Delta Modulation Quantizing Noise Analytical and Computer Simulation Results for Gaussian and Television Input Signals

By J. B. O'NEAL, Jr.

(Manuscript received August 24, 1965)

Quantizing noise in delta modulation systems falls into two categories, granular noise and slope overload noise. Granular noise exists because the decoded output signal can assume only a specified number of levels in the range of interest. Slope overload noise occurs when the slope of the input signal is greater than the delta modulator is capable of reproducing. When the S/N ratio is not too small, the noise power from these two sources is additive.

A formula for the slope overload noise power for Gaussian input signals is derived. This is used with an earlier result for the granular noise to give over-all signal-to-quantizing-noise ratios. It is shown by computer simulation that the assumptions employed in deriving these signal-to-quantizing-noise ratios are valid and that the analytical results provide good estimates of the true values of these ratios. Computer simulation of the Picturephone visual telephone, a low-bandwidth television system, illustrates that the formulas derived for Gaussian signals apply quite well to video signals. Estimates are given for the S/N ratios theoretically possible when a 4.5-mc monochrome television signal is transmitted by delta modulation.*

The characteristics of delta modulation quantizing noise may cause it to be subjectively less annoying than an equivalent amount of additive Gaussian noise.

I. INTRODUCTION

Delta modulation (ΔM) is one way in which an analog signal may be converted into pulses suitable for transmission over binary channels. Since the modulation process may be simpler and cheaper for ΔM than for standard pulse code modulation (PCM), there is considerable interest in determining how the performance of ΔM relates to that of

* *Picturephone* is a service mark of the Bell System.

standard PCM. Digital transmission of an analog signal is one way to effectively trade bandwidth for noise immunity in the transmission medium. Indigenous to this trade, however, is the introduction of quantizing noise in the encoding and decoding processes. Bennett[1] has studied this problem for standard PCM. De Jager[2] and Van de Weg[3] have studied quantizing noise for ΔM, but their results are somewhat restricted due to the difficulties encountered in analyzing a nonlinear feedback system such as ΔM. Zetterberg[4] discusses ΔM through the discipline of information theory.

This study was initiated primarily to determine what kind of performance could be expected if ΔM were used for narrow-band television signals such as *Picturephone* signals, but more general results are also presented. The metric used as a figure of merit is the ordinary signal-to-quantizing-noise ratio S/N. Recently, the concept of delta modulation has been subject to a rash of embellishments and modifications, some of which may improve its ability to transmit television signals and all of which add equipment complexity. This study, however, is limited to the simplest type of single-integration ΔM system with a uniform constant step size. Formulas for S/N ratio are found which apply when the S/N ratio is large, which is the region of greatest interest. The formulas derived apply to Gaussian signals with arbitrary spectra. The results are verified by simulating various ΔM systems on an IBM 7094 digital computer and analyzing their performance with Gaussian input signals. A practical ΔM system designed for the transmission of a *Picturephone* signal is simulated, and the results illustrate that the formulas derived for Gaussian signals apply quite well to monochrome television signals.

II. THEORY

Delta modulation is a simple type of predictive quantizing system.[5,6] This means that the value of the signal is predicted at each sample time and only the difference between the actual signal value and this predicted value is transmitted. At the receiver, the value of the decoded signal at every sample time is predicted to be the same as that at the previous sample time. At each sample time, the transmitted signal is simply a correction which, when added to the decoded signal at the previous sample time, gives (to an approximation) the signal at the current sample time. Systems of this kind are also called differential feedback PCM systems.

The basic single integration ΔM system is shown in Fig. 1. The adder and delay element in the feedback loop around the quantizer simply

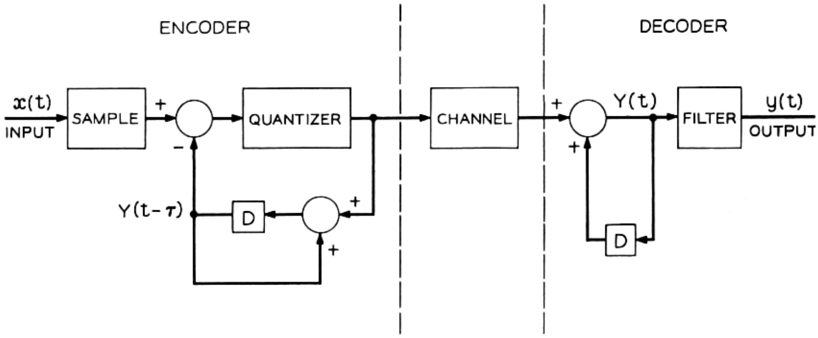

Fig. 1 — Basic binary ΔM system.

form an accumulator. The transfer function of the feedback loop is $D/(1 - D)$ where D is the unit delay operator $e^{-s\tau}$ and represents a delay of one sample interval τ. Since

$$D/(1 - D) = D + D^2 + D^3 \cdots ,$$

the signal being fed back is simply the sum of all previously transmitted samples. It is also identical to the decoded signal at the receiver and represents the signal value predicted for the next sample time. The difference between the input signal and this predicted signal is quantized and transmitted through a discrete channel. In practice, it is necessary to insert a leak, or amplifier with gain less than one, in the feedback loop. Systems with leak are discussed in Section VII.

For the most part we restrict ourselves to 2-level quantizers whose outputs can assume only the levels $\pm k$, where k is called the step size. In this paper, the term delta modulation (ΔM) when used without qualifications implies a 2-level quantizer. The extension of our results to multilevel quantizers is simple and is covered in Section X. Systems with 2-level quantizers are of greater practical interest because, in this case, the equipment required for ΔM is simple and cheap compared to standard PCM systems. Since the quantizer levels are $\pm k$, the decoded output before filtering $Y(t)$ can assume the values $\pm ik$, $i = 1, 2, \cdots$.

We assume that the input signal $x(t)$ has zero mean, unit variance, and is bandlimited to the frequency band $(0,f_o)$. The sampling rate f_s is typically many times the bandwidth and, for a 2-level system, is identical to the bit rate.

In ΔM systems there are two types of quantizing noise, granular noise and overload noise. Granular noise is similar to the quantizing noise of PCM. It is caused by the fact that the output samples can assume only discrete values which in ΔM are multiples of the step size k. Overload noise is a result of the fact that the maximum slope a ΔM system may reproduce is limited to kf_s.

Typical signals in a ΔM system are shown in Fig. 2. $Y(t)$ is the reconstructed output signal before filtering. The noise is defined as

$$n(t) = x(t) - y(t), \tag{1}$$

where $x(t)$ is the input signal and $y(t)$ is the output signal. In the example illustrated in Fig. 2 the noise $n(t)$ is granular before time t_o. At time t_o the slope of the input $x(t)$ exceeds that which the delta modulator is capable of transmitting. The period of slope overload in the case shown is from t_o to t_1 and the noise during this period is

$$n(t) = x(t) - [y(t_o) + (t - t_o)x_o'] \qquad t_o \leqq t \leqq t_1,$$

where $x_o' = kf_s$ is the maximum slope the delta modulator is capable of reproducing.

The quantizing noise is not independent of the signal. Fig. 2 illustrates that granular noise is determined by the instantaneous amplitude of the input signal and overload noise is determined by the slope of the input signal. For very large step sizes almost all of the noise is granular. As the step size is decreased the output signal loses its ability to rise and fall rapidly and overload noise becomes dominant. As the step size approaches zero so does the output $y(t)$. Due to the definition in (1) the noise $n(t)$ approaches the signal $x(t)$ and the S/N ratio, which is 10 log $\overline{x^2}/\overline{n^2}$, approaches zero db as the output $y(t)$ approaches zero.

