Ray Propagation in Beam-Waveguides with Redirectors

By E. A. J. MARCATILI

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In a beam-waveguide, wanted and unwanted lens displacements from a straight line cause the beam propagating in either direction to depart severely from the guide axis. By including beam redirectors at each lens, it is shown that it is possible to reduce those displacements by amounts which depend (i) on the relative position of the lens where the beam deflection is sampled and of the redirector into which that information is fed, and (ii) on whether redirectors are sensitive to one or both directions of propagation.

If transmission takes place in one direction only, it is better to place each redirector before its deflection sampling point (feed-forward control). If transmission occurs in both directions, it is advantageous to place each redirector after its deflection sampling point (feed-back control).

I. INTRODUCTION

Correlated and uncorrelated transverse displacements of the lenses of a beam-waveguide[1,2,3,4,5] cause a propagating light beam to deviate from the axis of the guide. Since that deviation can grow proportionally to the number of lenses, the tolerance requirements on the lenses alignment become quite severe when the lenses are closelo spaced, as in beam-waveguides made with gaseous lenses.[6,7]

In order to relax those tolerances, devices have been proposed that sense the position of the beam and introduce deflections tending to re-align the beam with the guide axis.*

In this paper, the steady-state ray trajectories of beams traveling in opposite directions through a sequence of misaligned lenses are derived for several possible arrangements of iterated redirectors.

These calculations answer the following questions:

(*i*) Considering a single beam and assuming that the displacement

* Redirectors of this nature were proposed long ago by R. Kompfner and L. U. Kibler for beam-waveguides with widely separated lenses.

of the ray from the center of each lens will be used as an error signal to deflect the beam, where is it more advantageous to introduce each deflection, immediately ahead of or behind the deflection sampling point?

(*ii*) How effective is each method?

(*iii*) If the displacements of a beam are kept within tolerable limits because of the use of redirectors, are the displacements of another beam traveling in the opposite direction also within those tolerable limits?

(*iv*) For beams traveling in opposite directions, what reduces more the displacements, redirectors sensitive to one of the beams or to both of them?

II. ANALYSIS OF THE PROBLEM

Consider a sequence of aberration-free cylindrical lenses (two-dimensional problem) of focal length f and spacing L, Fig. 1, arbitrarily aligned in the plane of the drawing. The normals at the center point of each segment connecting the centers of successive lenses determine the lengths R_n which characterize the lens positions. Superimposed with each lens are four variable prisms, though for simplicity we have indicated only those at the nth lens. Lenses and prisms are idealized in the sense that they introduce phase shift in infinitely small thickness.

In a practical case, it is not necessary to have separate lens and prisms; they may be one and the same device. For example, a lens laterally displaced from the axis of the beam-waveguide is equivalent to a centered lens plus a prism.

Prisms number 1 and 2 are controlled by beam sensors,* S_1 and S_2, which are sensitive only to an east-bound ray. Those prisms deflect the beam at the nth lens by an angle

$$A\,\frac{r_{n-1}}{L} + B\,\frac{r_{n+1}}{L},$$

which is proportional to the beam displacements r_{n-1} and r_{n+1} normalized to the lens spacing L. The constants of proportionality, A and B, are determined by the amplifiers connecting each sensor with each prism.

Similarly, wedges 3 and 4 are controlled by sensors S_3 and S_4 sensitive only to a west-bound ray. The angular deflections introduced by those prisms at the nth lens is $C(\rho_{n-1})/L + D(\rho_{n+1})/L$.

* By beam sensor I mean any photosensitive device that measures the beam displacement from the center of the lens.

