Approximation of the Error Probability in a Regenerative Repeater with Quantized Feedback

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I. INTRODUCTION

Recently, Zador[1] gave a clever functional iteration procedure for determining the error probability in a binary regenerative repeater with quantized feedback. Unfortunately, quantitative results for the long pulse sequences of interest are difficult to come by due to the prohibitive amount of computer time required to carry out the iterations. We have found a simple approximate procedure that breaks the computational bottleneck in all cases of practical interest. The crux of our approach is the approximation of the functional iteration by a difference equation. For clarity, we use only a few terms of a Taylor series in establishing the difference equation approximation. More terms can be used to obtain a better approximation if needed.

II. RECAP OF ZADOR'S WORK

In Ref. 1, Zador shows that the kth iterate of the transformation $Uf(x)$ — denoted by $U^k f(x) = U^{k-1}[Uf(x)]$ — when evaluated at $x = 0$ yields the average bit error probability $p(k)$, for the last bit in a random sequence of $k + 1$ bits processed by a regenerative repeater with quantized feedback. The transformation $Uf(x)$ is given by [Zador's (14)]

$$Uf(x) = p_1(x)f(rx - a) + p_2(x)f(rx) + p_3(x)f(rx + a), \quad (1)$$

where

$$\begin{aligned} p_1(x) &= p[N(-g_0 - x)] \\ p_2(x) &= 1 - p_1(x) - p_3(x) \\ p_3(x) &= q[1 - N(g_0 - x)] \\ f(x) &= p_1(x) + p_3(x) \\ N(x) &= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} \exp\,(-y^2/2\sigma^2)\, dy \\ 0 &< r < 1 \\ a &= -2g_1 . \end{aligned} \quad (2)$$

In (2) above, $r = \exp\,(-T/T_c) = \exp\,(-b)$ is the decrement of the simple RC low-pass filter in the feedback path of the regenerative repeater; T_c is the time constant of this filter or equivalently the reciprocal of the 3-dB point on the high-pass filter in the transmission line preceding the repeater. The quantity g_0 is the peak value of pulse response of the medium in response to the transmission of a $+1$, and $g_1 < 0$ is the value of the isolated pulse response of the medium one time slot (T) away from the peak. For convenience, we assume equally likely transmitted pulses, $p = q = \frac{1}{2}$.

III. APPROXIMATION

If we (*i*) substitute for $p_2(x)$ from (2) into (1), and (*ii*) expand $f(rx \pm a)$ in a Taylor's Series about rx, and (*iii*) retain terms in the series through a^2 we get the approximation

$$Uf(x) = f(rx) + a \left.\frac{df(y)}{dy}\right|_{y=rx} [p_3(x) - p_1(x)] + \frac{a^2}{2} \left.\frac{d^2f(y)}{dy^2}\right|_{y=rx} [p_3(x) + p_1(x)]. \tag{3}$$

When we note that

$$\left.\frac{d^nf(y)}{dy^n}\right|_{y=0} = 0 \quad \text{for} \quad n \text{ odd}$$

$$p_3(0) = p_1(0) \tag{4}$$

$$f(0) = p_3(0) + p_1(0) = p(0) = \frac{1}{2}\left[1 - \operatorname{erfc}\left(\frac{g_0}{\sigma\sqrt{2}}\right)\right],$$

then

$$p(1) = p(0)\left[1 + \frac{a^2}{2} \left.\frac{d^2f(y)}{dy^2}\right|_{y=0}\right]. \tag{5}$$

Proceeding through the functional iteration defined by (3), retaining terms involving a^2 and lower, and using (4) we obtain the difference equation

$$p(k) = p(k-1) + p(0)Kr^{(2k-1)}, \tag{6}$$

where $Z = g_0/\sigma$ is the peak signal to rms noise ratio for an isolated pulse, and

$$K = \frac{Z}{2\sqrt{2\pi}} \left(\frac{a}{\sigma}\right)^2 \exp\left(-\tfrac{1}{2}Z^2\right). \tag{7}$$

Equation (6) is easily solved to give

$$p(k) = p(0)\left[1 + K\sum_{j=1}^{k} r^{2(j-1)}\right]. \tag{8}$$

Then

$$\lim_{k\to\infty} p(k) = p(0)\left[1 + \frac{K}{1 - r^2}\right]. \tag{9}$$

The bracketed term in (8) or (9) gives the enhancement of error probability due to errors made on all previous pulses.

To compare results obtained by application of (9) with those obtained experimentally by R. D. Howson,[2] we take $b = \frac{1}{4}$, $g_0 = \exp(-\frac{1}{8})$, and $g_1 = -g_0(1 - \exp(-\frac{1}{4}))$. Over a wide range of S/N ratios of interest, analysis based upon (9) predicts about a 0.6-dB S/N penalty of this system over the ideal case of no low frequency cutoff. Agreement with experiment is excellent. It should be noted that the enhancement term in (9) is very close to unity and the S/N penalty is due essentially to the reduction of the pulse peak by the low-frequency cut-off.

In a future paper we will show (*i*) how Zador's approach can be extended to a wider class of systems, and (*ii*) how the approximation given herein can be used and improved when necessary, to arrive at meaningful quantitative results.

REFERENCES

1. Zador, P. L., Error Probabilities in Data System Pulse Regenerator with DC Restoration, B.S.T.J., *45*, July, 1966, pp. 979–984.
2. Flavin, M. A., Gerrish, A. M., Howson, R. D., and Salz, J., unpublished work.

Application of Automatic Transversal Filters to the Problem of Echo Suppression

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Long-haul voice communication has long been subject to the problem of returned echo. The advent of synchronous satellite communication introduces increased delay as a degrading factor in the overall quality of two-way conversation. This compounds the problem in that the echo