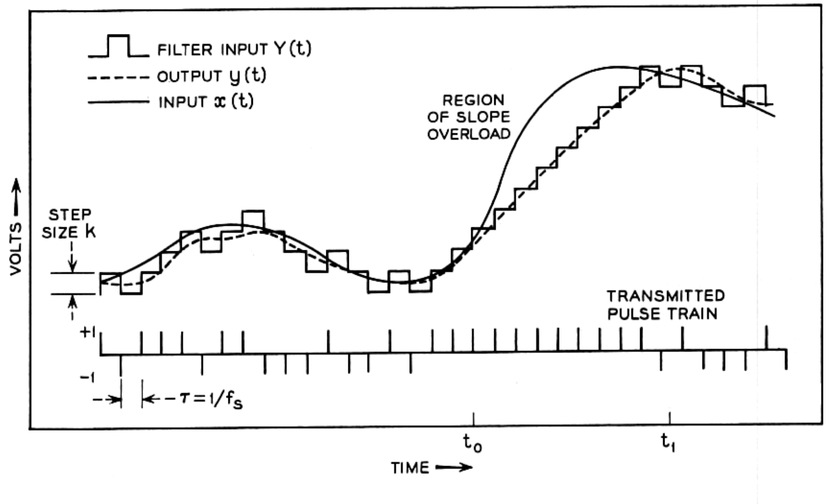

Fig. 2 — Signals in a ΔM system.

Clearly, this definition of the noise and of the S/N ratio must be used with discretion when the S/N ratio is small.

Any plot of S/N ratio versus step size will show a general tendency to approach the three asymptotes shown in Fig. 3. In Appendix A, S. O. Rice has computed an approximation for slope overload noise N_o for Gaussian input signals. This approximation is valid only when the maximum slope x_o' is somewhat larger than the rms value of $x'(t)$. In a previous paper[3] Van de Weg has found an approximation for the granular noise N_g, and his expression for N_g is used in this paper. If the overload noise is small it occurs in short bursts. During a burst, the overload noise is the dominant source of noise. When there is no overload burst all noise present is granular noise. The total noise power N can, therefore, be approximated by the sum of N_o and N_g. If we normalize our results so that the rms value of the input $x(t)$ is unity, then our approximation for the total noise power is

$$N = N_o + N_g, \tag{2}$$

$$N_o = \frac{3^5}{4\sqrt{2\pi}} \left(\frac{b_o^2}{b_2}\right) \left(\frac{f_s k}{\sqrt{b_o}}\right)^{-5} \exp\left(-\frac{f_s^2 k^2}{2b_o}\right), \tag{3}$$

$$N_g = \frac{8k^2}{\pi^2 F_s} \left[\frac{\pi^2}{12} + \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} (-1)^{nl} \frac{\sin 2\pi n/F_s}{2\pi n/F_s} \frac{1}{l^2} \exp\left(-\frac{\pi^2 l^2}{k^2}(1 - a_n)\right)\right], \tag{4}$$

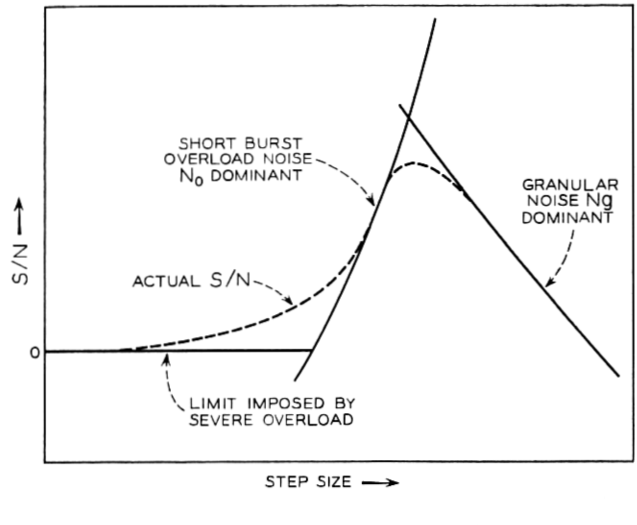

Fig. 3 — Asymptotes for S/N ratios in ΔM for $f_s \gg f_o$.

where

f_s = sampling frequency
$F_s = f_s/f_o$ = sampling frequency as a multiple of the bandwidth f_o
$a_n = \psi(n/f_s)$ where $\psi(\tau)$ is the autocovariance function of $x(t)$
k = step size
b_o = variance of $x'(t)$
b_2 = variance of $x''(t)$.

Equation (3) is obtained by substituting $f_s k$ for x_o' in (22) of Appendix A. Equation (4) is taken from page 382 of Ref. 3. These equations show that the noise power is a function of the power density spectrum of the input signal $x(t)$ since b_o, b_2, and $\psi(\tau)$ are determined by this spectrum. Since (2), (3), and (4) apply to an input signal with an rms value of 1, the rms signal to rms noise ratio in db is S/N = $-$ 10 $\log_{10} N$.

III. FLAT BANDLIMITED GAUSSIAN SIGNALS

For flat signals bandlimited to $(0,f_o)$ the expressions for b_o and b_2, which are computed from (17) in Appendix A, and $\psi(\tau)$ are

$$b_o = \frac{(2\pi f_o)^2}{3},$$

$$b_2 = \frac{(2\pi f_o)^4}{5},$$

$$\psi(\tau) = \frac{\sin 2\pi f_o \tau}{2\pi f_o \tau}, \qquad (5)$$

Fig. 4 shows plots of S/N ratio for a flat bandlimited Gaussian input signal. Points from some simulated delta modulation systems are also illustrated. Sampling rates of 4, 8, 16, 32, and 64 times the bandwidth are shown. The S/N ratios are plotted as a function of the quantity $kf_s/f_o = kF_s$ which, for any given sampling rate, may be thought of as the normalized step size. As an example of the use of these curves, we see that for a sampling rate of 32 times the bandwidth the maximum S/N ratio is about 29 db and it will occur when the step size is about 12/32. If the rms value of the input signal were not 1 but σ then, of course, the maximum S/N ratio of 29 db would occur when the step size is $12\sigma/32$. To the left of the S/N peaks the noise is primarily overload noise N_o while to the right of these peaks the noise is primarily granular noise N_g. The optimum value of kF_s maximizes the S/N ratio by providing the proper balance between N_o and N_g.

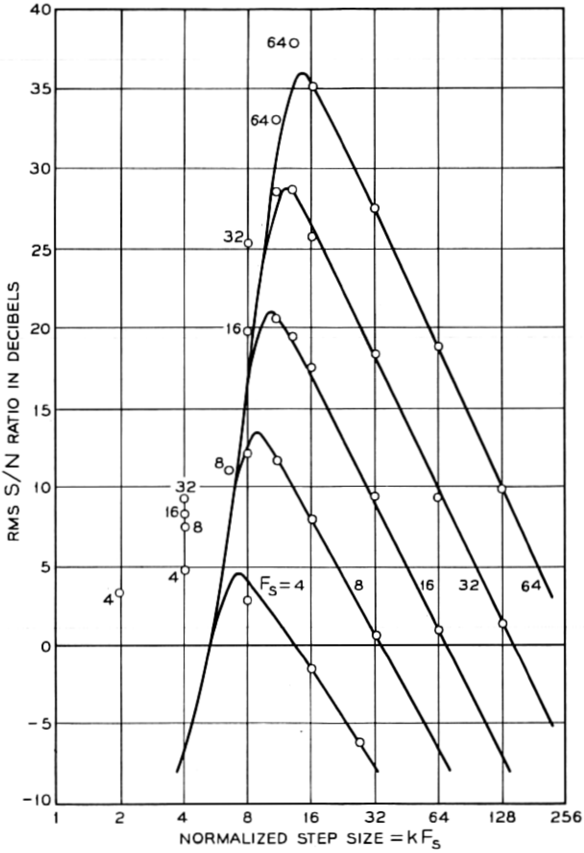

Fig. 4 — S/N ratio for bandlimited flat Gaussian signals showing theoretical curves and results of computer simulation. ($F_s = f_s/f_o$ is the ratio of sampling frequency to bandwidth.)