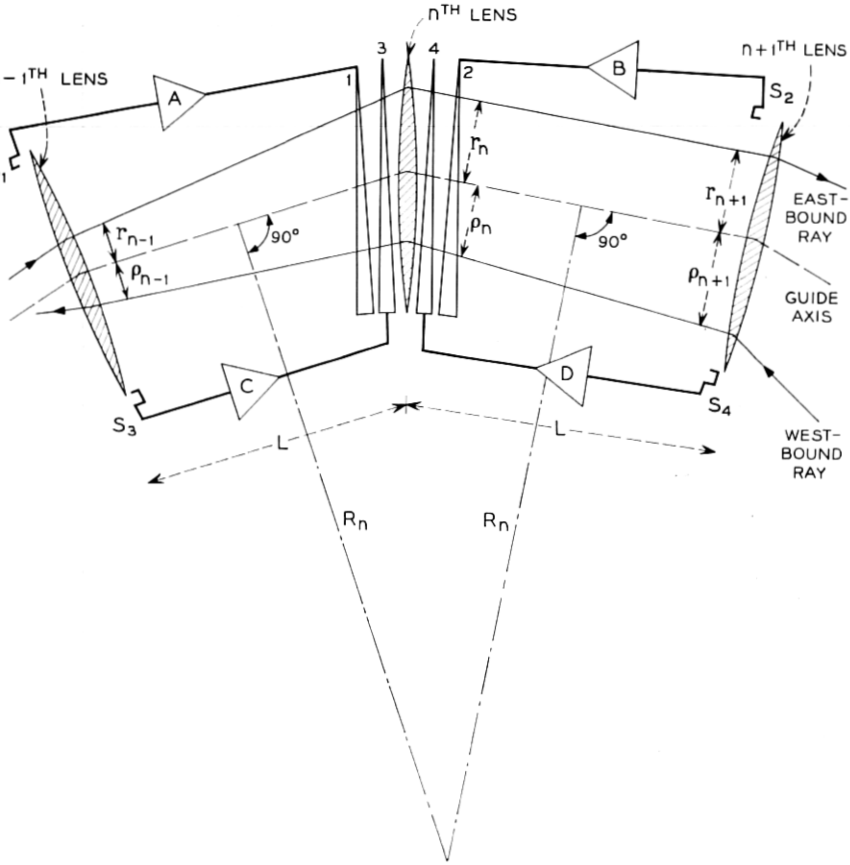

Fig. 1 — Sequence of misaligned lenses with redirectors.

The difference equation for the east-bound paraxial ray can be derived by inspection of Fig. 1,

$$r_{n+1} = r_{n-1} + \left(\frac{r_n - r_{n-1}}{L}\right) 2L - \frac{L}{f} r_n$$

$$- Ar_{n-1} - Br_{n+1} - C\rho_{n-1} - D\rho_{n+1} + \frac{L^2}{R_n}.$$

This equation is easy to interpret by noting that the deflection r_{n+1} at the $(n + 1)$th lens is equal to the sum of several partial deflections. The first two measure the deflection at the $(n + 1)$th lens as if the nth lens, prisms, and curvature of the guide axis did not exist; the third term is the deflection introduced by the nth lens; the fourth, fifth, sixth, and seventh terms are those introduced by the prisms; the last is the deflection introduced by the lenses misalignment. Cross product terms do not appear because the ray is considered paraxial.

Rearranging terms in that equation, introducing a similar expression for the west-bound ray and calling

$$1 - L/2f = \cos\theta, \tag{1}$$

two simultaneous difference equations are obtained

$$\begin{aligned} r_{n+1}(1 + B) - 2r_n \cos\theta + r_{n-1}(1 + A) + C\rho_{n-1} + D\rho_{n+1} &= \frac{L^2}{R_n} \\ \rho_{n-1}(1 + C) - 2\rho_n \cos\theta + \rho_{n+1}(1 + D) + Ar_{n-1} + Br_{n+1} &= \frac{L^2}{R_n} \end{aligned} \tag{2}$$

that permit the determination of r_n and ρ_n.

