Geometrical Representation of Gaussian Beam Propagation

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A single equation governing the various properties encountered in propagation of Gaussian beams is discussed. These characteristics may be graphically presented on an impedance chart such as a Smith chart or in the form of normalized curves. The geometrical representations highlight the relation between Gaussian mode theory and geometrical optics formulas.

The power coupling coefficient between two Gaussian modes suggests a complex mismatch coefficient whose geometrical representation is essentially the same as that of a complex reflection coefficient in transmission-line theory. Application of the Smith chart in determining a complex mismatch coefficient is illustrated by graphical solution of a beam-matching problem.

I. INTRODUCTION

The propagation of a Gaussian beam and its transformation through a lens has been well treated in previous literature.[1,2,3,4] This paper will show that a single, formally identical equation governs three properties of Gaussian beam propagation: (*i*) the phase front curvature and the beam radius in terms of the distance from the beam waist and the minimum beam radius; (*ii*) the propagation of a Gaussian beam in free space; (*iii*) the transformation of a Gaussian beam through a lens. Geometrical representations of these characteristics highlight the relationship between Gaussian beam propagation and geometrical optics.

Several recent papers[5,6,7,8] have been devoted to graphical solutions of Gaussian mode problems. One recalls that the Smith chart is a geometrical representation of complex reflection coefficient in transmission line theory. It seems logical, therefore, to look for the counterpart of a complex reflection coefficient in Gaussian mode theory in order that the full potential of the Smith chart may be realized in graphical solutions of Gaussian mode problems. In this paper, a complex mismatch coefficient will be defined such that the geometrical representation of this coefficient is essentially the same as that of a complex reflection coefficient in transmission line theory. This observation immediately suggests the application of the Smith chart of complex mismatch coefficients to a graphical solution of beam matching problems.

II. THE ANALOGY AMONG THREE PROPERTIES OF A GAUSSIAN BEAM

In order to discuss the Gaussian beam transformation it is convenient to summarize the relationships among the parameters of a Gaussian beam propagating along the z-axis first.[4,9]

$$w(z) = \bar{w} \sqrt{1 + \left(\frac{\lambda z}{\pi \bar{w}^2}\right)^2} \tag{1a}$$

$$r(z) = z\left[1 + \left(\frac{\pi \bar{w}^2}{\lambda z}\right)^2\right]. \tag{1b}$$

In (1), w is the beam radius at which the field amplitude has fallen to $1/e$ of its maximum value on the z-axis, $\bar{w}$ is the minimum beam radius (called the beam waist) where one has a plane phase front at $z = 0$, and r is the radius of curvature of the phase front at z. It should be noted that the phase front is not exactly spherical; therefore, its radius of curvature is exactly equal to r only on the z-axis. The z-coordinate, measured from the beam waist, is taken to be positive to the right and negative to the left; the parameters are illustrated in Fig. 1. Equation (1) may be solved for $\bar{w}$ and z as follows:

$$\bar{w} = \frac{w}{\sqrt{1 + \left(\frac{\pi w^2}{\lambda r}\right)^2}} \tag{2a}$$

$$z = \frac{r}{1 + \left(\frac{\lambda r}{\pi w^2}\right)^2}. \tag{2b}$$

Equations (1) and (2) may be transformed into a single equation of complex variables:

$$\frac{1}{\pi w^2/\lambda} - i\frac{1}{r} = \frac{1}{(\pi \bar{w}^2/\lambda) + iz}. \tag{3}$$

We are also interested in the transformation of a Gaussian beam going through a lens of focal length f as shown in Fig. 2. A beam with its minimum beam radius $\bar{w}_1$ located at d_1 will become another beam with its

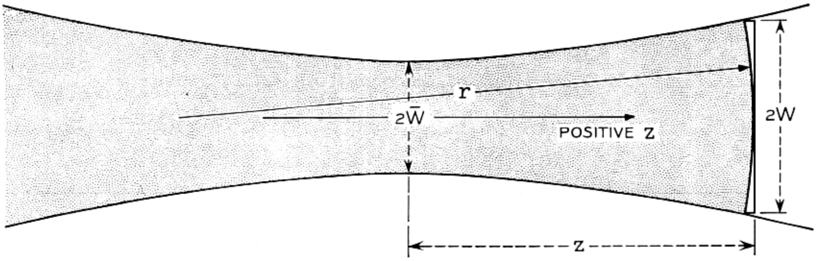

Fig. 1 — Parameters of a Gaussian beam.