IV. RC SHAPED GAUSSIAN SIGNALS

If a flat Gaussian signal bandlimited to the frequency range $(0,f_o)$ is passed through a low-pass RC filter, then the one-sided power spectrum of the output is

$$F(f) = \frac{2\pi\alpha}{\tan^{-1}\left(\dfrac{2\pi f_o}{\alpha}\right)} \times \frac{1}{(2\pi f)^2 + \alpha^2} \qquad 0 < f < f_o , \tag{6}$$

where we assume a mean square value of unity and $\alpha = 1/\text{RC}$. The RC spectrum given by (6) is relatively flat from $f = 0$ to a corner frequency $f_c = \alpha/2\pi$ at which it rolls off to assume a slope of -6 db/octave up to the cutoff frequency f_o. It is shown in Appendix B that if α is small enough then we may approximate the autocovariance function $\psi(\tau)$ by

$$\psi(\tau) = \frac{\frac{\pi}{2} e^{-\alpha\tau} - \alpha\tau\left(\frac{\cos 2\pi f_o\tau}{2\pi f_o\tau} + \mathrm{Si}\,(2\pi f_o\tau) - \frac{\pi}{2}\right)}{\frac{\pi}{2} - \frac{\alpha}{2\pi f_o}} \qquad \tau \geqq 0, \tag{7}$$

$$\psi(-\tau) = \psi(\tau).$$

The values of b_o and b_2 for this RC Gaussian signal are found from (17) to be

$$b_o = \frac{2\pi f_o\alpha}{\tan^{-1}\left(\frac{2\pi f_o}{\alpha}\right)} - \alpha^2, \tag{8}$$

$$b_2 = \frac{(2\pi f_o)^3\alpha - 6\pi f_o\alpha^3}{3\tan^{-1}\left(\frac{2\pi f_o}{\alpha}\right)} + \alpha^4. \tag{9}$$

Using the above expressions we can find the S/N resulting when an RC shaped signal is transmitted through a ΔM system. Figs. 5 and 6 show S/N versus $kf_s/f_o = kF_s$ for $\alpha = 0.25\, f_o$ and $\alpha = 0.068\, f_o$, respectively. These values for α were chosen because this causes the signal spectra to have roughly the same shape as the envelope of a *Picturephone* signal (using current standards) and a black and white entertainment TV signal (FCC standard), respectively. We assume here that the *Picturephone* signal would be bandlimited to 375 kc before encoding. Computer analysis of a composite video signal with current *Picturephone* standards indicates that the *Picturephone* spectrum has a corner at about

$$0.25 \times 375 \text{ kc}/2\pi \cong 15 \text{ kc}.$$

In the U.S., standard entertainment black and white TV is approximately bandlimited to 4.5 mc and has a corner at about 0.068×4.5 mc$/2\pi \cong 49$ kc.* These corner frequencies are slightly dependent on picture material. Although television signals have strong periodic components, the frequencies involved are so low that their periodic nature does not influence the S/N ratio. The spectrum of the quantizing noise, however, is affected by this periodicity, for, in general, periodic sampling of periodic signals produces quantizing noise which has periodic com-

* An unpublished study by W. N. Toy shows that this corner frequency may lie between 40 and 60 kc depending on picture material.

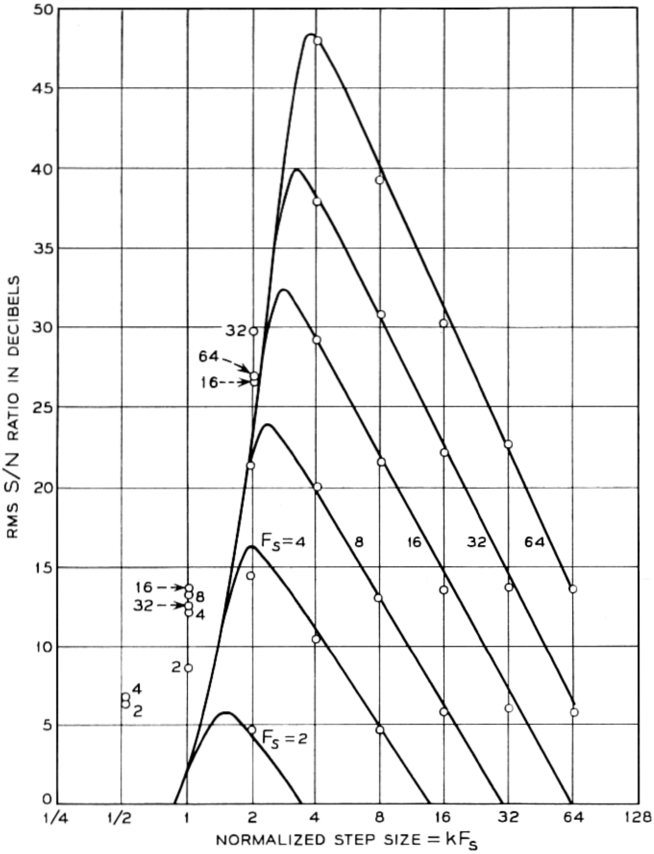

Fig. 5 — S/N for RC shaped Gaussian signals with $\alpha = 0.25 f_o$.

ponents. Although Figs. 5 and 6 apply specifically to Gaussian input signals, they give estimates of the performance of basic ΔM systems when used to transmit *Picturephone* signals and standard TV signals respectively. Fig. 11 which compares four of the curves in Fig. 5 with a computer simulation of an actual *Picturephone* visual telephone system using typical picture material shows that these estimates are good ones.

V. COMPARISON OF PCM AND ΔM FOR GAUSSIAN INPUTS

In Fig. 7, the S/N ratios for standard PCM and optimum basic ΔM systems with Gaussian input signals are compared. Bennett[1] showed that, under conditions usually encountered in practical systems, the rms

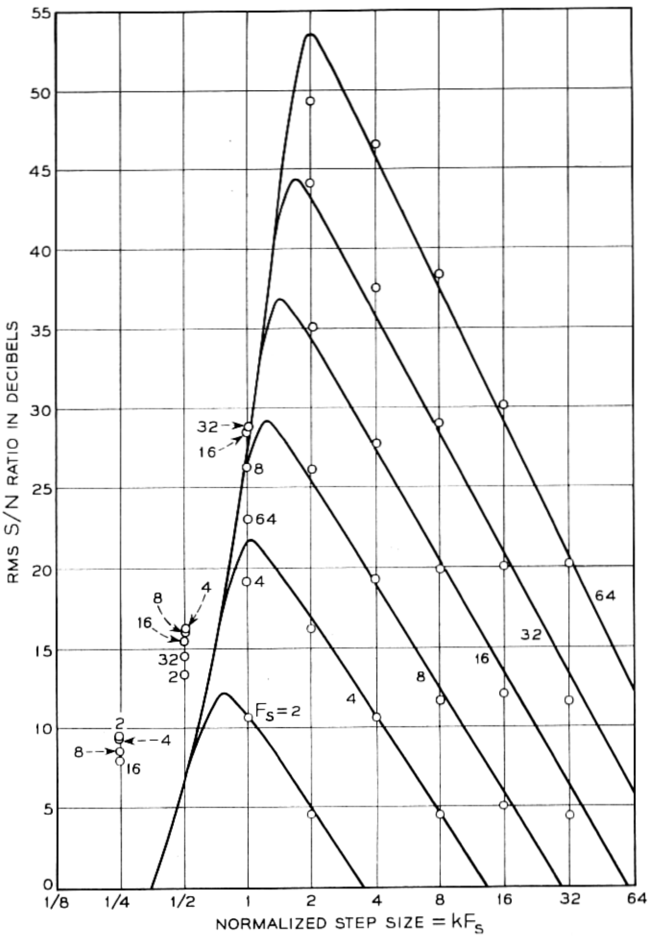

Fig. 6 — S/N for RC shaped Gaussian signals with $\alpha = 0.068 f_o$.