We are not interested in the general solution of (2), but rather in four particular cases. Before considering them, and to compare results, we solve first the set of equations (2) assuming no redirectors, $A = B = C = D = 0$. Using standard techniques to solve difference equations,[8]

$$\begin{aligned} r_n &= M \cos n\theta + N \sin n\theta + \frac{L^2}{\sin\theta} \sum_{m=0}^{n-2} \frac{\sin(n - m - 1)\theta}{R_{m-1}} \\ \rho_n &= P \cos n\theta + Q \sin n\theta - \frac{L^2}{\sin\theta} \sum_{m=n+2}^{\nu} \frac{\sin(n - m + 1)\theta}{R_{m-1}}. \end{aligned} \tag{3}$$

M, N, P, and Q are constants of integration determined by the initial conditions of the east- and west-bound rays. The lenses have been numbered from 0 to ν inclusively. Assuming further that the curvature of the guide axis $1/R_n$ has a Fourier component of amplitude $1/R$ and period θ, we can neglect the other spectral components[1] and in (3) replace the function $1/R_k$ by $(\sin k\,\theta)/R$, after which, keeping only the terms proportional to n, we obtain

$$\begin{aligned} r_n &\cong -n \frac{L^2}{2R} \frac{\cos n\theta}{\sin\theta} \\ \rho_n &\cong (\nu - n) \frac{L^2}{2R} \frac{\cos n\theta}{\sin\theta}. \end{aligned} \tag{4}$$

If there are no redirectors, both rays increase their deflections proportionally to the number of lenses. These results, which already have been found by other people,[1,2,3] will be compared with those obtained when redirectors are included.

Now let us consider four particular cases.

First case: only east-bound correction; each beam deflection is fed to a correcting prism at the next lens following the direction of the ray. This implies

$$A \neq 0 \qquad B = C = D = 0$$

and (2) reduce to

$$\begin{aligned} r_{n+1} - 2r_n \cos\theta + (1 + A)\, r_{n-1} &= \frac{L^2}{R_n} \\ \rho_{n-1} - 2\rho_n \cos\theta + \rho_{n+1} + A\, r_{n-1} &= \frac{L^2}{R_n}. \end{aligned} \tag{5}$$

Using standard techniques[8] to solve difference equations, the general solution of (5) is found to be

$$r_n = M_1(\cos\theta + S)^n + N_1(\cos\theta - S)^n + \frac{L^2}{2S}\sum_{m=0}^{n-2}\frac{(\cos\theta + S)^{n-m-1} - (\cos\theta - S)^{n-m-1}}{R_{m-1}} \tag{6}$$

$$\rho_n = P_1 \cos n\theta + Q_1 \sin n\theta - \frac{L^2}{\sin\theta}\sum_{k=n+2}^{\nu}\left(\frac{1}{R_{k-1}} - \frac{A r_{k-2}}{L^2}\right)\sin\,(n - k + 1)\theta \tag{7}$$

where

$$S = \sqrt{-A - \sin^2\theta};$$

$\nu + 1$ is the total number of lenses numbered from 0 to ν, and M_1, N_1, P_1, and Q_1 are constants of integration to be determined by the initial conditions of the east- and west-bound rays. The east-bound ray, r_n, has minimum deflection from the guide axis if

$$\begin{aligned} A &= -1 \\ \cos\theta &= 1 - \frac{L}{2F} = 0. \end{aligned} \tag{8}$$

The first equation implies that for the east-bound beam, the deflection introduced by prism number 1 at the $(n + 1)$th lens must be equal to minus the deflection of the beam at the $(n - 1)$th lens; the second equation indicates that the sequence of lenses must be confocal. In order to see what is the effect of small departures from conditions (8), we assume

$$\begin{aligned} A &= -1 - \delta \\ \cos\theta &= \varepsilon \end{aligned} \tag{9}$$

with $|\delta| \ll 1$ and $|\varepsilon| \ll 1$, and then recalculate the deflections of the opposite traveling beams by substituting (9) in (6) and (7). Neglecting terms with powers of δ and ε bigger than one leads to