minimum beam radius $\bar{w}_2$ located at d_2. Since the beam radius remains the same in passing through the lens we obtain from (1a)

$$\bar{w}_1 \sqrt{1 + (\lambda d_1/\pi \bar{w}_1^2)^2} = \bar{w}_2 \sqrt{1 + (\lambda d_2/\pi \bar{w}_2^2)^2}. \tag{4}$$

The thin lens formula states that the change of the phase front curvature may be approximated by the reciprocal of the focal length. Thus, using (1b), one obtains

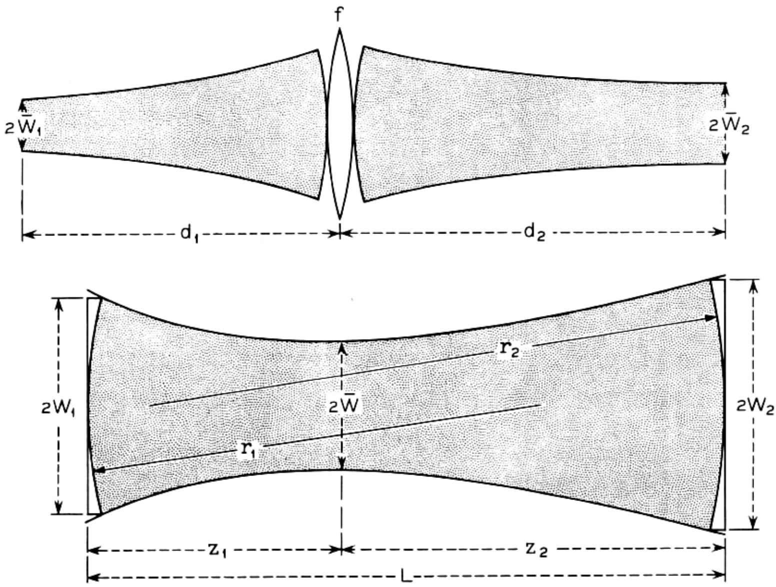

Fig. 2 — The analogy between the Gaussian beam transformation through a lens and the Gaussian beam propagation in free space.

$$\frac{1}{d_1[1 + (\pi \bar{w}_1^2/\lambda d_1)^2]} + \frac{1}{d_2[1 + (\pi \bar{w}_2^2/\lambda d_2)^2]} = \frac{1}{f} \tag{5}$$

where the sign of the second term on the left side is positive because d_2 has been taken to be positive to the left of the beam waist following the convention of Goubau[1] and Kogelnik.[2] Straightforward algebra will lead to the following solutions for (4) and (5):

$$\frac{d_2}{f} - 1 = \frac{\dfrac{d_1}{f} - 1}{\left(\dfrac{d_1}{f} - 1\right)^2 + \left(\dfrac{\pi \bar{w}_1^2}{\lambda f}\right)^2} \tag{6}$$

$$\frac{\bar{w}_2^2}{\bar{w}_1^2} = \frac{1}{\left(\dfrac{d_1}{f} - 1\right)^2 + \left(\dfrac{\pi \bar{w}_1^2}{\lambda f}\right)^2}. \tag{7}$$

The above two expressions are essentially rearrangements of formulas obtained by Goubau.[1] When $\pi \bar{w}_1^2/\lambda \ll | d_1 - f |$, (6) and (7) approach the thin lens formula and the magnification formula of geometrical optics.* When $\pi \bar{w}_1^2/\lambda f \to \infty$ the condition for geometrical optics focusing of parallel rays is obtained. These two equations have been plotted in Figs. 3 and 4 for various values of $p = \pi \bar{w}_1^2/\lambda f$. There it is seen that the singularity at $d_1 = f$ in geometrical optics is eliminated in Gaussian mode transformations. The maxima and minima in Fig. 3 may be easily found by differentiating (6) with respect to d_1/f, and they are given by

$$\frac{d_2}{f} = 1 \pm \frac{f}{2\pi \bar{w}_1^2/\lambda} \quad \text{when} \quad \frac{d_1}{f} = 1 \pm \frac{\pi \bar{w}_1^2}{\lambda f}. \tag{8}$$

