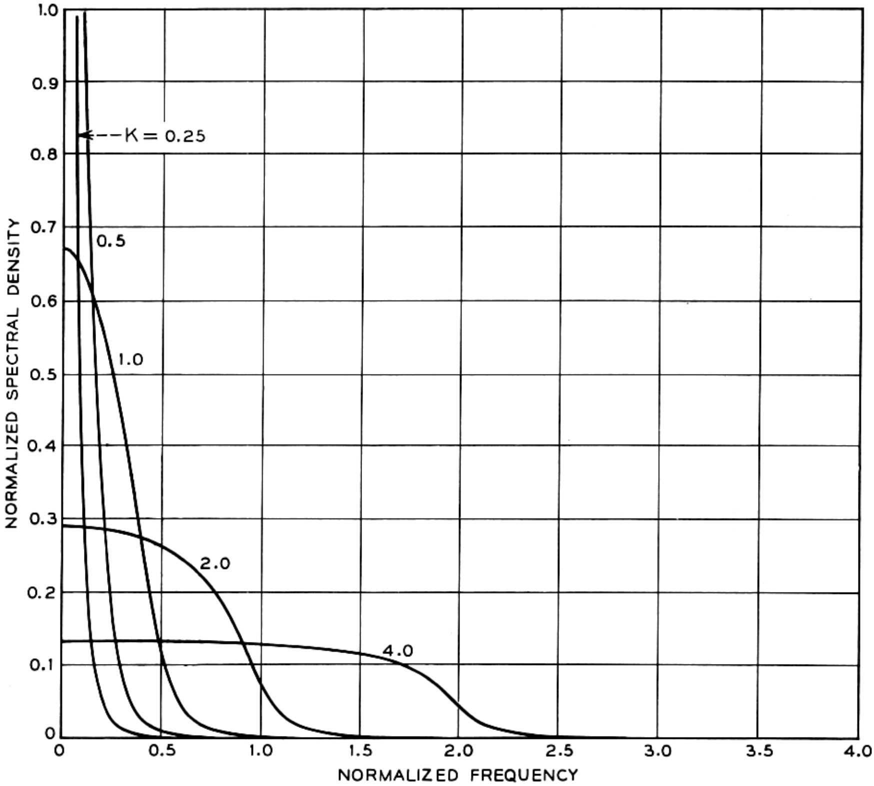

Fig. 4 — Spectra for continuous distribution FM FAX signals.

Inserting this into (29) we obtain

$$\frac{G_+(\beta)}{A^2} = \frac{R^2 + (\varphi_0 + 2\pi K)\varphi_0}{R^2 + (\varphi_0 + 2\pi K)^2}, \tag{41}$$

where

$$R = \ln \left[\frac{1 + (2\pi\beta - \pi K)^2}{1 + (2\pi\beta + \pi K)^4} \right]^{\frac{1}{2}}$$

$$\varphi_0 = \arctan\,(2\pi\beta - \pi K) - \arctan\,(2\pi\beta + \pi K).$$

The normalized frequency deviation K is now modified to include r, namely

$$K = r\omega_d/\pi\nu. \tag{42}$$

For this continuous case, spectra for various K are shown on Fig. 4. The shape of these is very nearly rectangular, with height of $1/2K$ and width of $K/2$, for the displayed positive spectrum.

Considering the above results, together with results from a previous study on digital FM,[1] it is interesting to observe that in all the plots the shape of the spectrum is approximately the same as the first order probability density function of the baseband modulation process when K is large, in accordance with the adiabatic theorem.[2]

REFERENCES

1. Anderson, R. R., and Salz, J., Spectra of Digital FM, B.S.T.J., *44,* July, 1965, pp. 1165–1189.
2. Blachman, N. M., Limiting Frequency-Modulation Spectra, Information and Control *1,* 1957, pp. 26–37.