Mutual Synchronization of Geographically Separated Oscillators

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A control scheme for synchronizing the frequencies of geographically separated oscillators connected by communication links consists of averaging the phases received at each station from remote oscillators, comparing the result with the local phase, and applying the filtered error signal as a correction to the local oscillator frequency. The system was studied by V. E. Beneš who found a sufficient condition for the stability of the system using advanced mathematical techniques. In this paper, the stability condition is derived (for a slightly more general control scheme) using only the transfer function concept of linear systems and some properties of determinants. A practical difficulty regarding the final frequency of the oscillators is discussed and a modification of the control scheme is shown to alleviate the difficulty. Also examined are the questions of sensitivity to parameter changes, the effect of jitter noise on the performance of the system, and the effect of failure of an oscillator or transmission link.

I. INTRODUCTION

Consider a network of N geographically separated stations that are connected by directed communication links. A local clock, or oscillator, is situated at each station. The problem of synchronizing the frequencies of the oscillators is of considerable practical interest for continental pulse code modulation (PCM) systems.

The local oscillators have frequencies which may be altered in proportion to a control signal. In the absence of external control, each oscillator operates at a different frequency. The network is "connected" in the sense that from any station to any other station there is either a direct transmission link or an indirect path via one or more intermediate stations. A fixed time delay is associated with each transmission link.

In an important but unpublished paper, V. E. Beneš[1] has examined a linear control scheme in which each station receives the phases of neighboring stations, i.e., those stations connected to it by direct transmission links. The phases are averaged and compared with the local phase; the error is filtered and applied as a correction to the frequency of the oscillator. Similar schemes were also proposed by Runyon.[2] Beneš has proved that under suitable conditions the system is stable, i.e., the oscillators asymptotically settle to a common frequency and the phase differences have finite asymptotic values. He also finds explicit formulae for the final frequency and asymptotic phase differences. To obtain these results, he resorted to the mathematical techniques of renewal theory and Tauberian theory. By assuming the stability of the system, as proved by Beneš, A. J. Goldstein[3] has rederived the expressions for final frequency and phase differences in a more direct manner. Bonomi, La Marche, and Varaiya[4] improved the treatment of the stability problem and suggested some avenues of approach for the study of transient response. In each case, the authors relied on the mathematical theory of Markoff chains and stochastic matrices.

M. Karnaugh[5] has formulated a more realistic and more sophisticated nonlinear control model. Broad stability conditions for this model are not yet known; however, certain special cases resemble the Beneš model.

In this paper, the stability conditions and the expression for final frequency for a slightly more general version of the Beneš model are derived in a simple manner using only the transfer function concept of linear systems and elementary properties of determinants. This approach permits a clearer intuitive understanding and should be readily comprehensible to the non-mathematician. The sensitivity of the system to parameter changes is also examined and certain questions regarding the final frequency of the oscillators are clarified.

In Section II we give a formulation of the problem and obtain the basic equations describing the system. In Section III certain crucial properties of the matrix of averaging coefficients are derived which result from the topological constraint that the network is connected. Stability is proved in Section IV and an expression for the final frequency is obtained. Section V considers some practical questions with regard to how the final frequency is related to the free-running frequencies of the oscillators. Section VI examines the questions of sensitivity to parameter changes, the effect of failure of an oscillator or transmission link, and the effect of jitter noise.

II. FORMULATION

Let f_i be the frequency of the ith oscillator in the absence of external control, and $r_i(t)$ the control signal applied to the ith oscillator at time t.

If $p_i(t)$ denotes the total cyclical phase of the ith oscillator, then the actual frequency at time t is given by

$$\dot{p}_i(t) = f_i + r_i(t) \tag{1}$$

where the dot denotes the time derivative.