$$r_n = L^2\left(\frac{1}{R_{n-1}} + \frac{2\varepsilon}{R_{n-2}} + \frac{\delta}{R_{n-3}}\right) \tag{10}$$

$$\begin{aligned}\rho_n &= P_1 \cos n\,\theta + Q_1 \sin n\,\theta \\ &\quad - L^2 \sum_{k=n+2}^{\nu} \left(\frac{1}{R_{k-1}} + \frac{1+\delta}{R_{k-3}} + \frac{2\varepsilon}{R_{k-4}} + \frac{\delta}{R_{k-5}}\right) \sin\,(n - k + 1)\theta.\end{aligned} \tag{11}$$

The east-bound ray displacement, r_n, does not grow proportionally to n as it would if the redirectors were not present (4). The small departure ε and δ, from the ideal conditions (8) change only slightly the minimum possible beam displacement, $r_{n\ \min} = L^2/R_{n-1}$.

Furthermore, assuming as before that the curvature of the guide axis $1/R_k$ is Fourier-analyzed and we replace its value in (11) by the component $(\sin k\,\theta)/R$, the resulting west-bound ray displacement

$$\rho_n = P_1 \cos n\,\theta + Q_1 \sin n\,\theta \tag{12}$$

not only does not grow but besides, is insensitive to a first order, to the departures δ and ε from the ideal conditions (8).

The only drawback of this system is that the amplifiers connecting sensors with prisms must be stable to maintain $A \cong -1$.

Second case: east- and west-bound correction; the beam deflection at each lens is fed to a correcting prism at the next lens following the ray. These conditions require

$$A = D \qquad C = B = 0.$$

The governing equations are obtained by replacing these values in (2)

$$r_{n+1} - 2r_n \cos\theta + r_{n-1}\,(1 + A) + A\,\rho_{n+1} = L^2/R_n \tag{13}$$

$$\rho_{n-1} - 2\rho_n \cos\theta + \rho_{n+1}\,(1 + A) + A\,r_{n-1} = L^2/R_n. \tag{14}$$

Calculating ρ_n from (13) and substituting in (14) we derive a difference equation for r_n,

$$\begin{aligned} r_{n+2} &- 2\frac{2+A}{1+A} r_{n+1} \cos\theta + 2\left(1 + \frac{2\cos^2\theta}{1+A}\right) r_n \\ &- 2\frac{2+A}{1+A} r_{n-1} \cos\theta + r_{n-2} = \frac{L^2}{1+A}\left[\frac{1}{R_{n+1}} + \frac{1}{R_{n-1}} - \frac{2\cos\theta}{R_n}\right], \end{aligned} \tag{15}$$

the solution of which is

$$r_n = M_2 \cos n\,\theta + N_2 \sin n\,\theta + P_2 \cos n\,\alpha + Q_2 \sin n\,\alpha + \frac{L^2}{(1 + A) \sin \alpha} \sum_{m=0}^{n-2} \frac{\sin (n - m - 1)\alpha}{R_{m+1}}. \tag{16}$$

M_2, N_2, P_2, and Q_2 are constants of integration that depend on initial conditions of the east- and west-bound rays, and

$$\alpha = \cos^{-1} \frac{1 - L/2f}{1 + A}. \tag{17}$$

The solution for the west-bound ray is similar to (16).

Minimum deflection of the east-bound beam, r_n, is obtained by selecting the amplifiers such that

$$|A| \gg 1. \tag{18}$$

The deflection, r_n, deduced from (16) and (18) is then

$$r_n \cong M_2 \cos n\,\theta + N_2 \sin n\,\theta + P_2 \cos n\,\alpha + Q_2 \sin n\,\alpha + \frac{L^2}{A} \sum_{m=0}^{n-2} \frac{\sin (n - m - 1)\alpha}{R_{m+1}}. \tag{19}$$

If the curvature of the guide axis, $1/R_m$, is replaced by the Fourier component $(\sin m\,\alpha)/R$, the deflection r_n for $n \gg 1$ is

$$r_n \cong -\frac{nL^2}{2AR} \cos n\alpha. \tag{20}$$

Therefore, deflections are A times smaller than in the absence of redirectors (4), but much larger than in the previous case of redirectors sensitive only to one direction of propagation (10), (12).

