The Effect of a Finite-Width Decision Threshold on Binary Differentially Coherent PSK Systems

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A binary PCM regenerator must decide which of two signal states was transmitted. This decision is normally based on whether some particular voltage is above or below a certain threshold at some particular time. In a real regenerator, the voltage in question must differ from this threshold value by some finite amount in order to cause the device to respond properly. The purpose of this calculation is to examine the effect of this "dead zone" on the probability of error in a differentially coherent PSK system both in the case where the signal is limited in amplitude (after the noise is added) and when it is not.

It is found that when the expected value of signal power, S, exceeds the threshold value, T, by more than 6 db and 10 db in the limited and unlimited cases, respectively, the effect on error probability is less than the effect of a 1-db degradation in signal-to-noise ratio. However, for smaller values of signal-to-threshold (S/T) the degradation becomes large. Numerical values for error probability as a function of signal-to-noise ratio are presented for $S/T = \infty$, 12, 9, 6, 4, and 3 db for both cases.

I. INTRODUCTION

A binary differentially coherent phase shift keyed (DCPSK) system is one in which the two (binary) states — "mark" and "space" — are transmitted as phase changes between adjacent time slots. In such a system, optimum results are obtained when the expected values of the two possible signal states in a given time slot differ in phase by π radians. For clarity it will be assumed in the following discussion that the expected value of phase change is 0 or π, respectively, according to whether a space or a mark was transmitted, even though phase changes of $\theta + 0$ and $\theta + \pi$ (θ an arbitrary but fixed angle) are equally suitable.

Such a system usually consists of a limiter, followed by a storage mechanism and a product demodulator, as shown in Fig. 1. If the signals in adjacent time slots have amplitudes A and B, respectively, and a phase difference ψ, the output of the product demodulator is proportional to

$$v = AB \cos \psi. \tag{1}$$

If the limiter is ideal, $A = B = 1$. The ideal regenerator samples the sign of v and regenerates a mark if $v < 0$ and a space if $v > 0$.

The probability that, due to additive Gaussian noise, $v > 0$ if a mark was sent or equivalently the probability that $v < 0$ if a space was sent has been calculated by many authors.[1-4] This probability is given by

$$\prod = \tfrac{1}{2} \exp \left(-\mathrm{S/N}\right) \tag{2}$$

where S/N is the signal-to-noise ratio.

In a practical system, the regenerator requires some finite value of v in order to regenerate a mark or a space reliably. While it is true that the effects of this finite-width decision level can be made arbitrarily small by providing sufficient gain ahead of the regenerator and sufficient dynamic range for the regenerator, it is often difficult — especially with some solid-state devices — to achieve either this gain or the necessary dynamic range. For this reason, it may be desirable to operate a regenerator under conditions where the decision threshold is important. In this paper we consider the following model:

(*i*) An error is made if $|v| > \varepsilon$ and, due to additive Gaussian noise, v has the wrong sign.

(*ii*) No error is made if $|v| > \varepsilon$ and v has the correct sign.

(*iii*) There is a 0.5 probability of error if $|v| < \varepsilon$, regardless of the sign of v.

The probability of error in such a system is clearly dependent on both the signal-to-noise ratio S/N and the signal-to-threshold ratio S/T. The latter is defined as the ratio of the expected value of signal power

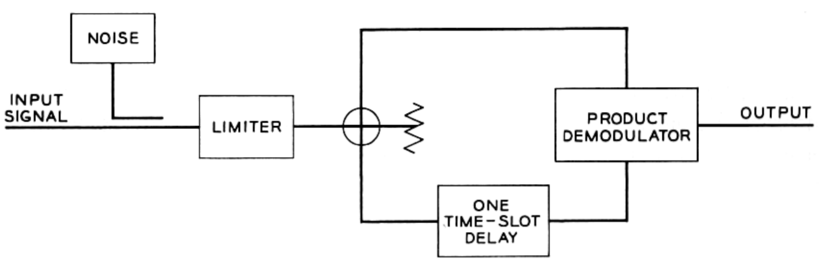

Fig. 1 — DCPSK receiver.

to the minimum value which causes the output of the product demodulator, v, to exceed ε in magnitude.