The control scheme at the ith oscillator is shown in Fig. 1. The phases of all neighboring stations are transmitted to the ith station. The transmission delay associated with the path from station j to station i is denoted as τ_{ij}. Each phase received at station i is compared with the local phase; the differences are weighted with the nonnegative averaging coefficients a_{ij} and summed. The weighted sum of phase

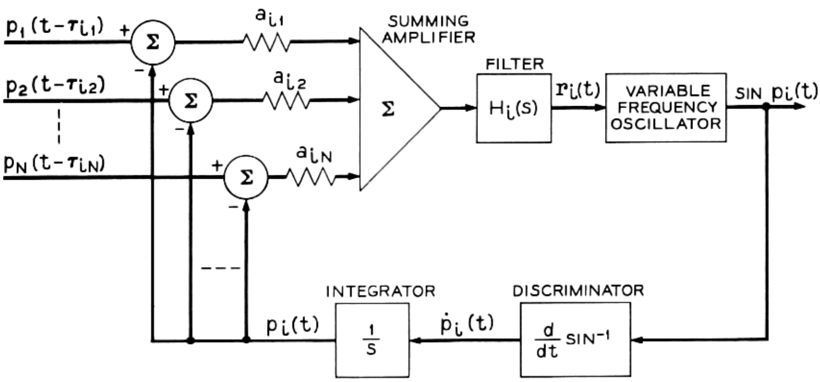

Fig. 1 — Station i of the phase averaging control system.

differences is applied to the filter with transfer function $H_i(s)$, and the filter output is the frequency correction term $r_i(t)$. Thus, we have

$$r_i(t) = h_i(t) * \sum_{j=1}^{N} a_{ij}[p_j(t - \tau_{ij}) - p_i(t)], \tag{2}$$

where $h_i(t)$ is the impulse response of the filter at the ith station and the asterisk denotes convolution.

We assume the filters have three simple properties: (i) causality, i.e., the response at any instant does not depend on the future of the input, (ii) stability, in the sense that a bounded input always produces a bounded output, and (iii) positive dc gain, i.e.,

$$H_i(0) \equiv \lambda_i > 0. \tag{3}$$

Without loss in generality we may assume that the averaging coefficients sum to unity, i.e.,

$$\sum_{j=1}^{N} a_{ij} = 1. \tag{4}$$

Clearly a scaling of all the coefficients a_{ij} for fixed i is equivalent to a change in gain factor of the ith filter.

If there is no direct transmission path from the jth to the ith station, then the coefficient a_{ij} is presumed to be zero. Thus, the $N \times N$ matrix, A, whose ijth element is a_{ij}, contains all the topological information about the network of communication links. In order that mutual synchronization be possible, it is certainly necessary that the network be connected so that from any station to any other station there is either a direct or indirect transmission path. The resulting properties of the averaging matrix A imposed by this topological constraint play a vital role in the proof of stability for the system.

In agreement with the Beneš model we consider the starting conditions where the oscillators are assumed to have been free-running for an indefinitely long time prior to $t = 0$, and at $t = 0$ the control signals r_i are connected to the oscillators. Thus, we have

$$p_i(t) = f_i t + p_i(0), \qquad t < 0 \tag{5}$$

where $p_i(0)$ is the phase at $t = 0$, and from (1), (2), and (4), the frequency of the ith oscillator when the controls are operating is

$$\dot{p}_i(t) = h_i(t) * \sum_j a_{ij}[p_j(t - \tau_{ij}) - p_i(t)] + f_i , \qquad t \geqq 0. \tag{6}$$

Equations (5) and (6) for $i = 1, 2, \cdots, N$ completely characterize the behavior of the system under the particular starting conditions of interest. Taking the ordinary Laplace transform of (6), we obtain

$$sP_i = H_i \sum_j \hat{a}_{ij} P_j - H_i P_i + \frac{1}{s} f_i + p_i(0) + Q_i , \tag{7}$$

where

$$\hat{a}_{ij} = a_{ij} \exp(-s\tau_{ij}),$$

$$Q_i(s) = H_i(s) \sum_j \hat{a}_{ij} \int_{-\tau_{ij}}^{0} p_j(t) \exp(-st)\, dt,$$

and $P_i(s)$ is the Laplace transform of $p_i(t)$. The term $Q_i(s)$ is the contribution to the ith oscillator frequency after $t = 0$ due to the contents of the transmission links at $t = 0$. Using (5), $Q_i(s)$ can be evaluated explicitly, but for our purposes it is sufficient to note that

$$sQ_i(s) \to 0 \quad \text{as} \quad s \to 0. \tag{8}$$

The transformed equations (7) can, in principle, be solved for the phases $p_i(t)$ for $t \geqq 0$. The desired stability information can be obtained directly from these equations. We shall, however, obtain this information in a somewhat indirect but more profitable way by defining an associated linear time-invariant system with N inputs and N outputs.