value of the quantizing noise for Nyquist rate sampled PCM is $E_o/\sqrt{12}$ where E_o is the step size. If n, the number of quantizing digits, is not too small, then E_o is equal to $E_p/2^n$, where E_p is the peak-to-peak value of the incoming signal. Therefore, the rms S/N ratio in db for PCM is

$$D = 20 \log \left(\frac{\sigma}{E_p} \sqrt{12} \right) + 20 \log 2^n,$$

where σ/E_p is the ratio of the signal's rms value to its peak value. If we encode only the values of the signal which lie between -4σ and

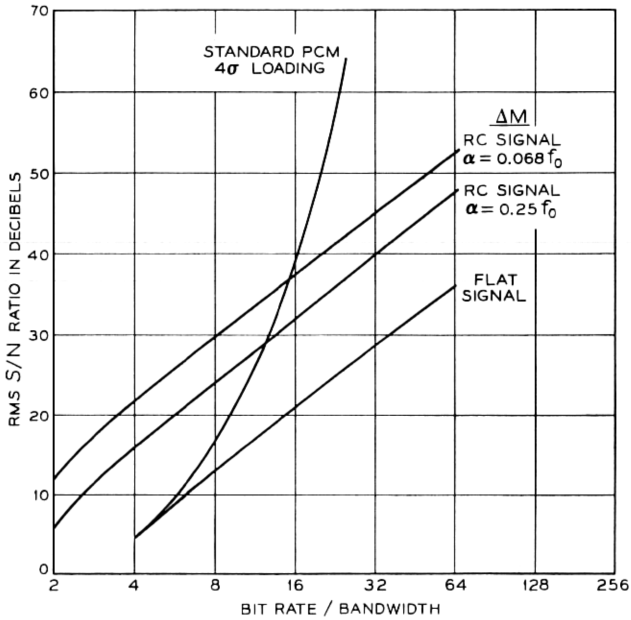

Fig. 7 — Comparison of ΔM and standard PCM for bandlimited Gaussian signals.

4σ and neglect values outside of this range then $E_p = 8\sigma$ and the formula for S/N ratio in db becomes

$$D \cong -7.3 + 6n.$$

This is called 4σ loading. Although 4σ loading is used for comparison in Fig. 7, this is not optimum loading. For PÇM, the loading which gives the highest value of S/N is dependent on the number of digits, n, used to encode each sample value. PCM with a fixed amount of loading (regardless of n) provides a realistic comparison with ΔM in systems, such as television, where no amplitude overload is allowed. For other applications, however, restricting the PCM system to a fixed amount of loading may not provide a valid comparison between PCM and ΔM.

Although the curve shown in Fig. 7 for PCM is continuous, it is actually defined only when the bit rate is an even multiple of the bandwidth because we have assumed that the sampling rate is always twice the bandwidth.

VI. COMPUTER SIMULATION WITH GAUSSIAN SIGNALS

The basic ΔM system of Fig. 1 was simulated on an IBM 7094 digital computer and the resulting S/N ratios are shown as points on Figs. 4, 5, and 6. Flat bandlimited signals sampled at the Nyquist rate were easily simulated by simply using independent random numbers with a Gaussian distribution and unit variance. For convenience, we assumed that $f_0 = 1$. A flat signal sampled at R times the Nyquist rate is simulated by filtering the random samples with a digital filter whose cutoff frequency is $1/R$. A discussion of sampled data filters is given in Ref. 7. Digital sharp cutoff filters, like their continuous counterparts, are not easily realized. The filters used all had the general shape given in Fig. 8. The filter shown is down 3 db at 0.25 and we call the bandwidth 0.25 for this reason. This filter would be used when simulating ΔM systems whose sampling frequency is 8 times the bandwidth (4 times the Nyquist rate). Other filters used had exactly the same shape when plotted on a log scale but, of course, had different 3-db points. These low-pass filters were nonrecursive filters which were simulated by finding a sequence of numbers which represented the filter's impulse response. Filtering was

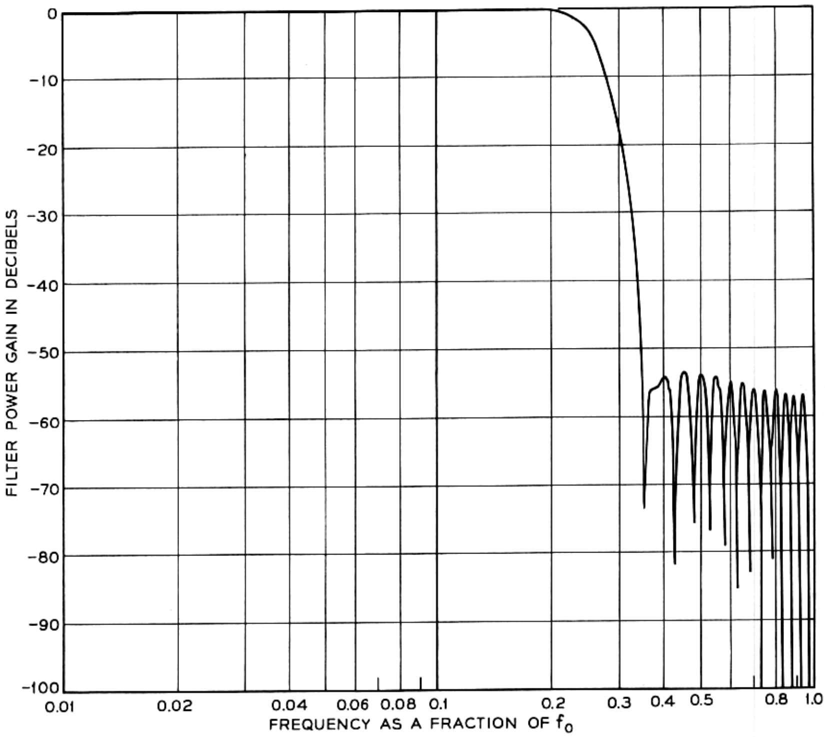

Fig. 8 — Power spectrum of low-pass filter with a bandwidth of $\frac{1}{4} f_o$ or $\frac{1}{8} f_s$.

accomplished by convolving the samples representing the signal with this sequence.

The RC shaped signals were simulated by passing the random samples representing flat noise through a digital simulation of a low-pass RC network.

VII. SIMULATION OF THE *PICTUREPHONE* SYSTEM

The basic ΔM system shown in Fig. 1 is impractical because errors introduced in the transmission medium will not decay out. To prevent the accumulation of errors from severely degrading the signal and to improve the operation of the system even when no errors are present, it is necessary to place a leak in the feedback loop represented by an amplifier with gain β less than unity. The ΔM system with leak added is shown in Fig. 9. This system was simulated on the computer and a signal similar to the *Picturephone* signal was encoded and decoded by it.