II. SYSTEMS WITHOUT A LIMITER

If the regenerator were ideal ($\varepsilon = 0$), the performance would be independent of whether or not the limiter in Fig. 1 is included, since the sign of v is independent of the product AB (both A and B are positive numbers). However, for the case where $\varepsilon > 0$, the limiter plays an important role under certain conditions. In this section, we consider the case where the limiter is omitted. The received signal can then be thought of as the vector sum of a unit vector representing the transmitted signal and a noise vector with Gaussian-distributed x and y components. The angle between the two vectors representing received signals one time slot apart together with their magnitudes A and B determine v. This is illustrated in Fig. 2 for a "mark". From Fig. 2, it is apparent that

$$\begin{aligned} v = AB \cos \psi = \mathbf{A}\cdot\mathbf{B} &= A_xB_x + A_yB_y \\ &= (A_o + a_x)(B_o + b_x) + a_yb_y \end{aligned} \tag{3}$$

where $A_o = +1$ or -1 and $B_o = +1$ or -1 depending on the message. Cos ψ has the wrong sign if the sign of v differs from the sign of A_oB_o.

There will be two probability density functions $p_-(v)$ and $p_+(v)$, respectively, for the cases $A_oB_o = -1$ and $A_oB_o = +1$. From symmetry, it is apparent that the error probability is the same in these two cases The probability of error can thus be determined from either

$$\Pi = \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} p_-(v)dv + \int_{\varepsilon}^{\infty} p_-(v)dv \tag{4}$$

or

$$\Pi = \frac{1}{2}\int_{-\varepsilon}^{\varepsilon} p_+(v)dv + \int_{-\infty}^{-\varepsilon} p_+(v)dv. \tag{5}$$

Bennett and Salz[3] derive the density functions $p_\pm(v)$ for the quadratic form of (3) where a_x, a_y, b_x, and b_y are independent Gaussian variables of zero mean and variance σ^2. They are given (in our notation) by

$$\begin{aligned} p_\pm(v) = \frac{1}{(2\pi)^{3/2}\sigma^3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} &\frac{\exp[-(x^2+y^2)/2\sigma^2]}{\sqrt{(1+x)^2+y^2}} \\ &\cdot\exp\left(-\frac{[v \mp (1+x)]^2}{2\sigma^2[(1+x)^2+y^2]}\right) dx\, dy. \end{aligned} \tag{6}$$

σ^2 is the mean noise power.

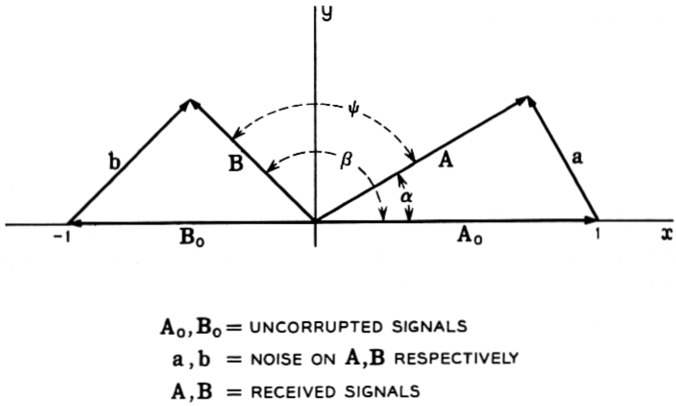

Fig. 2 — Phasor diagram of a pair of received signals.

In Appendix A it is shown that substituting (6) into (4) or (5) gives

$$\Pi = \frac{1}{8\pi\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-1)^2+y^2}{2\sigma^2}\right)\left\{\operatorname{erfc}\frac{\varepsilon+x}{\sqrt{2}\,\sigma\sqrt{x^2+y^2}} + \operatorname{erfc}\frac{-\varepsilon+x}{\sqrt{2}\,\sigma\sqrt{x^2+y^2}}\right\}dx\,dy. \tag{7}$$

This integral must be computed by numerical methods. The results of these computations are shown in Fig. 3.