The points of inflection are

$$\frac{d_1}{f} = 1 \quad \text{and} \quad 1 \pm \frac{\sqrt{3}\pi \bar{w}_1^2}{\lambda f}.$$

Equations (6) and (7) can be combined into one equation of complex variables:

$$\left(\frac{\pi \bar{w}_2^2}{\lambda f}\right) - i\left(\frac{d_2}{f} - 1\right) = \frac{1}{\left(\dfrac{\pi \bar{w}_1^2}{\lambda f}\right) + i\left(\dfrac{d_1}{f} - 1\right)}. \tag{9}$$

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* Simple algebra may easily reduce them to the more familiar forms, $\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}$ and $\frac{\bar{w}_2^2}{\bar{w}_1^2} = \frac{d_2^2}{d_1^2}$.

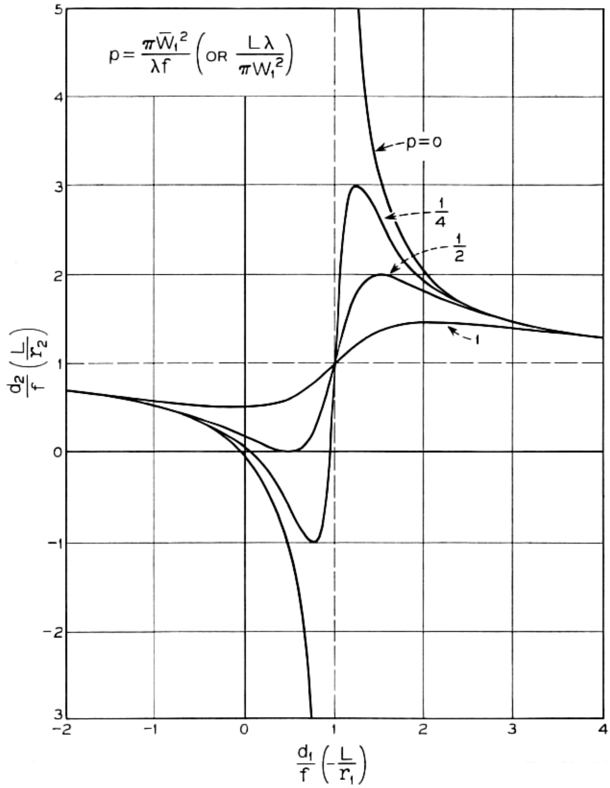

Fig. 3 — The graphical presentation of (5) and (14).

Next, let us consider the propagation of a Gaussian beam along the z-axis. One would like to express the beam radius w_2 and the curvature $1/r_2$ of the phase front at any given point in terms of the beam radius w_1 and the curvature $1/r_1$ at some other point along the axis. Since any Gaussian beam has a single beam waist at a definite location, one obtains from (2) the following conditions:

$$\frac{w_1}{\sqrt{1 + (\pi w_1^2/\lambda r_1)^2}} = \frac{w_2}{\sqrt{1 + (\pi w_2^2/\lambda r_2)^2}} \tag{10}$$

$$\frac{r_2}{1 + (\lambda r_2/\pi w_2^2)^2} - \frac{r_1}{1 + (\lambda r_1/\pi w_1^2)^2} = z_2 - z_1 . \tag{11}$$

Let $L = z_2 - z_1$. Rearranging (10) and (11) yields

$$\frac{\dfrac{1}{\pi w_1^2/\lambda}}{\left(\dfrac{1}{\pi w_1^2/\lambda}\right)^2 + \left(\dfrac{1}{r_1}\right)^2} = \frac{\dfrac{1}{\pi w_2^2/\lambda}}{\left(\dfrac{1}{\pi w_2^2/\lambda}\right)^2 + \left(\dfrac{1}{r_2}\right)^2} \tag{12a}$$

and

$$-\frac{\dfrac{1}{r_1}}{\left(\dfrac{1}{\pi w_1^2/\lambda}\right)^2 + \left(\dfrac{1}{r_1}\right)^2} + \frac{\dfrac{1}{r_2}}{\left(\dfrac{1}{\pi w_2^2/\lambda}\right)^2 + \left(\dfrac{1}{r_2}\right)^2} = L. \tag{12b}$$