The input signal used was a television-like signal obtained by scanning a square slide with a slow-speed flying spot scanner, sampling the output, and encoding the samples into 11-bit PCM. The PCM samples were recorded on magnetic tape suitable as an input to the IBM 7094 digital computer.[8] The sampling frequency was synchronous with the line frequency. The slide used to produce the signal was a head and shoulders view of a girl similar to what might be expected on a *Picturephone* call. In this way, a signal was produced which represented a *Picturephone* signal. Current standards for the *Picturephone* system produce a 271-line picture with 2-to-1 interlace. The frame rate is 30 per second. The horizontal and vertical blanking pulses are 21 and 1000 μsec, respectively. The simulated signal was subjected to filtering so

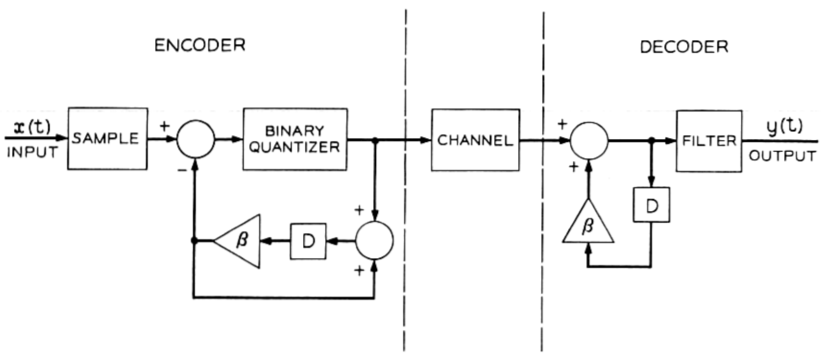

Fig. 9 — ΔM system with leak β.

that it was equivalent to a Picturephone signal which had been passed through a sharp cutoff filter with a bandwidth of 375 kc. The bandwidth then is considered to be 375 kc and sampling at R times the bandwidth means sampling at a rate of $R \times 375 \times 10^3$ samples per second.

The power spectrum of the composite signal (video and blanking pulses) on tape was found by computing the Fourier transform of its autocovariance function. The envelope of the power spectrum could be roughly approximated by the spectrum of (4) with an α of about 0.25. The spectrum, however, had the irregularities typical of a TV signal.

The slope overload produced by ΔM for smaller step sizes, occurs primarily during blanking intervals, and is not visible on the picture tube. For this reason, S/N ratios were found both for the composite signal and for the video part of the signal. In the region of interest, the S/N ratios for the video part of the signal are a few db above those for the composite signal.

Fig. 10 shows S/N ratios obtained from the simulation using the *Picturephone* signal on magnetic tape. The scale on the left of Fig. 10 gives rms composite signal to rms noise ratios. The scale on the right of Fig. 10 gives peak-to-peak composite signal to rms noise ratios because this is the quantity usually measured in television systems. The peak-to-peak composite signal to rms noise ratios are 11.6 db greater than the rms signal to rms noise ratios because the peak-to-peak value of the composite signal used is 11.6 db greater than its rms value. The relationship between three metrics associated with the input are as follows:

	relative value	db
peak-to-peak of composite signal	3.79	11.6
rms of composite signal	1	0
rms of video part of signal	0.512	−5.8

For each sampling rate in Fig. 10, there are two curves. In the solid curves the rms value of the noise in only the video part of the signal is used. The dotted curves use the rms value of the noise in the whole composite signal (video plus sync). Using only the noise in the video part leads to a better metric of picture quality because, as long as sync is maintained, noise in the sync pulses does not degrade the resulting television picture.

In Fig. 10 the actual bit rates are shown. For example, a bit rate of 6 megabits means sampling at 16 times the bandwidth since $16 \times 375 \times 10^3 = 6 \times 10^6$. The S/N ratios of the video signal varied only slightly with leak factor, β, as long as β was near 1. The values of β used were those which gave the best S/N ratios. These values are shown on Fig. 10

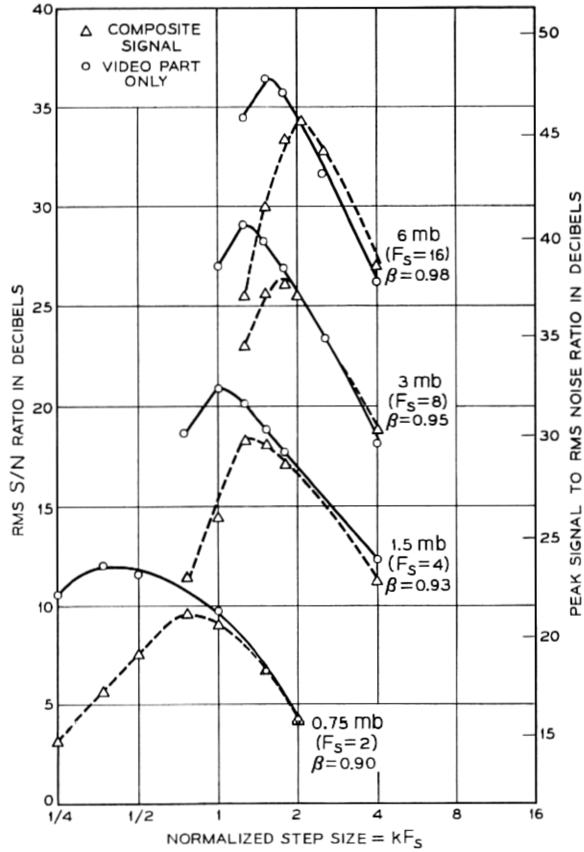

Fig. 10 — Simulated *Picturephone* systems. (Scale on the right assumes a signal whose peak-to-peak value is 11.6 db higher than its rms value.)

and also apply to the points on Fig. 11. It will be shown in a subsequent paper[9] how the optimum value of β is related to the covariance between adjacent sample values.

For the video, the optimum step sizes yielded peak-to-peak signal to rms noise ratios of 48.0, 40.8, 32.5, and 23.7 db for sampling rates of 16 f_o(6 mb), 8 f_o(3 mb), 4 f_o(1.5 mb), and 2 f_o(0.75 mb), respectively.

Fig. 11 was plotted to show the close agreement of the simulated *Picturephone* video results (with leak) to the theoretical results (without leak) for RC Gaussian signals with the same spectrum envelope as a *Picturephone* signal ($\alpha = 0.25 f_o$). The points in Fig. 11 are the same as those in Fig. 10 connected by the solid curves. They have been trans-

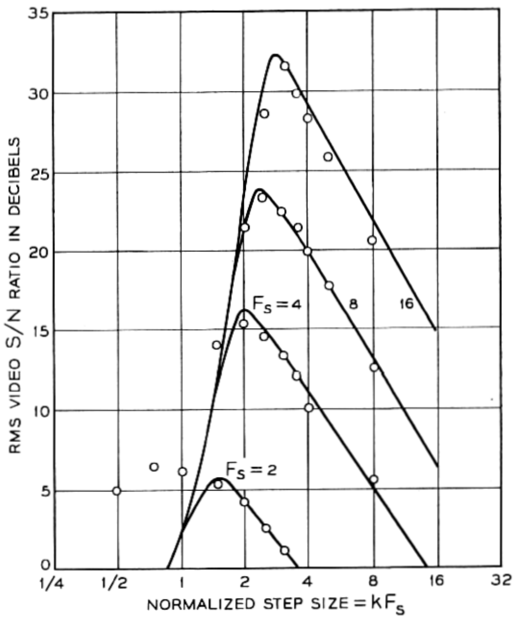

Fig. 11 — Comparison of simulated *Picturephone* video signals (points) to theoretical Gaussian results (solid curves) for $\alpha = 0.25 f_o$.

lated vertically by 5.8 db because the rms value of the video is 5.8 db less than the rms value of the composite signal, and they have been shifted to the right by a factor of 1/0.512 because the normalized step size is expressed in terms of the rms value of the signal. In previous figures, the rms value of the signal meant the rms value of the composite signal which is 1, but in Fig. 11 we are dealing with the rms value of the video which is 0.512.