III. SYSTEMS WITH AN IDEAL LIMITER

If an ideal limiter* is included in the system as shown in Fig. 1, (3) reduces to

$$v = \cos\psi. \tag{8}$$

In this case, it is simpler to perform the calculation of error probability from a consideration of the probability density of ψ itself. From Fig. 4 and the criteria for error listed in Section I, one obtains

$$\Pi = \frac{1}{2}\int_{\pi+\cos^{-1}\varepsilon}^{\pi+\cos^{-1}(-\varepsilon)} p(\psi)d\psi + \int_{\pi+\cos^{-1}(-\varepsilon)}^{\cos^{-1}\varepsilon} p(\psi)d\psi + \frac{1}{2}\int_{\cos^{-1}\varepsilon}^{\cos^{-1}(-\varepsilon)} p(\psi)d\psi \tag{9}$$

* By an ideal limiter is meant a device which removes all amplitude variation from the signal without affecting the phase. That is, it transforms the signal $A(t)e^{j\varphi(t)}$ into the signal $A_0e^{j\varphi(t)}$.

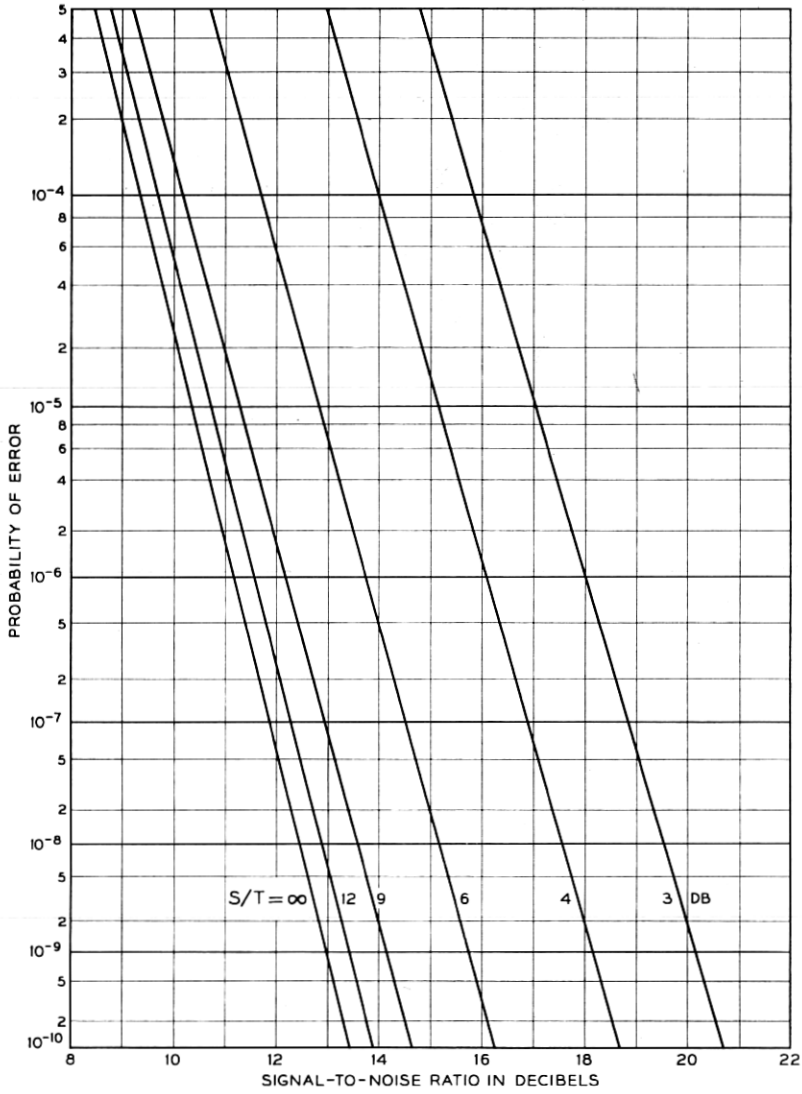

Fig. 3 — Probability of error in an unlimited DCPSK system.