Consider the same control arrangement for the N interconnected oscillators. Instead of the former starting conditions, suppose the control paths have always been connected and that each oscillator can be activated by an arbitrary frequency "input" as shown in Fig. 2. Then the actual frequency of the ith oscillator at time t is the sum of the basic frequency input $v_i(t)$ and the correction component $r_i(t)$ leaving the filter. The phases $p_i(t)$ are considered the "outputs" of the linear system. When $v_i(t) \equiv 0$ for each i, the system is at rest and all outputs $p_i(t)$ are identically zero.

The importance of the associated linear system is that any desired starting conditions in the physical model can be treated by an equivalent set of inputs to the linear system. To clarify this, note that the system of Fig. 2 is characterized by the equations

$$\dot{p}_i(t) = h_i(t) * \sum_j a_{ij}[p_j(t - \tau_{ij}) - p_i(t)] + v_i(t), -\infty < t < \infty. \quad (9)$$

Formally taking the exponential (two-sided Laplace) transform of (9) we obtain

$$sP_i = H_i \sum_j \hat{a}_{ij} P_j - H_i P_i + V_i, \quad (10)$$

where $P_i(s)$ and $V_i(s)$ are, respectively, the exponential transforms of $p_i(t)$ and $v_i(t)$. Equation (10) implicitly characterize the associated linear system whose inputs are $v_i(t)$ and outputs $p_i(t)$ as long as $v_i(t)$ has an exponential transform. Comparing (7) and (10) we see that the

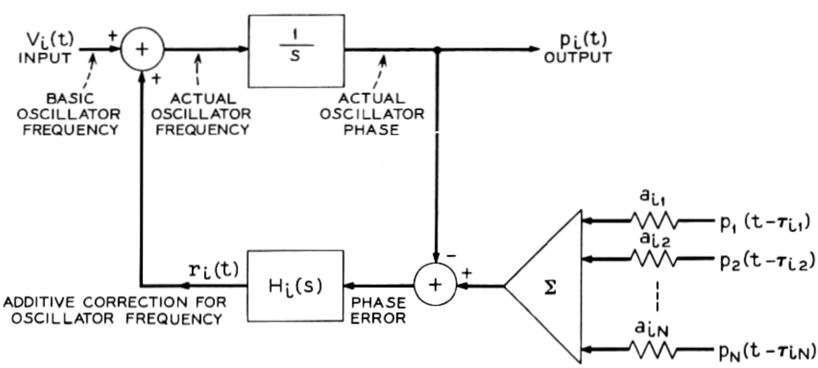

Fig. 2 — Model of station i of the associated linear system.

phase responses of the physical model with the corresponding starting conditions will be the outputs of the associated linear system for $t \geqq 0$ if we select the inputs to be

$$V_i(s) = \frac{1}{s} f_i + p_i(0) + Q_i(s). \tag{11}$$

In the time domain, these inputs are

$$v_i(t) = f_i u(t) + p_i(0)\delta(t) + q_i(t), \tag{12}$$

where $q_i(t)$ is the response of the filter $H_i(s)$ to a time-limited input which begins at time $t = -\max \tau_{ij}$ and ending at $t = 0$, $\delta(t)$ is the unit impulse function and $u(t)$ is the unit step function. From (8) and the final value theorem it follows that $q_i(t) \to 0$ as $t \to \infty$. It is important to note that the phase responses to the inputs (12) will be the same as the phase responses of the physical model only for $t \geqq 0$. For $t < 0$ the responses of the associated linear system do not correspond to the physical model.