If we also rearrange (4) and (5) as follows:

$$\frac{\pi \bar{w}_1^2/\lambda}{(\pi \bar{w}_1^2/\lambda)^2 + d_1^2} = \frac{\pi \bar{w}_2^2/\lambda}{(\pi \bar{w}_2^2/\lambda)^2 + d_2^2} \tag{13a}$$

$$\frac{d_1}{(\pi \bar{w}_1^2/\lambda)^2 + d_1^2} + \frac{d_2}{(\pi \bar{w}_2^2/\lambda)^2 + d_2^2} = \frac{1}{f} \tag{13b}$$

the analogy* between (12) and (13) now emerges with the following one-to-one correspondences:

$$\frac{\pi \bar{w}_1^2}{\lambda} \leftrightarrow \frac{1}{\pi w_1^2/\lambda}$$

$$\frac{\pi \bar{w}_2^2}{\lambda} \leftrightarrow \frac{1}{\pi w_2^2/\lambda}$$

$$d_1 \leftrightarrow -\frac{1}{r_1}$$

$$d_2 \leftrightarrow \frac{1}{r_2}$$

$$\frac{1}{f} \leftrightarrow L.$$

Using the above analogy, one may immediately write down the solutions of (12) for w_2 and $1/r_2$

$$\frac{L}{r_2} - 1 = \frac{-\dfrac{L}{r_1} - 1}{\left(-\dfrac{L}{r_1} - 1\right)^2 + \left(\dfrac{L}{\pi w_1^2/\lambda}\right)^2} \tag{14}$$

* The analogy would look even better, if r_1 had been assumed positive to the left of the beam waist. This fact, of course, results from the sign convention of d_1, which is positive to the left of the lens.

$$\frac{L}{\pi w_2^2/\lambda} = \frac{L/\frac{\pi w_1^2}{\lambda}}{\left(-\frac{L}{r_1} - 1\right)^2 + \left(\frac{L}{\pi w_1^2/\lambda}\right)^2}. \tag{15}$$

Except for slightly different normalizations, these two expressions coincide with two formulas obtained by Rowe[10] using rather involved algebra. The curves in Figs. 3 and 4 represent (14) and (15), if the coordi-