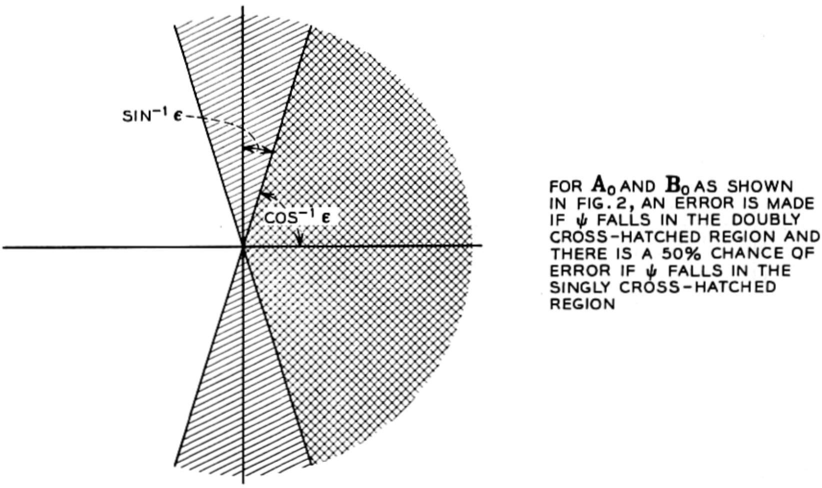

Fig. 4 — Geometric interpretation of the limits of integration over $p(\psi)$.

where $p(\psi)$ is the probability density function of ψ. The function $p(\psi)$ is derived in Appendix B. It is given by:

$$\begin{aligned} p(\psi) = & -\frac{1}{2\pi}\exp\left(-\frac{1}{\sigma^2}\right) + \frac{1}{\pi}\exp\left(\frac{1}{2\sigma^2}\right) \\ & - \frac{1}{4\pi\sigma^2}\int_{-(\pi/2)}^{\pi/2} \cos\alpha\cos(\alpha+\psi) \\ & \cdot \exp\left(-\frac{\sin^2\alpha + \sin^2(\alpha+\psi)}{2\sigma^2}\right) d\alpha \\ & + \frac{1}{4\pi\sigma^2}\int_{-(\pi/2)}^{\pi/2} \cos\alpha\cos(\alpha+\psi) \\ & \cdot \exp\left(-\frac{\sin^2\alpha + \sin^2(\alpha+\psi)}{2\sigma^2}\right) \operatorname{erf}\frac{\cos\alpha}{\sqrt{2}\,\sigma} \\ & \cdot \operatorname{erf}\frac{\cos(\alpha+\psi)}{\sqrt{2}\,\sigma}\, d\alpha. \end{aligned} \tag{10}$$

The probability of error is derived in Appendix C by substituting (10) into (9). The result is

$$\Pi = \frac{1}{\pi}\int_0^{\pi/2} \exp\left[-\frac{(1-\varepsilon^2)(1/2\sigma^2)}{1-\varepsilon\sin 2\theta}\right] d\theta. \tag{11}$$

Again the integral must be computed numerically. The results are shown in Fig. 5.

IV. CONCLUSIONS

From the form of (2) one sees that for the ideal case (no dead zone) the probability of error, $\Pi(S/N)$, will plot as a straight line if the scale of the abscissa of the graph paper is proportional to the logarithm of the logarithm of the scale of the ordinate.* A grid of this type is used in Figs. 3 and 5. From these figures one sees that the graph of Π vs S/N is very nearly linear and very nearly parallel to the graph of the ideal case even when the dead zone is significant.

Since the family of curves of $\Pi(S/N)$ for various values of S/T is a set of approximately parallel lines, it is convenient to define a "threshold effect noise figure," N_T, as the change of S/N which would give a degradation in error-probability equivalent to that due to the dead zone. Since these curves are not exactly parallel lines, N_T will be a function of the value of Π at which it is determined, but this effect is quite small over the range of values plotted in Figs. 3 and 5.

The values of N_T as a function of S/T (evaluated at $\Pi = 10^{-9}$) are plotted in Fig. 6 for both the limited and the unlimited cases. From Fig. 6 it is observed that $N_T \leqq 1$ db for S/T $\geqq$ 9.4 db and 5.8 db in the unlimited and limited cases, respectively. However, as S/T becomes smaller the effect of N_T on error-probability becomes quite important.

The difference in the value of N_T for the limited case and the unlimited case gives a measure of the improvement gained from the ideal limiting process. This improvement is very small for large S/T; however, for S/T = 8 db it already amounts to about 1-db improvement in N_T, for S/T = 6 db the improvement is about 2 db, and for S/T = 3 db this improvement exceeds 4 db.

V. ACKNOWLEDGMENTS

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APPENDIX A

Derivation of the Error-Rate Equation for Unlimited Systems

Substituting (6) into (4) or (5) and making the change of variable

* It may be worth mentioning that for 1 db $\leqq$ S/N $\leqq$ 10 db one can calculate the error rate for the ideal case directly on a slide rule by reading 2Π on the LL3 scale directly opposite S/N (in bels) on the L scale.