Equation (10) may be expressed in the form

$$P_i = \beta_i(s) \sum_{j=1}^{N} \hat{a}_{ij} P_j + \left(\frac{1}{s + H_i}\right) V_i, \tag{13}$$

where

$$\beta_i(s) = \frac{H_i(s)}{s + H_i(s)}. \tag{14}$$

The simplified model of the linear system, corresponding to (13), is shown in Fig. 3 where $\beta_i(s)$ is the transfer function of the feedback configuration as shown. Thus, the operation of the ith station is to average the incoming phases, apply the average to the filter $\beta_i(s)$,

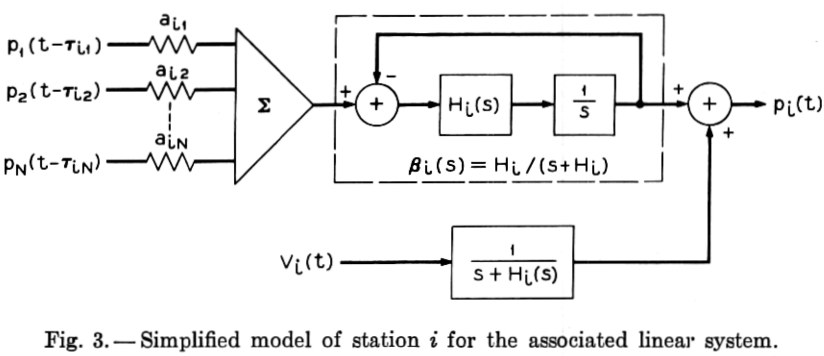

Fig. 3. — Simplified model of station i for the associated linear system.

and finally add a filtered input component to produce the phase response $p_i(t)$. We shall see in Section IV that the condition for stability of the system is simply that the filter $\beta_i(s)$ have a gain less than unity for sinusoidal inputs.

Equations (13) for $i = 1, 2, \cdots, N$ can be formally solved for the phase responses with the help of matrix notation. Let $B(s)$ be the $N \times N$ matrix whose ijth component is

$$b_{ij}(s) = \beta_i(s)\, \hat{a}_{ij}$$

and let $C(s)$ be the diagonal $N \times N$ matrix whose iith element is

$$c_{ii}(s) = \frac{1}{s + H_i(s)}.$$

Note that for $s = 0$ we have $c_{ii}(0) = 1/\lambda_i$ and $B(0) = A$, where λ_i is the dc gain of $H_i(s)$ and A is the averaging matrix, both defined earlier. Let $P(s)$ and $V(s)$ be the N component column matrices whose ith elements are, respectively, $P_i(s)$ and $V_i(s)$. Then (13) becomes

$$[I - B]\, P = CV \tag{15}$$

or

$$P = KV, \tag{16}$$

where

$$K(s) = [I - B(s)]^{-1}\, C(s) \tag{17}$$

is the matrix transfer function of the linear system. Thus, each element $K_{ij}(s)$ of $K(s)$ is the scalar transfer function relating the output $p_i(t)$ to the input $v_j(t)$ when all other inputs are zero. In Section IV we shall determine certain key properties of the singularities of $K_{ij}(s)$. In order to examine the behavior of $I - B(s)$ in the neighborhood of $s = 0$, certain important properties of the averaging matrix A will be needed. In the next section these properties are derived.

III. PROPERTIES OF THE AVERAGING MATRIX

As a result of (4), the averaging matrix A has row sums equal to one. From the requirement that the network be connected, certain restrictions are placed on which combinations of elements of A may be zero. These two characteristics of A imply certain essential properties of the matrix $I - A$ where I is the identity matrix.

Theorem 1: If A is the averaging matrix of a connected network of N stations then the matrix $I - A$ has rank $N - 1$.

Proof: Since the equation

$$(I - A)x = 0 \tag{18}$$

is satisfied by any vector x with all components equal, the matrix $I - A$ is singular and must, therefore, have rank less than N. Suppose that its rank is less than $N - 1$. Then (18) has at least two nontrivial solutions that are linearly independent. Therefore, there exists a nontrivial solution, u, whose components are not all equal. Now let w be the solution vector with each component equal to the negative of the smallest component of u. Then $y = u + w$ is a nontrivial solution of (18) with all components non-negative and at least one component equal to zero. Let $\mathcal{A} = \{i_1, i_2, \cdots, i_k\}$ be the set of indices for which $y_i = 0$ and $\mathcal{B} = \{i_{k+1}, i_{k+2}, \cdots, i_N\}$ the set of indices for which y_i is positive. Since y satisfies (18) we have

$$y_i - \sum_{j=1}^{N} a_{ij}y_j = 0 \qquad i = 1,2, \cdots, N$$

and so

$$\sum_{j \varepsilon \mathcal{B}} a_{ij}y_j = 0, \qquad \text{for } i \varepsilon \mathcal{A}.$$