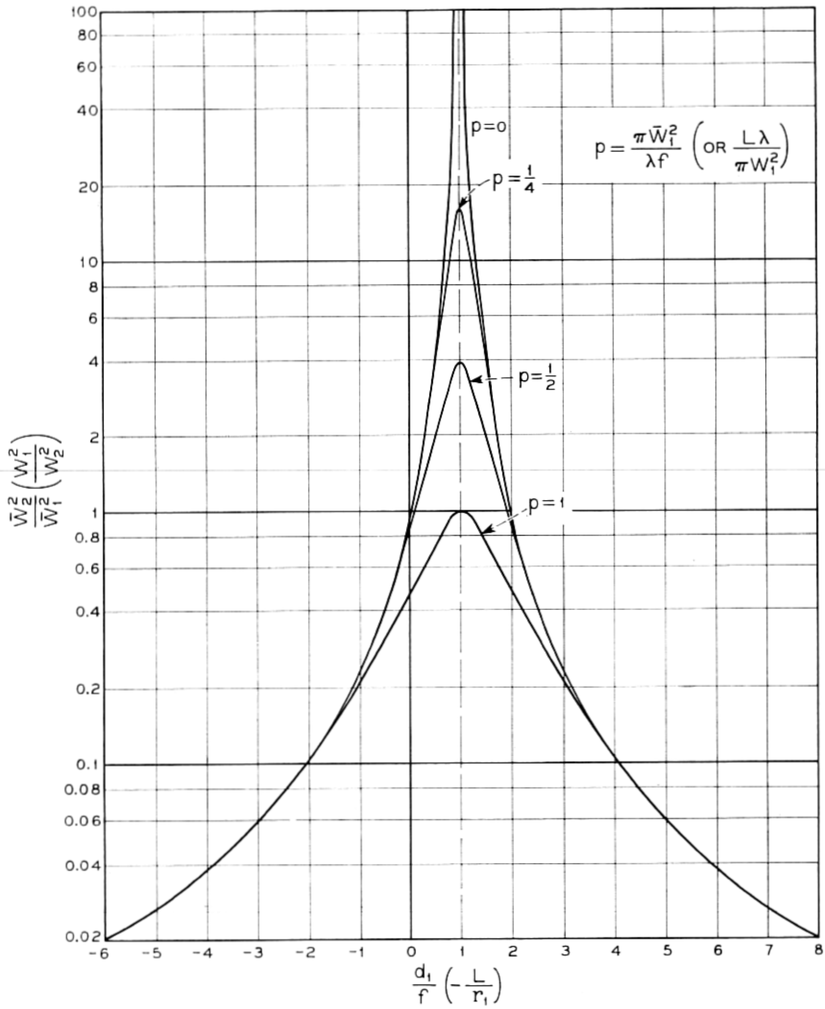

Fig. 4 — The graphical presentation of (6) and (15).

nates and the parameters are replaced by those shown in parentheses in the figures. When the wavelength approaches zero, the Gaussian beam becomes the spherical wave of geometrical optics. The singularity of the spherical wave is eliminated in the Gaussian beam representation. Equations (14) and (15) can also be combined into one equation of complex variables:

$$\frac{L}{\pi w_2^2/\lambda} - i\left(\frac{L}{r_2} - 1\right) = \frac{1}{\left(\frac{L}{\pi w_1^2/\lambda}\right) + i\left(-\frac{L}{r_1} - 1\right)}. \tag{16}$$

It is interesting to observe that (3), (9), and (16) are formally identical, and they are amenable to graphical representation on an impedance chart, e.g., a Smith chart. Thus, the circumference of the unit circle on a Smith chart corresponds to geometrical optics, while the interior describes the properties of a Gaussian beam. The transformation may be performed by simply taking diametrically opposite points on a Smith chart. It should be noted that while (3) has not been normalized, (9) and (16) are dimensionless. Representation of (3) by the Smith chart has also been proposed by Deschamps and Mast;[7] the analogy between (9) and (16) corresponds to the dual forms of the cartesian Gaussian beam chart.[5,6] Rearrangement of (14) and (15) will yield equations similar to those discussed in Ref. 8.

III. THE COMPLEX MISMATCH COEFFICIENT DIAGRAM

Using some of the results of the previous section, a graphical solution of mode matching problems using a properly defined complex mismatch coefficient will now be discussed. Equation (3) can be rewritten as

$$\frac{\pi \bar{w}^2}{\lambda} - iz = \frac{1}{\left(\frac{1}{\pi w^2/\lambda}\right) + i\frac{1}{r}} \tag{19}$$

and one may adopt the following notations:

$$\frac{1}{\pi w^2/\lambda} + i\frac{1}{r} = R + iX = Z \tag{20a}$$

$$\frac{\pi \bar{w}^2}{\lambda} - iz = G - iB = Y. \tag{20b}$$

Thus, a Gaussian beam may be characterized by either a beam impedance Z or a beam admittance Y. Now the following interesting identity can easily be verified.

$$\left|\frac{Z_1 - Z_0}{Z_1 + Z_0^*}\right|^2 = \left|\frac{Y_1 - Y_0}{Y_1 + Y_0^*}\right|^2 = 1 - \tau \tag{21}$$

where the subscript 1 represents the incoming beam, 0 represents the fundamental mode of a receiving system,* denotes complex conjugate, and τ is the power coupling coefficient[11] between two Gaussian modes. The first two expressions in (21) certainly look like reflection coefficients in transmission lines. This observation suggests the designation of $(Z_1 - Z_0)/(Z_1 + Z_0^*)$ and $(Y_1 - Y_0)/(Y_1 + Y_0^*)$ as complex mismatch coefficients. They are indeed identical to reflection coefficients in form except for the complex conjugate in the denominator. If Z_1 (or Y_1) is normalized with respect to a real Z_0 (or a real Y_0), then the complex mismatch coefficient is formally identical to a complex reflection coefficient. Both the complex mismatch coefficient and the complex reflection coefficient are bilinear transformations of the impedance or admittance.