In Fig. 12, the peak-to-peak composite signal to rms noise ratios for the simulation of ΔM are compared with those for standard PCM systems. For n-digit standard PCM, the ratio of peak-to-peak signal to rms noise in db is

$$D_{pp} = 20 \log \sqrt{12} + 20 \log 2^n$$
$$\approx 10.8 + 6n.$$

In PCM, about 3 db can be gained by assigning only one or two levels to the sync pulses and using the remaining available levels to encode the video part of the signal. This has not been taken into consideration in the above formula or in Fig. 12.

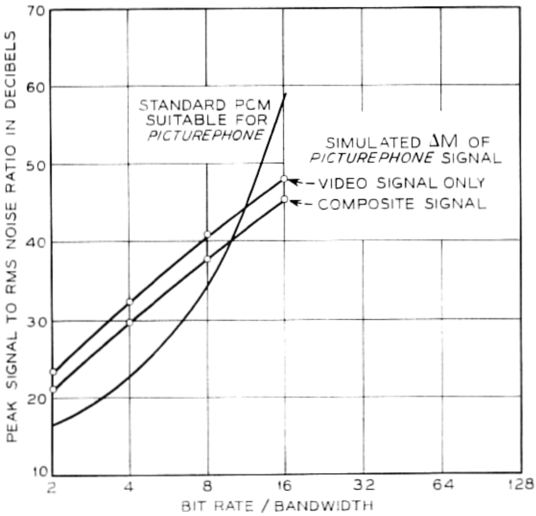

Fig. 12 — Comparison of simulated ΔM *Picturephone* system to standard PCM.

VIII. ESTIMATES OF S/N RATIOS FOR ΔM OF ENTERTAINMENT TELEVISION

The curves of Fig. 6 allow us to obtain a rough estimate of the S/N ratios possible for monochrome entertainment television systems (FCC standard). For the *Picturephone* signal used in the simulation, the rms video was 17.4 db below the peak-to-peak value of the composite signal. A comparison between *Picturephone* standards and entertainment television standards suggests that this is also a reasonable number for entertainment television. Moreover, since this number is dependent on picture material, we can rest comfortably in the knowledge that we can surely find a picture whose rms video is 17.4 db below its peak-to-peak value. It should be possible to transmit such a picture with peak-to-peak composite signal to video rms noise ratios of about 54, 46, 38, and 29 db at bit rates of 16, 8, 4, and 2 times the bandwidth, respectively. These ratios were found by adding 17.4 db to the peak values of the curves in Fig. 6.

IX. INTERPRETATION OF S/N RATIOS FOR TELEVISION SYSTEMS

It is well known that the quality of a television picture cannot be judged by its S/N ratio alone. In fact, such a judgment may be quite misleading. The quality of a picture contaminated with additive flat Gaussian noise may be quite different from that of the same picture contaminated by ΔM quantizing noise even if the S/N ratios are identical.

Quantizing noise in standard PCM exhibits itself in television pictures as contouring. If the number of digits used is high enough, typically 7 or 8 for monochrome entertainment television, then the eye cannot detect contouring and no degradation due to quantizing noise is noticeable. In ΔM, quantizing noise in television signals exhibits itself subjectively in four ways — as grainy noise, slope overload, contouring, and edge busyness. The fact that the noise is present in four subjectively different forms may be a significant advantage for ΔM. An observer normally judges the quality of a television picture by the most severe degradation that he sees rather than by jointly considering several different types of degradation. ΔM takes advantage of this. In a subjectively optimum ΔM system the step size, sampling rate, and filter characteristics are chosen to spread the degradation into these four categories to produce the best subjective picture. In earlier computations, we considered all noise to be either granular noise N_g or overload noise N_o. In this section, we have further divided granular noise into three different categories — grainy noise, contouring, and edge busyness—because this N_g exhibits itself as three subjectively distinguishable phenomena. Granular quantizing noise tends to have a rectangular amplitude distribution and is likely to be less annoying than an equivalent amount of additive noise with a Gaussian amplitude distribution.

Slope overload noise will appear subjectively like a loss of bandwidth since the rise time of white to black transitions is increased. The snow or salt and pepper effect so ubiquitous in TV systems is what we are calling grainy noise. Contouring occurs since there are only a finite number of steady levels or flat tones which can be established by a ΔM system (if the leak β is unity the number of levels is one less than the peak-to-peak value of the signal divided by the step size). Edge busyness exhibits itself as a busyness at vertical edges, i.e., vertical white to black transitions, when the sampling rate and line rate are not synchronized. Due to the lack of synchronization and the finite time between samples these transitions do not occur in a straight line on successive lines and frames.

X. MULTI-LEVEL QUANTIZING

If the quantizer shown in Fig. 1 is not a 2-level device, then the formula for N_o in (3) must be modified. If we define an n-digit uniform quantizer to be one which quantizes its input into the nearest one of the

2^n levels $\pm k$, $\pm 3k$, $\cdots$, $\pm(2^n - 1)k$, then overload noise does not occur until $|x'(t)|$ becomes greater than $(2^n - 1)kf_s$. To modify N_o in (3) then to accommodate a uniform n-level quantizer we simply replace k by $(2^n - 1)k$. The expression for N_g does not need to be modified.

XI. SUMMARY AND CONCLUSIONS

An analytical solution for the S/N ratio as a function of step size and bit rate is presented for a signal with an arbitrary spectrum and Gaussian amplitude distribution. The correctness of the solution has been demonstrated by computer simulation of some delta modulation systems. Computer simulations with low bandwidth television signals have shown that the formula is accurate for actual video signals whose amplitude distribution is not necessarily Gaussian. These analytical results will allow us to predict what values of S/N ratio may be obtained at various bit rates for many types of signals.

Some computer simulations demonstrated that properly designed ΔM systems can transmit 375-kc *Picturephone* signals with peak signal to rms noise ratios as high as 24, 32, 40, and 48 db for bit rates of 2, 4, 8, and 16 times the bandwidth, respectively. Estimates of the S/N ratios possible for 4.5-mc black and white entertainment television are about 29, 38, 46, and 54 db for the above bit rates, respectively. These S/N ratios are somewhat dependent on picture material. Although S/N ratios are an important indicator of system performance, final judgments about a system's ability to transmit television pictures must be based on subjective viewing tests.

The quantizing noise in television ΔM systems distributes itself into 4 visually distinguishable categories. This suggests that ΔM quantizing noise may be subjectively less objectionable than an equivalent amount of noise introduced by other means.

Any comparison between the relative performance of ΔM and standard PCM depends on the characteristics of the signal to be transmitted. Generally, however, for signals whose spectra decrease with frequency, ΔM is superior by the S/N criterion to standard PCM when low quality reproduction of the signal is allowed. For high-quality systems, standard PCM gives better results than ΔM. This is due to the fact that in ΔM doubling the bit rate gives only about an 8-db gain in S/N ratio. In standard PCM, increasing the number of bits per sample by 1 increases the S/N ratio by 6 db. The relative performance of ΔM and standard PCM for two types of signals is illustrated in Figs. 7 and 12.