But this is only possible if

$$a_{ij} = 0, \qquad i \varepsilon \mathcal{A} \quad \text{and} \quad j \varepsilon \mathcal{B},$$

which implies that there is no transmission path from any station with identifying index in $\mathcal{B}$ to any station with identifying index in $\mathcal{A}$. Consequently, the network of N stations is not connected, which is a contradiction. Therefore, $I - A$ must have rank $N - 1$.

Theorem 2: If A is the averaging matrix of a connected network, the cofactors of all the elements in any given row of $I - A$ are equal and positive. Specifically, if M_{ij} is the cofactor of the ijth element of $I - A$, then

$$M_{ij} = M_{ik} > 0$$

for $i, j, k = 1, 2, \cdots, N$.

Proof: Since $I - A$ has rank $N - 1$, the solutions of

$$(I - A)y = 0$$

satisfy[6]

$$\frac{y_j}{y_k} = \frac{M_{ij}}{M_{ik}}, \qquad i = 1,2, \cdots, N.$$

But the only solutions y are those with all components equal. Therefore,

$$M_{ij} = M_{ik} \quad \text{all} \quad i,j,k. \tag{19}$$

Let $R(\varepsilon) = I - \varepsilon A$ and let $M_{ij}(\varepsilon)$ be the cofactor of the ijth element of $R(\varepsilon)$. For $0 \leqq \varepsilon < 1$, each principal minor of $R(\varepsilon)$ is the determinant of a diagonally dominated matrix (see Appendix), so that

$$M_{ii}(\varepsilon) \neq 0 \qquad 0 \leqq \varepsilon < 1.$$

Since $M_{ii}(0) = 1$, it follows by continuity that $M_{ii}(1) \geqq 0$. Hence, from (19)

$$M_{ij} = M_{ik} \geqq 0 \quad \text{all} \quad i,j,k. \tag{20}$$

Now $(I - A)'$, where the prime denotes the transpose, must also have rank $N - 1$. Thus, solutions of

$$(I - A)' z = 0 \tag{21}$$

satisfy

$$\frac{z_j}{z_k} = \frac{M_{ji}}{M_{ki}} \qquad i = 1,2, \cdots, N. \tag{22}$$

Equations (20) and (22) imply that the nonzero components of z must have the same sign. Suppose a solution z of (21) has at least one component zero and nonzero components positive. Then the same argument used in Theorem 1 leads to the conclusion that the network is disconnected, which is a contradiction. Therefore, there is a solution z with all components positive and consequently (22) implies that all cofactors M_{ij} are positive, which completes the proof.

IV. ANALYSIS

With the help of the preceding results, we are now in a position to prove stability and determine the expression for final frequency. These results will be obtained under the assumption that $\beta_i(s)$, for each i, satisfies the condition

$$|\beta_i(j\omega)| < 1, \qquad \omega \neq 0. \tag{23}$$

In Appendix B we show, with the help of the Nyquist criterion, that condition (23) implies the stronger condition

$$|\beta_i(s)| < 1 \quad \text{for} \quad s \quad \text{in} \quad \mathcal{R}, \tag{24}$$

where $\mathcal{R}$ is the right half and imaginary axis of the s plane excluding the point $s = 0$.

In Section II we saw that the associated linear system is characterized by the matrix transfer function $K(s)$ given by

$$K(s) = [I - B(s)]^{-1} C(s). \tag{25}$$

Now, since $\beta_i = H_i/(s + H_i)$, it follows from (24) that

$$c_{ii}(s) = 1/(s + H_i)$$

has no singularities in $\mathcal{R}$. Furthermore, under condition (24) the matrix $I - B(s)$ is diagonally dominated (see Appendix A) for all s in $\mathcal{R}$. Thus, the determinant $|I - B(s)|$ is nonzero for all s in $\mathcal{R}$, and so we conclude that each component transfer function $K_{ij}(s)$ is analytic in the region $\mathcal{R}$.