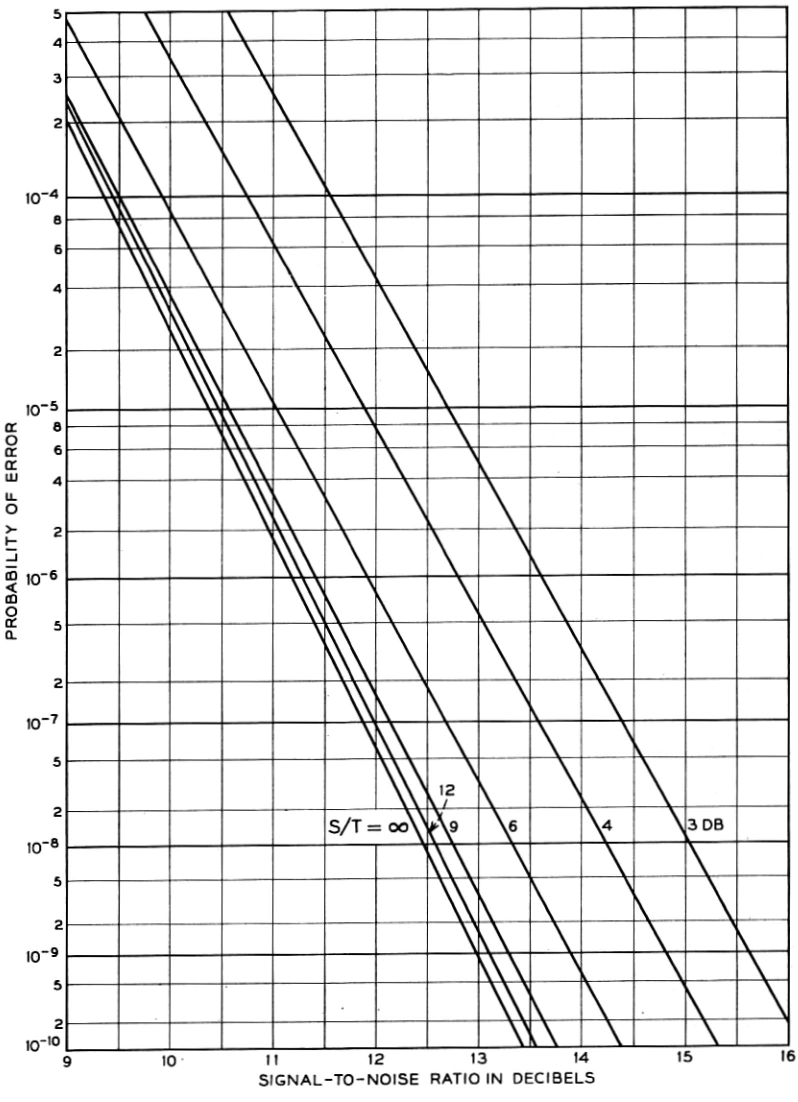

Fig. 5 — Probability of error in a limited DCPSK system.

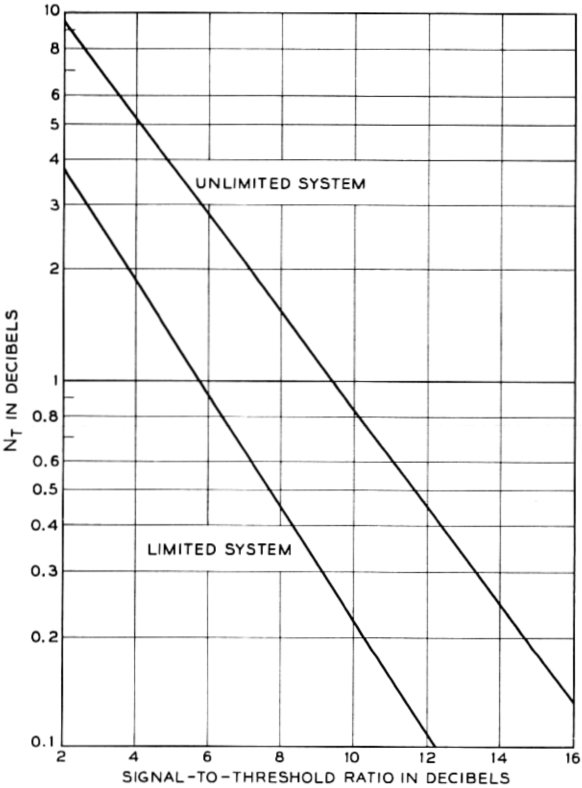

Fig. 6 — Threshold effect noise figure vs signal-to-threshold ratio.