XII. ACKNOWLEDGMENTS

The author is pleased to acknowledge the help of S. O. Rice. Not only is his Appendix A a fundamental part of this paper, he also made helpful contributions in other parts of the text. This ΔM problem was originally suggested by T. V. Crater and discussions with him were invaluable. Also contributing to the preparation of this article were J. E. Abate, W. H. Jules, J. R. Gray, and J. H. Davis.

APPENDIX A*

Noise Burst, Steep-Slope Approximation to Δ-Modulation Noise

When the step size is not too large, most of the output noise in a delta modulation system is produced in bursts. The bursts occur when the time derivative of the input exceeds the maximum slope permitted by the system. An approximation for the average value of the power in this type of noise is given here. It turns out that the analysis is quite similar to that used to study the distribution of fade lengths in radio transmission (see (1) of Ref. 10).

The material given here was worked out jointly by Mr. O'Neal and myself. The concepts and approximations associated with the delta modulation system are due to him. My part consisted chiefly in supplying the steps suggested by the analogy with the fading problem.

The problem is the following. Let the input voltage be a Gaussian process $x(t)$ with (one-sided) power spectrum $F(f)$. Let the output of the system be $y(t)$. Most of the time $y(t)$ will be equal to $x(t)$. During intervals of this sort, the absolute value of $x'(t) = dx(t)/dt$ is less than some given positive quantity x_o'. The intervals during which $y(t) \neq x(t)$ start at the instants $|x'(t)|$ increases through the value x_o'. Let t_o denote the starting time of a particular interval of this type and suppose, for convenience, that $x'(t_o)$ is positive so that $x'(t_o) = x_o'$. Throughout the interval $y(t)$ is defined to be $x(t_o) + (t - t_o)x_o'$, and increases linearly with time. The interval lasts as long as $x(t)$ exceeds $y(t)$. It ends at time t_1 when $x(t)$ and $y(t)$ again become equal;

$$x(t_1) = y(t_1) = x(t_o) + (t_1 - t_o)x_o'. \tag{10}$$

For intervals starting with $x_o'(t_o) = -x_o'$, $y(t)$ is defined similarly by $y(t) = x(t_o) - (t - t_o)x_o'$.

The output noise is defined to be

$$n(t) = x(t) - y(t). \tag{11}$$

* This appendix was written by S. O. Rice.

Since $n(t)$ will be zero much of the time, the noise tends to occur in short bursts. The average noise power is

$$\overline{n^2(t)} = \lim_{T\to\infty} \frac{1}{T} \int_0^T n^2(t)\, di. \tag{12}$$

We seek an approximation for $\overline{n^2(t)}$ which holds when x_o' is large compared to the rms value of $x'(t)$.

When x_o' is large, the interval $t_1 - t_o$ tends to be small and an expression for its length may be obtained by considering the expansion of $x(t)$ about t_o. Here, and in much of the following, we take $x'(t_o)$ to be positive:

$$\begin{aligned} x(t) &= x(t_o) + (t - t_o)x_o' + \frac{(t - t_o)^2}{2!} x''(t_o) \\ &\quad + \frac{(t - t_o)^3}{3!} x'''(t_o) + \cdots \end{aligned} \tag{13}$$

$$0 = \frac{(t_1 - t_o)^2}{2!}\left(x''(t_o) + \frac{t_1 - t_o}{3} x'''(t_o) \cdots\right) \tag{14}$$

$$t_1 - t_o = -\frac{3x''(t_o)}{x'''(t_o)} + \cdots. \tag{15}$$

In going from the first equation to the second we have set $t = t_1$ and used the end-of-interval relation (10). When x_o' is large we can expect $x'''(t_o)$ to be large and negative while $x''(t_o)$ remains of order 1. This is shown in the following paragraph.

To begin with, consider the special case $x = \sin \omega t$ with t near zero. Then x and its time derivatives are such that $x \approx 0$, $x' \approx \omega$, $x'' \approx 0$, $x''' \approx -\omega^3$, etc. Our Gaussian $x(t)$ and its derivatives behave in somewhat the same way when $x'(t)$ is large. Let $\xi = x'(t)$, $\eta = x''(t)$, $\zeta = x'''(t)$ denote the first three derivatives at time t of an $x(t)$ chosen at random from the ensemble of $x(t)$'s. It may be shown that their joint probability density is

$$\frac{(2\pi)^{-3/2}}{\sqrt{b_2 B}} \exp\left[-\frac{\xi^2}{2b_o} - \frac{\eta^2}{2b_2} - \frac{(\zeta + b_2 b_o^{-1}\xi)^2}{2Bb_o^{-1}}\right], \tag{16}$$

where b_o and b_2 are the respective variances of $x'(t)$ and $x''(t)$, and

$$\begin{aligned} b_n &= \int_0^\infty (2\pi f)^{n+2} F(f)\, df, \qquad n = 0, 2, 4 \\ B &= b_o b_4 - b_2^2 \end{aligned} \tag{17}$$

where $F(f)$ is the power spectrum of $x(t)$. When $x'(t)$ is held fixed at the large value x_o', (16) shows that the average value of $x'''(t)$ is $-b_2x_o'/b_o$. If x_o' is so large that this average value is large compared to the standard deviation $\sqrt{B/b_o}$, we are led to approximate $x'''(t)$ in expression (15) in the interval $t_1 - t_o$ by its average value. Making the further approximation of neglecting the higher order terms in (15) gives

$$t_1 - t_o \approx \frac{3b_o x''(t_o)}{b_2 x_o'} \tag{18}$$

for the length of the interval during which $x(t)$ exceeds the output $y(t)$.

The noise energy in the burst corresponding to the interval $t_1 - t_o$ is

$$\begin{aligned} \int_{t_o}^{t_1} n^2(t)\,dt &= \int_{t_o}^{t_1} [x(t) - y(t)]^2\,dt \\ &= \int_{t_o}^{t_1} [x(t) - x(t_o) - (t - t_o)x_o']^2\,dt \\ &\approx \int_{t_o}^{t_1} \left[\frac{(t - t_o)^2}{2!} x''(t_o) + \frac{(t - t_o)^3}{3!} x'''(t_o)\right]^2 dt \\ &\approx \frac{81}{140}\left(\frac{b_o}{b_2 x_o'}\right)^5 [x''(t_o)]^7. \end{aligned} \tag{19}$$

In going to the last line, the approximation (18) for $t_1 - t_o$ is used and $x'''(t_o)$ is replaced by $-b_2x_o'/b_o$. It is helpful to make the change of variable $t - t_o = (t_1 - t_o)z$.

Expression (19) gives the noise energy in a burst for which the initial value of $x''(t)$ is equal to $x''(t_o)$. As we go from burst to burst, $x''(t_o)$ will fluctuate but it will always be positive for the type of interval ($x'(t_o) = x_o' > 0$) we are considering. This is because $x'(t)$ is increasing as it passes upward through the value x_o' at time t_o.