At $s = 0$, the matrix $I - B(s)$ reduces to $I - A$ which is singular according to Theorem 1. Thus, the determinant $|I - B(s)|$ has a zero at $s = 0$. To show that it is only a simple zero we find an asymptotic expression* for the determinant in the neighborhood of $s = 0$. In the matrix $I - B(s)$, we replace the elements $b_{ij}(s)$ by their asymptotic expressions

$$b_{ij}(s) \sim a_{ij}\left[1 - \left(\tau_{ij} + \frac{1}{\lambda_i}\right)s\right], \qquad s \to 0$$

where we have used the relations $\exp(-s\tau) \sim 1 - s\tau$ and $H_i/(s + H_i) \sim 1 - s/\lambda_i$. Without changing the value of the determinant, we may replace the first column by the sum of all the columns. The first column then becomes

$$s\left(\tau_1 + \frac{1}{\lambda_1}\right), s\left(\tau_2 + \frac{1}{\lambda_2}\right), \cdots, s\left(\tau_n + \frac{1}{\lambda_n}\right),$$

where

$$\tau_i = \sum_{j=1}^{N} a_{ij}\tau_{ij} \tag{26}$$

is an average of the transmission delays of links arriving at the ith

* The technique for finding the asymptotic expression is due to A. J. Goldstein.

station. Now we expand the determinant about the first column and obtain

$$| I - B(s) | \sim s \sum_{i=1}^{N} \left(\tau_i + \frac{1}{\lambda_i} \right) M_{i1}, \qquad s \to 0, \tag{27}$$

where M_{ij} is the cofactor of the ijth element of $I - B(0)$, as defined in Section III.

Since M_{i1} is positive (from Theorem 2), it follows from (27) that $| I - B(s)|$ has only a simple zero at $s = 0$. But $c_{ii}(0) = 1/\lambda_i$ is finite, so that from (25) we conclude that each $K_{ij}(s)$ has a simple pole at $s = 0$. Using (25) and the asymptotic expression (27) it follows that

$$K_{ij}(s) \sim \gamma_j/s, \qquad s \to 0, \tag{28}$$

where

$$\gamma_j = \frac{M_{ji}/\lambda_j}{\sum_l \left(\tau_l + \frac{1}{\lambda_l} \right) M_{l1}}.$$

Note that γ_j is positive and independent of i because $M_{ji} = M_{j1} > 0$, according to Theorem 2. Thus, letting

$$d_j = \frac{M_{j1}/\lambda_j}{\sum_l M_{l1}/\lambda_l}, \tag{29}$$

we have

$$\gamma_j = \frac{d_j}{1 + \sum_l \tau_l d_l \lambda_l} \tag{30}$$

with $0 < d_j < 1$ and $\sum d_j = 1$.

We have, therefore, shown that each transfer function $K_{ij}(s)$ is analytic in the right half and on the imaginary axis of the s plane except at $s = 0$ where it has a simple pole with positive residue independent of i. The impulse response $k_{ij}(t)$, associated with $K_{ij}(s)$, will, therefore, consist of exponentially decaying sinusoids and a step function of height γ_j.

To determine the stability of the original model under the particular starting conditions, we examine the asymptotic behavior of the phase responses of the associated linear system when subjected to the inputs given by (11). From (16), we have

$$P_i(s) = \sum_{j=1}^{N} K_{ij}(s) \left[\frac{1}{s} f_j + p_j(0) + Q_j(s) \right]. \tag{31}$$

From the known properties of $K_{ij}(s)$, it follows that the phase response $p_i(t)$ for $t > 0$ will be the sum of terms decaying exponentially to zero plus a term of the form $ft + \eta_i$ where f and η_i are obtained by the residue theorem according to

$$f = \lim_{s \to 0} s^2 P_i(s)$$

and

$$\eta_i = \lim_{s \to 0} \frac{d}{ds} [s^2 P_i(s)].$$

The final frequency f of the ith oscillator is, therefore, given by

$$f = \sum_{j=1}^{N} \gamma_j f_j \tag{32}$$

which is independent of i. Thus, we have proved stability of the system since the frequency of each oscillator has been shown to asymptotically approach the common frequency f and the phase differences clearly approach finite values. From (30) the expression for the final frequency can be written as

$$f = \frac{\sum_j d_j f_j}{1 + \sum_i \tau_i d_i \lambda_i} \tag{33}$$

which, with the help of (29), shows the dependence of f on the delays τ_{ij} and the dc gains λ_i.