Third case: only east-bound correction; each beam deflection is fed back into a correcting prism at the preceding lens. This means

$$A = C = D = 0, \qquad B \neq 0 \tag{21}$$

and (2) reduce to

$$\begin{aligned} (1 + B)\, r_{n+1} - 2r_n \cos \theta + r_{n-1} &= L^2/R_n \\ \rho_{n+1} - 2\rho_n \cos \theta + \rho_{n-1} + B\, r_{n+1} &= L^2/R_n . \end{aligned} \tag{22}$$

Their solutions are

$$r_n = \frac{M_3 \sin n\,\beta + N_3 \cos n\,\beta}{(1 + B)^{n/2}} + \frac{L^2}{\sin\beta} \sum_{m=0}^{n-2} \frac{\sin(n - m - 1)\,\beta}{R_{m+1}(1 + B)^{n-m/2}}$$

$$\rho_n = P_3 \sin n\,\theta + Q_3 \cos n\,\theta + \frac{L^2}{\sin\theta} \sum_{k=n+2}^{\nu} \left[\frac{1}{R_{k-1}} - \frac{B r_k}{L^2}\right] \sin(n - k + 1)\,\theta \qquad (23)$$

where, similarly to the previous cases, $\nu + 1$ is the total number of lenses, M_3, N_3, P_3, and Q_3 are integration constants and

$$\beta = \cos^{-1} \frac{1 - L/2f}{\sqrt{1 + B}}. \qquad (24)$$

East- and west-bound rays have small deflections if one chooses the proportionality constant of the amplifier

$$|B| \gg 1 \qquad (25)$$

and also lenses far from plane or concentric

$$\frac{L}{2f} \neq \begin{cases} 0 \\ 2 \end{cases}. \qquad (26)$$

Under these conditions, deflections (23) become

$$r_n \cong \frac{M_3 \sin n\,\beta + N_3 \cos n\,\beta}{B^{n/2}} + \frac{L^2}{R_{n-1}\,B}$$

$$\rho_n \cong P_3 \sin n\,\theta + Q_3 \cos n\,\theta + \frac{L^2}{B \sin\theta} \sum_{k=n+2}^{\nu} \frac{\sin(n - k + 1)\,\theta}{R_{k-1}}. \qquad (27)$$

Assuming perfect beam launching from both ends, $M_3 = N_3 = P_3 = Q_3 = 0$, and, as in previous cases, replacing $1/R_k$ by the Fourier component $(\sin k\,\theta)/R$, both rays are described by

$$r_n = \frac{L^2}{BR_{n-1}}$$

$$\rho_n = -\frac{(\nu - n)L^2 \cos n\,\theta}{2BR \sin\theta}. \qquad (28)$$

Deflections in the east-bound ray, r_n, do not grow and they are B times smaller than in the first case (10). West-bound ray deflections, ρ_n, are only B times smaller than those in the absence of redirectors (4).

Fourth case: east- and west-bound correction; the deflection of each beam is fed into a correcting prism at the preceding lens. To achieve these conditions we must choose

$$A = D = 0, \qquad B = C \neq 0, \qquad (29)$$

and (2) become

$$r_{n+1}\,(1 + B) - 2r_n \cos\theta + r_{n-1} + B\,\rho_{n-1} = L^2/R_n \tag{30}$$

$$\rho_{n+1} - 2\rho_n \cos\theta + \rho_{n-1}\,(1 + B) + B\,r_{n+1} = L^2/R_n\,. \tag{31}$$