$x \to x - 1$ gives

$$\begin{aligned}\Pi_\pm = {} & \frac{1}{(2\pi)^{3/2}\sigma^3}\,\frac{1}{2}\int_{-\varepsilon}^{\varepsilon}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left(-\dfrac{(x-1)^2+y^2}{2\sigma^2}\right)}{\sqrt{x^2+y^2}} \\ & \cdot\exp\left(-\frac{[v \mp x]^2}{2\sigma^2[x^2+y^2]}\right)dx\,dy\,dv \\ & + \frac{1}{(2\pi)^{3/2}\,\sigma^3}\int_{\varepsilon}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\exp\left(-\dfrac{(x-1)^2+y^2}{2\sigma^2}\right)}{\sqrt{x^2+y^2}} \\ & \cdot\exp\left(-\frac{[\mp v + x]^2}{2\sigma^2[x^2+y^2]}\right)dx\,dy\,dv.\end{aligned} \tag{12}$$

Making the substitution

$$u = \frac{v + x}{\sqrt{2}\,\sigma\,\sqrt{x^2 + y^2}}$$

and changing the order of integration gives

$$\prod_{\pm} = \frac{1}{4\pi\sigma^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left(-\frac{(x-1)^2 + y^2}{2\sigma^2}\right)\left\{\frac{1}{2}\operatorname{erf}\frac{\varepsilon + x}{\sqrt{2}\,\sigma\,\sqrt{x^2+y^2}} - \frac{1}{2}\operatorname{erf}\frac{-\varepsilon + x}{\sqrt{2}\,\sigma\,\sqrt{x^2+y^2}} + \operatorname{erfc}\frac{\varepsilon + x}{\sqrt{2}\,\sigma\,\sqrt{x^2+y^2}}\right\}dx\,dy \tag{13}$$

where erf (z) and erfc (z) have the usual definitions

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z \exp\,(-t^2)\,dt, \qquad \operatorname{erfc}\,(z) = 1 - \operatorname{erf}\,(z). \tag{14}$$

Equation (13) is readily simplified by means of (14) to give (7).

APPENDIX B

Derivation of the Probability Density $p(\psi)$

Bennett[5] and Davenport and Root[6] give the probability density of α (see Fig. 2) as

$$p(\alpha) = \frac{1}{2\pi}\exp\left(-\frac{1}{2\sigma^2}\right) + \frac{1}{2}\frac{1}{\sqrt{2\pi}\,\sigma}\cos\alpha\,\exp\left(-\frac{\sin^2\alpha}{2\sigma^2}\right)\left[\operatorname{erf}\frac{\cos\alpha}{\sqrt{2}\sigma} + 1\right]. \tag{15}$$

A similar argument gives

$$p'(\beta) = \frac{1}{2\pi}\exp\left(-\frac{1}{2\sigma^2}\right) + \frac{1}{2}\frac{1}{\sqrt{2\pi}\,\sigma}\cos\beta\,\exp\left(-\frac{\sin^2\beta}{2\sigma^2}\right)\left[\operatorname{erf}\frac{\cos\beta}{\sqrt{2}\sigma} - 1\right]. \tag{16}$$

From these results, one can evaluate $p(\psi)$.

$$\begin{aligned} p(\psi) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\delta(\psi + \alpha - \beta)p(\alpha)p'(\beta)d\alpha d\beta \\ &= \int_{-\pi}^{\pi} p(\alpha)p'(\alpha + \psi)d\alpha. \end{aligned} \tag{17}$$