V. REMARKS ON THE FINAL FREQUENCY

From the results of the preceding section it is clear that the final frequency can be below even the lowest oscillator free-running frequency. In fact, it is evident from (33) that the final frequency is a monotonically decreasing frunction of the system gain-delay products. Thus, the controls may bring the system to a frequency outside its practical operating range.

The final frequency reduction is a consequence of the fact that the frequency control of each station varies directly with the differences of *total* phase. The interstation delays introduce phase lags which drive down the frequency of each station. This point is made somewhat clearer by considering a system in which all the oscillators have the same frequency f and the same initial phase. When the controls are applied at $t = 0$ an average phase "error" $(-f\tau_i)$ is applied to the control path of each oscillator i. This "error" causes a simultaneous reduction in the oscillator frequencies from which the system never completely recovers.

As a conceptual solution to this difficulty, suppose the system of Fig. 1 is modified so that the local phase at the output of the integrator is passed through a delay line before being compared with the incoming phases. This local delay at station i is chosen equal to τ_i, the average delay of links terminating at station i, as defined in (26).

In the previous model, the error signal was determined by a comparison of the local phase at the present time with the remote phases of earlier times. In this modified system, however, the comparison is made between phases which on the average occur at the same time. Thus, the undesired component of the error signal due to interstation delays is eliminated.

Using an argument which parallels the development of Sections II, III, and IV, it may readily be shown that the final frequency for the modified system is given by

$$f = \sum_j d_j f_j. \tag{34}$$

In contrast with the original system, it is evident that the final frequency of the modified system is always an average of the free-running frequencies.

The Beneš formulation (Fig. 1) may be viewed as a simplified abstraction of the more complex practical systems that have been proposed.[2,5] Both the Beneš formulation and the modified system contain a total phase comparator which is an impractical element. Karnaugh[5] has shown that an important linear subclass of the more realistic class of systems he has proposed obey equations of the same form (6) as in the Beneš model. This more realistic formulation also fits the linear system model with modified frequency "inputs" that depend in a different manner on the initial conditions. It is, therefore, subject to the stability condition (23).

Moreover, it has a different final frequency which approaches an average of the free-running frequencies as the interstation delays become large.

In short, the Beneš formulation was sufficient to provide the important stability criterion, but neglected factors affecting the final frequency. The linear system model developed here is general enough to be applicable to both the Beneš formulation and a linear subclass of the more realistic Karnaugh formulation.

VI. SENSITIVITY AND RELATED QUESTIONS

Suppose that the system has been operating in a synchronized steady-state condition for a long time, and at some instant, say $t = t_0$, a sudden change is made in one or more parameters of the system. The subsequent phase responses are determined by considering the new associated linear system subjected to suitable inputs equivalent to the pertinent starting conditions. These inputs will have exactly the same form as (11) but now the term $Q_i(s)$ will be evaluated using the past history of the phases given by

$$p_i(t) = f_s t + \eta_i \qquad t < t_0 ,$$

where f_s is the synchronous frequency prior to the parameter change. If the stability condition (23) is satisfied for the new system and if the parameter change does not reduce any λ_i or a_{ij} to zero, the new system will also be stable. Consequently, after $t = t_0$ the frequencies of the oscillators will asymptotically resynchronize to the new final frequency determined by (33) or (34) using the changed parameter values. From these arguments we can also deduce that the effect of a slowly time-varying parameter on the system operation is to cause a corresponding slowly varying synchronous frequency. By "slow" time variations we mean that the time for a noticeable change in a parameter value to occur is much longer than the time constants associated with the transient response of each $K_{ij}(s)$.

By similar arguments, it is easily seen that failure of a transmission link will lead to resynchronization if the remaining network is still connected. Also, in the case of oscillator failure, the remaining $N - 1$ oscillators will resynchronize to a new frequency if the resulting network of $N - 1$ stations is still connected after removal of all transmission links entering or leaving the inoperative station. In each case, the final frequency can be computed from (33) or (34) using the appropriate parameter values. To prove these results, the nonzero averaging weights can be rescaled so that A has row sums unity; the filter gains λ_i are assumed to be correspondingly rescaled. The characterizing equations for the new system then has the required form and so resynchronization will occur.