The Smith chart is a geometrical representation of the complex reflection coefficient. The same Smith chart may also represent the complex mismatch coefficient. With respect to a reference mode, any phase front of a Gaussian beam may be represented by two diametrically opposite points on a Smith chart, i.e., the normalized admittance and the normalized impedance. The propagation of a Gaussian beam corresponds to travel along a conductance circle, while imposing change of radius of curvature by a lens, say, corresponds to travel along a resistance circle. The pair of two diametrically opposite points may be reduced to one point by imposing a flip-over Smith chart on the original Smith chart, however, overcrowded coordinate lines are not desirable in practical graphical solutions. This latter version of geometrical representations corresponds to a bilinear transformation of the Gaussian beam chart discussed in Ref. 5, and is the same as that suggested in Ref. 7. It should be noticed that Collins' Gaussian beam chart[5] is a geometrical representation of the beam impedance, while Li's dual form[6] is that of the beam admittance. Here, however, the Smith chart is utilized as a complex mismatch coefficient diagram and its relationship to the power coupling coefficient between two Gaussian modes is identified.

IV. A NUMERICAL EXAMPLE OF BEAM MATCHING

In order to illustrate the application of the Smith chart, consider the problem of matching the output beam of an optical maser to an interferometer as shown in Fig. 5. The maser resonator consists of mirrors of 5 and 10-meter radius of curvature separated by 1 meter, and the interferometer consists of a pair of 10-meter mirrors separated by 10 cm.

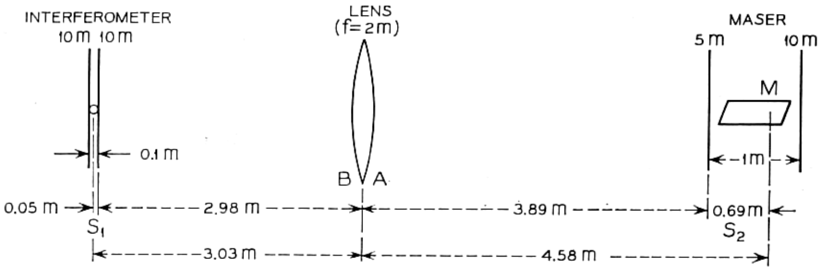

Fig. 5 — The geometry of a beam matching problem.

The maser beam exits from the 5-meter mirror and a lens with a focal length of 2 meters is available for matching. First, one calculates the minimum beam half-widths and their locations for both resonators by the following formulas:[4]

$$\frac{\pi \bar{w}^2}{\lambda} = \frac{\sqrt{d(R_1 - d)(R_2 - d)(R_1 + R_2 - 2d)}}{R_1 + R_2 - 2d} \tag{22}$$

$$S = d\,\frac{R_2 - d}{R_1 + R_2 - 2d} \tag{23}$$

where R_1 and R_2 are the radii of curvature of the mirrors, d is mirror separation, $\bar{w}$ is the minimum beam half-width, and S is the location of the beam waist from the mirror of radius R_1. Now we readily get $\pi\bar{w}_1^2/\lambda = 0.705$ m, $S_1 = 0.05$ m for the interferometer, and $\pi\bar{w}_2^2/\lambda = 1.727$ m, $S_2 = 0.692$ m for the maser resonator. If all the parameters are normalized with respect to $\pi\bar{w}_1^2/\lambda$, then the fundamental mode of the interferometer is represented by the unity conductance circle passing through the center of Smith chart in Fig. 6 where 0 corresponds to the beam wasit for the interferometer. The maser output beam is represented by the 2.45 ($= \bar{w}_2^2/\bar{w}_1^2$) conductance circle passing through M which corresponds to the beam waist for the maser resonator. The inversions of these two circles with respect to the center 0 give the beam radius and the phase front curvature along the propagation path of the beam.

Let it be required that insertion of a 2-meter focal length lens changes the phase-front curvature of the maser output beam to that of the interferometer beam at the point where the two beams meet. Since the beam radius remains constant going through the lens, application of the thin lens formula demands that a segment of the beam resistance circle between the inverted maser and interferometer circles be equal to the length $\widehat{A'B'} = \pi\bar{w}_1^2/\lambda f = (\pi\bar{w}_1^2/\lambda)(1/r_1 - 1/r_2) = 0.3523$. After deter-