A difference equation in r_n exclusively is obtained by calculating ρ_n from (30) and substituting the result in (31). Thus,

$$\begin{aligned} r_{n+2} - 2\frac{2+B}{1+B}r_{n+1}\cos\theta + 2\left(1 + \frac{2\cos^2\theta}{1+B}\right)r_n - 2\frac{2+B}{1+B}r_{n-1}\cos\theta \\ + r_{n-2} = \frac{L^2}{1+B}\left(\frac{1}{R_{n-1}} + \frac{1}{R_{n+1}} - \frac{2\cos\theta}{R_n}\right). \end{aligned} \tag{32}$$

Its solution, similar to that of the second case, is

$$\begin{aligned} r_n = M_4 \cos n\,\theta + N_4 \sin n\,\theta + P_4 \cos n\,\beta + Q_4 \sin n\,\beta \\ + \frac{L^2}{(1+B)\sin\beta}\sum_{m=0}^{n-2}\frac{\sin\,(n-m-1)\,\beta}{R_{m+1}}. \end{aligned} \tag{33}$$

M_4, N_4, P_4, and Q_4 are the constants of integrations and β has been defined in (24). The west-bound ray is described by an expression similar to (33).

The beam displacement, r_n, is minimized by selecting

$$|\,B\,| \gg 1.$$

As before, if the curvature of the axis, $1/R_m$, has a Fourier component, $(\sin m\,\beta)/R$, the deflection, r_n, of the east-bound beam grows proportionally to the number of lenses n traversed by the beam. For $n \gg 1$, and neglecting terms not proportional to n, expression (33) becomes

$$r_n \cong -nL^2/(2BR)\cos n\,\beta. \tag{34}$$

Deflections are B times smaller than those in the absence of redirectors (4).

III. CONCLUSIONS

In a beam waveguide with misaligned lenses, large beam departure from the guide axis can be drastically reduced by selecting a confocal sequence of lenses and by using a beam redirector at every lens sensitive to only one direction of propagation and such that it displaces the beam at the following lens by a length equal and of opposite sign to the beam displacement on the preceding lens. Under these conditions, the waveguide axis is rectified and, for the beam that sensitizes the redirectors, the ray displacement at the nth lens is

$$r_n = \frac{L^2}{R_{n-1}}$$

where L equals the separation between lenses and $1/R_{n-1}$ equals the curvature of the guide axis at the $n - 1$th lens. For the opposite beam, the displacement at the same lens is

$$\rho_n = P_1 \cos \frac{n\pi}{2} + Q_1 \sin \frac{n\pi}{2}$$

where P_1 and Q_1 are constants that define the input of this beam.

These results are not substantially varied if the conditions of confocality and deflection per redirector are not fulfilled rigorously (see (10) and (12)).

The feed-forward control system appears so effective that a guide with small beam deflections should be obtained by using redirectors not continuously but only at discrete intervals.

If the redirectors are sensitive to one direction of propagation and each deflects the beam proportionally to the beam displacement in the following lens, the displacements of the beam are small indeed,

$$r_n = L^2/BR_{n-1}$$

where $B \gg 1$ is the constant of proportionality provided by the feed-back system. On the other hand, the ray traveling in the opposite direction behaves only as if the radii of curvature of the guide axis had been increased B times, (28). Therefore, only if the guide must transmit in a single direction is the feed-back control system more effective than the previous one.

No improvement is obtained by providing redirectors sensitive to both directions of propagation or by making the deflections proportional to the beam displacement at the lens where the redirector is located.

All we have said is applicable to steady-state trajectories, but we don't know if those trajectories are stable. Without studying any transient, though, we can conclude that in the first case, where sensitivity exists for one direction of propagation and each redirector is located after the sampling point, the trajectories are indeed stable. In effect, any displacement in the $(n - 1)$th lens, will produce a change in prism 1, Fig. 1, but since there is no loop to feed back this change to the sensor S_1, there is no loop where instability can exist.

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