The effect of independent jitter noise on the frequency of each oscillator may be considered by including a noise term $n_i(t)$ in each "input" $v_i(t)$. By superposition, the effect of noise can be considered separately. Thus, each phase response $p_l(t)$ will consist of the response in the absence of noise plus a noise component whose power density spectrum is

$$\sum_{\nu=1}^{N} | K_{l\nu}(j\omega) |^2 S_\nu(\omega),$$

where $S_\nu(\omega)$ is the power density spectrum of $n_\nu(t)$. Consequently, if the input noise jitter has zero mean and finite variance the output noise components will also have zero mean and finite variance. We conclude, therefore, that in the presence of noise jitter each oscillator will asymptotically have a common frequency with a random perturbation. The perturbations will be correlated but, in general, will not be identical. Furthermore, small jitter noise implies proportionately small perturbations.

VII. CONCLUDING REMARKS

We have seen that the transfer function approach has permitted a simple treatment of a rather complicated control system. Further studies regarding transient response or bounds on the size of perturbations due to jitter noise can be made for particular topological configurations by determining more information about the transfer functions $K_{ij}(s)$ with the help of (25). The linear system approach together with the added generality of having different filters at each station has made it possible to consider the effect of parameter changes or oscillator failure on the behavior of the system.

VIII. ACKNOWLEDGMENT

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APPENDIX A

A square matrix A is said to be *diagonally dominated* when for each row the sum of the magnitudes of the off-diagonal elements is less than the magnitude of the diagonal element, i.e.,

$$| a_{ii} | > \sum_{j \neq i} | a_{ij} |, \qquad \text{each } i.$$

Theorem: If A is diagonally dominated it is nonsingular.

Proof: Suppose the contrary. Then there exists a nontrivial solution $\{x_i\}$ satisfying

$$\sum_j a_{ij} x_j = 0 \qquad \text{each } i.$$

Let r be one of the indices for which $|x_i|$ is a maximum. Then

$$a_{rr}x_r = -\sum_{k \neq r} a_{rk}x_k$$

so that

$$|a_{rr}||x_r| \leqq \sum_{k \neq r} |a_{rk}||x_k| \leqq \sum_{k \neq r} |a_{rk}||x_r|$$

which is a contradiction. Hence, the theorem is proved.

APPENDIX B

On the Boundedness of $\beta_i(s)$

Theorem: If $\beta_i(s)$ is bounded by unity on the $j\omega$ axis then it is bounded by unity in the entire right-half plane.

Proof: Since

$$\beta_i(s) = \frac{H_i(s)/s}{1 + H_i(s)/s},$$

the condition $|\beta_i(j\omega)| < 1$ is equivalent to

$$|A| < |1 + A \exp(i\varphi)|, \tag{35}$$

where

$$A \exp(i\varphi) = H_i(j\omega)/j\omega.$$

But (35) is equivalent to

$$A \cos \varphi > -\tfrac{1}{2}$$

so that the locus of $H(j\omega)/j\omega$, as ω increases from $-\infty$ to ∞, cannot encircle the point -1. Hence, by the Nyquist stability criterion, $\beta_i(s)$ is analytic in the right-half plane. Furthermore, $\beta_i(\infty) = 0$. Thus, it follows that $|\beta_i(s)| < 1$ in the right-half plane according to the maximum modulus theorem.

REFERENCES

1. Beneš, V. E., unpublished work, 1959.
2. Runyon, J. P., Reciprocal Timing of Time Division Switching Centers, U.S. Patent No. 3,050,586, August 21, 1962.
3. Goldstein, A. J., unpublished work, 1963.
4. Bonomi, M. J., LaMarche, R. E., and Varaiya, P. P., unpublished work, 1963.
5. Karnaugh, M., A Model for the Organic Synchronization of Communication Systems, B.S.T.J., this issue, pp. 1705–1735.
6. Guillemin, E. A., *Mathematics of Circuit Analysis,* John Wiley & Sons, Inc., New York, 1949, p. 17.