Cahn[7] gives numerical values for $p(\psi)$ for a wide range of values of S/N. However, an expression for $p(\psi)$ suitable for calculation of error-probabilities is obtained as follows. Substituting (15) and (16) into (17) and performing the indicated multiplication gives

$$\begin{aligned} p(\psi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \Bigg\{ & \frac{1}{2\pi}\exp\left(-\frac{1}{\sigma^2}\right) + \frac{1}{2}\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{1}{2\sigma^2}\right)\cos\alpha \\ & \cdot\exp\left(-\frac{\sin^2\alpha}{2\sigma^2}\right)\operatorname{erf}\frac{\cos\alpha}{\sqrt{2}\sigma} + \frac{1}{2}\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{1}{2\sigma^2}\right) \\ & \cdot\cos(\alpha+\psi)\exp\left(-\frac{\sin^2(\alpha+\psi)}{2\sigma^2}\right)\operatorname{erf}\frac{\cos(\alpha+\psi)}{\sqrt{2}\,\sigma} \\ & + \frac{1}{4\sigma^2}\cos\alpha\cos(\alpha+\psi) \\ & \cdot\exp\left(-\frac{\sin^2\alpha+\sin^2(\alpha+\psi)}{2\sigma^2}\right) \\ & \cdot\operatorname{erf}\frac{\cos\alpha}{\sqrt{2\pi}\,\sigma}\operatorname{erf}\frac{\cos(\alpha+\psi)}{\sqrt{2}\,\sigma} - \frac{1}{4\sigma^2}\cos\alpha\cos(\alpha+\psi) \\ & \cdot\exp\left(-\frac{\sin^2\alpha+\sin^2(\alpha+\psi)}{2\sigma^2}\right)\Bigg\}\,d\alpha \end{aligned} \tag{18}$$

where the other terms vanish because they are odd with respect to $\alpha \to \alpha + \pi$.

Consider

$$I = \int_{-\pi}^{\pi} \cos(\alpha+\psi)\exp\left(-\frac{\sin^2(\alpha+\psi)}{2\sigma^2}\right)\operatorname{erf}\frac{\cos(\alpha+\psi)}{\sqrt{2}\,\sigma}\,d\alpha. \tag{19}$$

We observe that the integrand is periodic in period π and the result is therefore independent of ψ. Thus, we can write

$$I = 2\int_{-(\pi/2)}^{\pi/2} \cos\alpha\exp\left(-\frac{\sin^2\alpha}{2\sigma^2}\right)\operatorname{erf}\frac{\cos\alpha}{\sqrt{2}\,\sigma}\,d\alpha$$

which is easily integrated by means of the change of variable $x = \sin\alpha$ and the definition of the error function. The result is

$$I = 2\sqrt{2\pi}\,\sigma\left[1 - \exp\left(-\frac{1}{2\sigma^2}\right)\right]. \tag{20}$$

Substituting (20) into (18) and observing that all of the terms of the integrand of (18) are even with respect to $\alpha \to \alpha + \pi$, gives (10).

APPENDIX C

Derivation of the Error-Rate Equation for a Limited System

Designate the four terms on the right of (10) p_1, p_2, p_3, and p_4, respectively. We observe that

$$p_i(\psi) = p_i(\psi + \pi) \quad \text{for} \quad i = 1,2,4$$

$$p_3(\psi) = -p_3(\psi + \pi).$$

From this we have

$$\Pi = \int_{\pi+\cos^{-1}\varepsilon}^{\cos^{-1}\varepsilon} \{p_1(\psi) + p_2(\psi) + p_4(\psi)\} d\psi + \int_{\pi+\cos^{-1}(-\varepsilon)}^{\cos^{-1}\varepsilon} p_3(\psi) d\psi. \tag{21}$$

The integration of p_1 and p_2 is trivial. p_4 can be integrated by changing the order of integration and applying (20) twice. Performing the first integral in (21) and making a change of variable in the second integral gives

$$\Pi = \frac{1}{2} - \frac{1}{4\pi\sigma^2} \int_{-1}^{1} \int_{-\sqrt{1-x^2}\cos\varphi + x\sin\varphi}^{\sqrt{1-x^2}\cos\varphi + x\sin\varphi} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) dx dy, \tag{22}$$

where

$$\varphi = \sin^{-1} \varepsilon.$$

Inspection of the limits of integration in (22) reveals that the integration is to be performed over an ellipse centered at the origin. Since the integrand is spherically symmetric we can integrate over any quarter of the ellipse — say the first quadrant. Then

$$\Pi = \frac{1}{2} - \frac{1}{\pi\sigma^2} \int_0^{\pi/2} \int_0^{\frac{\cos\varphi}{\sqrt{1-\sin\varphi\sin^2\theta}}} \exp\left(-\frac{r^2}{2\sigma^2}\right) r dr d\theta,$$

where we have made the change of variable $x = r\cos\theta$, $y = r\sin\theta$. This can be easily integrated over r to give (11).

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