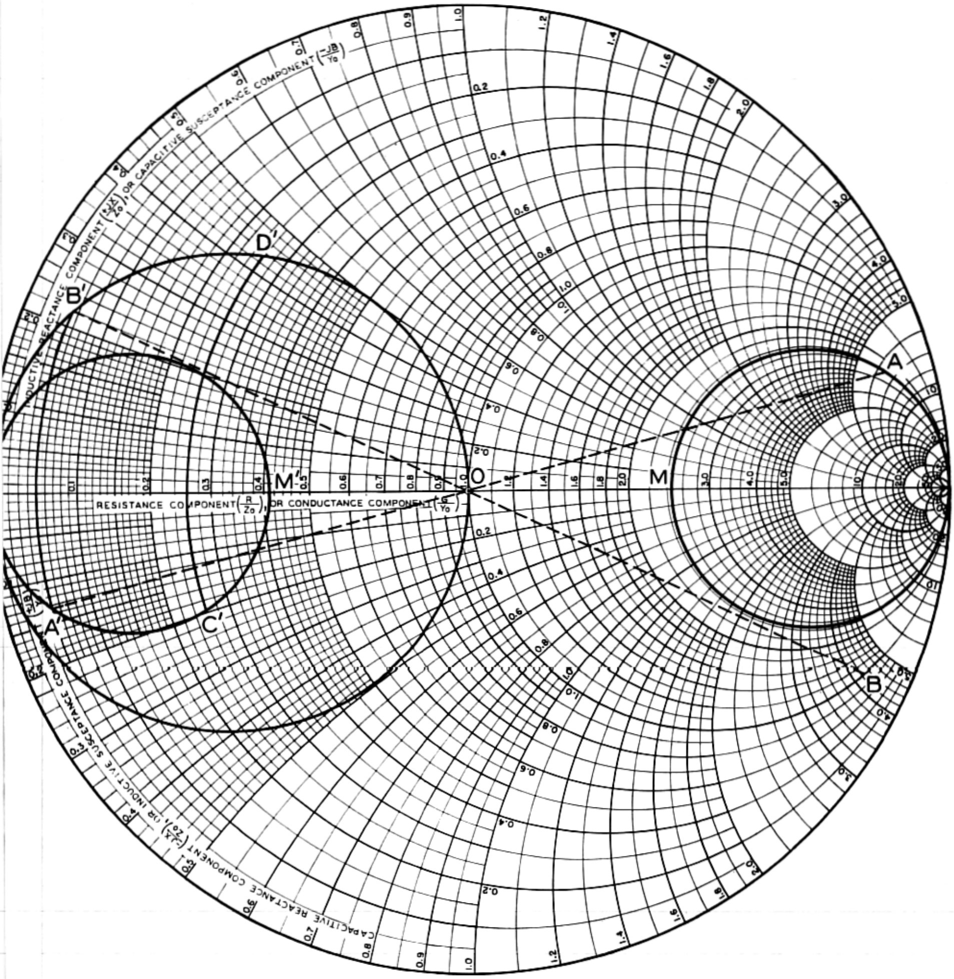

Fig. 6 — The graphical solution of a beam matching problem.

mining the two pairs of diametrically opposite points A–A' and B–B' (Fig. 6), the distances from the lens to the interferometer beam waist and to the maser beam waist can be read off the chart as $\widehat{OB} = 4.3 \times 0.705 = 3.03$ m and $\widehat{MA} = 6.5 \times 0.705 = 4.58$ m. The distances from the lens to the interferometer input mirror and to the maser output mirror are $3.03 - 0.05(S_1) = 2.98$ m and $4.58 - 0.69(S_2) = 3.89$ m, respectively.

V. DISCUSSION

The advantage of this complex mismatch coefficient diagram over the Gaussian beam charts discussed in Refs. 5 and 6, is similar to that of the Smith chart over cartesian impedance and admittance charts. Neither of the latter can represent an arbitrary transformation of a Gaussian beam, whereas the present mismatch coefficient chart, at least in principle, encompasses all possible transformations. Curvilinear coordinates are