The next step is to find the average noise energy in a burst. When a member $x(t)$ is picked at random from the ensemble of $x(t)$'s, the chance that $x'(t)$ will increase through the value x_o' during t_o, $t_o + dt$ with slope between x'' and $x'' + dx''$ is $x''p(x_o', x'')dx''\,dt$, where

$$p(\xi, \eta) = \frac{(2\pi)^{-1}}{\sqrt{b_o b_2}} \exp\left[-\frac{\xi^2}{2b_o} - \frac{\eta^2}{2b_2}\right]$$

is the joint probability density of $x'(t)$ and $x''(t)$ (see Section 3.5 of Ref. 11). When the ensemble of $x(t)$'s consists of an extremely large number M of members, the number which have the above behavior is approximately $(M\,dt)x''p(x_o', x'')dx''$. The total number M_o which pass upward through x_o' during t_o, $t_o + dt$ is obtained by integrating from $x'' = 0$ to $x'' = \infty$:

$$M_o = \frac{(M\ dt)}{2\pi}\left(\frac{b_2}{b_o}\right)^{\frac{1}{2}} \exp\left(-\frac{x_o'^2}{2b_o}\right).$$

Since the noise energy in a burst is proportional to x''^7 we average it over the M_o members:

$$\begin{aligned} \text{ave } x''^7 &= \frac{1}{M_o}\int_0^\infty x''^7(M\ dt)[x''p(x_o',x'')\ dx''] \\ &= (2b_2)^{7/2}\Gamma(9/2). \end{aligned}$$

Combining this with (19) gives

$$\text{ave} \int_{t_o}^{t_1} n^2(t)\ dt \approx \frac{\sqrt{2\pi}}{8}\left(\frac{3b_o}{x_o'}\right)^5 b_2^{-3/2} \tag{20}$$

for the average noise energy in a burst.

Our final assumption is that a noise burst occurs every time $x'(t)$ increases through x_o' or decreases through $-x_o'$. The true number of bursts tends to be less than this because such crossings can occur when a burst is in progress. From noise theory,[11] the expected number of such crossings in one second is

$$2\,\frac{1}{2\pi}\left(\frac{b_2}{b_o}\right)^{\frac{1}{2}} \exp\left(-\frac{x_o'^2}{2b_0}\right). \tag{21}$$

Multiplying by the average burst energy (20) then gives

$$\overline{n^2(t)} \approx \frac{1}{4\sqrt{2\pi}}\left(\frac{b_o^2}{b_2}\right)\left(\frac{3b_o^{\frac{1}{2}}}{x_o'}\right)^5 \exp\left(-\frac{x_o'^2}{2b_o}\right) \tag{22}$$

which is the approximation sought for the average noise power.

The right-hand side of (22) has the dimension of $(\text{volt})^2$ since the dimensions of x_o', b_o, b_2, are volt/sec, $(\text{volt/sec})^2$, $(\text{volt/sec}^2)^2$, respectively.

APPENDIX B

Computation of $\psi(\tau)$ for a Bandlimited RC Signal

By a bandlimited RC signal we mean a signal which will result if white noise is passed through a low-pass RC filter and then bandlimited to the interval $(0,f_0)$. The one-sided power spectrum of such a signal with unit variance is

$$F(f) = \frac{2\pi}{\alpha \tan^{-1}\left(\dfrac{2\pi f_o}{\alpha}\right)} \times \frac{\alpha^2}{(2\pi f)^2 + \alpha^2} \qquad 0 < f < f_o$$

$$F(f) = 0 \qquad f \geqq f_o.$$

where $\alpha = 1/RC$.

The autocovariance function of a bandlimited RC signal with unit variance is

$$\psi(\tau) = \int_0^\infty F(f) \cos 2\pi f\tau \, df$$

$$\psi(\tau) = K\left[\int_0^\infty \frac{\alpha^2}{(2\pi f)^2 + \alpha^2} \cos 2\pi f\tau \, df - \int_{f_o}^\infty \frac{\alpha^2}{(2\pi f)^2 + \alpha^2} \cos 2\pi f\tau \, df\right], \tag{23}$$

where

$$K = \frac{2\pi}{\alpha \tan^{-1}(2\pi f_o/\alpha)}.$$

The first integral can be evaluated by contour integration and is found to be $(\alpha/4) \exp(-\alpha \mid \tau \mid)$. If α is small enough then the second integral can be approximated by

$$\int_{f_o}^\infty \frac{\alpha^2}{(2\pi f)^2 + \alpha^2} \cos 2\pi f\tau \, df \cong \alpha^2 \int_{f_o}^\infty \frac{1}{(2\pi f)^2} \cos 2\pi f\tau \, df \cong \frac{\alpha^2 \mid \tau \mid}{2\pi}\left[\frac{\cos 2\pi f_o \mid \tau \mid}{2\pi f_o \mid \tau \mid} + \text{Si}\,(2\pi f_o \mid \tau \mid) - \frac{\pi}{2}\right]. \tag{24}$$

Therefore,

$$\psi(\tau) \cong K\left[\frac{\alpha}{4} e^{-\alpha|\tau|} - \frac{\alpha^2 \mid \tau \mid}{2\pi}\left(\frac{\cos 2\pi f_o \mid \tau \mid}{2\pi f_o \mid \tau \mid} + \text{Si}\,(2\pi f_o \mid \tau \mid) - \frac{\pi}{2}\right)\right],$$

where Si is the sine integral function. The above approximation of $\psi(\tau)$ does not have $\psi(0) = 1$ which is the necessary requirement of unit variance. To restore this requirement we divide the above expression by $\psi(0)$. Therefore, the autocovariance function for a bandlimited RC signal with unit variance may be approximated by

$$\psi(\tau) \cong \frac{\frac{\pi}{2} e^{-\alpha|\tau|} - \alpha \mid \tau \mid \left(\frac{\cos 2\pi f_o \mid \tau \mid}{2\pi f_o \mid \tau \mid} + \text{Si}\,(2\pi f_o \mid \tau \mid) - \frac{\pi}{2}\right)}{\frac{\pi}{2} - \frac{\alpha}{2\pi f_o}}.$$

Although the above approximation is convenient for computational purposes, an exact expression for $\psi(\tau)$ can be found. Instead of approximating the second integral on the right-hand side of (23), it can be integrated exactly giving

$$\frac{\alpha}{4\pi} \operatorname{Im} [e^{-\alpha\tau} E_1(-\alpha\tau - i2\pi f_o\tau) - e^{\alpha\tau} E_1(\alpha\tau - i2\pi f_o\tau)],$$

where E_1 is the exponential integral. An alternate way of arriving at the approximation in (24) is to expand the E_1 functions in the above expression about $-i2\pi f_o\tau$ and take the limit as α becomes small.

REFERENCES

1. Bennett, W. R., Spectra of Quantized Signals, B.S.T.J., *27*, July, 1948, pp. 446–472.
2. de Jager, F., Delta Modulation, A method of PCM Transmission Using a 1-Unit Code, Philips Res. Rep. *7*, 1952, pp. 442–466.
3. Van de Weg, H., Quantizing Noise of a Single Integration Delta Modulation System with an N-Digit Code, Philips Res. Rep. *8*, 1953, pp. 367–385.
4. Zetterberg, L., A Comparison Between Delta and Pulse Code Modulation, Ericsson Technics, *2*, No. *1*, 1955, pp. 95–154.
5. Cutler, C. C., Differential Quantization of Communication Signals, Patent No. 2,605,361, July 29, 1952.
6. Graham, R. E., Predictive Quantizing of Television Signals, IRE Wescon Convention Record, Part 4, 1958, pp. 147–157.
7. Kaiser, J. F., Design Methods for Sampled Data Filters, Proceedings of the First Allerton Conference on Circuit and System Theory, J. B. Cruz, editor, U. of Illinois Press, Urbana, Ill., 1963.
8. Graham, R. E. and Kelly, J. L., Jr., A Computer Simulation Chain for Research on Picture Coding, IRE Wescon Convention Record, Part 4, Aug., 1958, pp. 41–46.
9. O'Neal, J. B., Jr., Predictive Quantizing Systems (Differential Pulse Code Modulation) for Television Signals, to be published.
10. Rice, S. O., Distribution of Fades in Radio Transmission, B.S.T.J., *37*, May, 1958, pp. 581–635.
11. Rice, S. O., Mathematical Analysis of Random Noise, B.S.T.J., *23*, 1944, pp. 282–332; *24*, 1945, pp. 46–156.