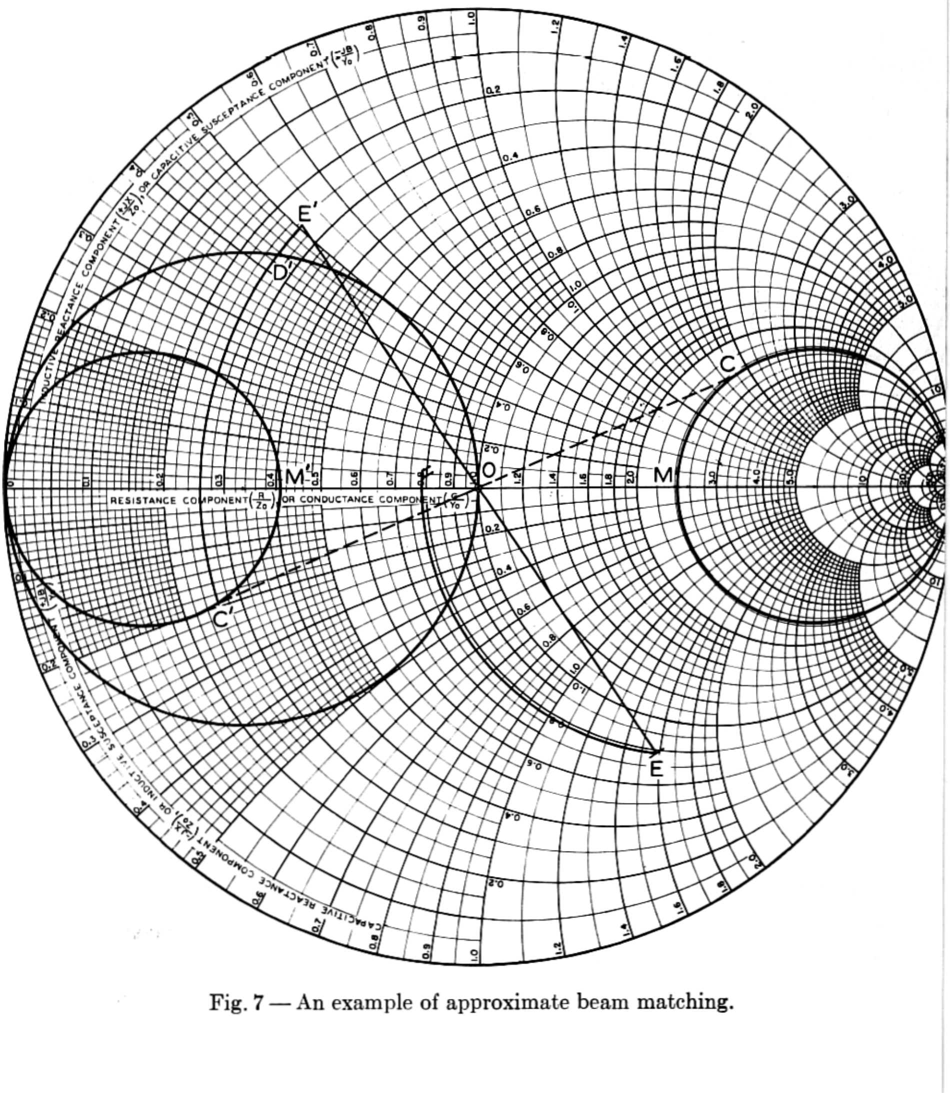

Fig. 7 — An example of approximate beam matching.

not a problem here because a good network of circles is readily available in the form of Smith charts. Sometimes it may be impossible to achieve perfect beam matching due to limitations imposed by the inadequacy of a lens or of space. One is able to minimize the mismatch using the procedure discussed here.

In the above example, there exists a minimum focal length of the lens beyond which perfect matching is not possible. In Fig. 6, this minimum is represented by the maximum arc length $\widehat{C'D'}$ (0.64) of the beam resistance circle between the inverted maser and interferometer circles which corresponds to a focal length of $\pi \bar{w}_1^2/\lambda \widehat{C'D'} = 0.705/0.64 = 1.102$ m. If one had only a one meter focal length lens available, $\widehat{C'D'}$ would be extended to E' as shown in Fig. 7 and the reciprocals of C' and E' are then found to be C and E. The distances from the lens to the maser output mirror and to the interferometer input mirror would be $\widehat{CM} - S_1 = 1.8 \times 0.705 - 0.05 = 1.22$ m and $\widehat{EF} - S_2 = 1.56 \times 0.705 - 0.69 = 0.41$ m for a resulting mismatch loss $|\overline{OF}|^2 = 0.015$.

VI. ACKNOWLEDGMENT

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